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Higher-Order Jacobsthal–Lucas Quaternions

Mine Uysal [1] and Engin Özkan [2,*]

[1] Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Erzincan 24100, Turkey
[2] Department of Mathematics, Faculty of Arts and Sciences, Erzincan Binali Yıldırım University, Erzincan 24100, Turkey
* Correspondence: eozkan@erzincan.edu.tr

Abstract: In this work, we define higher-order Jacobsthal–Lucas quaternions with the help of higher-order Jacobsthal–Lucas numbers. We examine some identities of higher-order Jacobsthal–Lucas quaternions. We introduce their basic definitions and properties. We give Binet's formula, Cassini's identity, Catalan's identity, d'Ocagne identity, generating functions, and exponential generating functions of the higher-order Jacobsthal–Lucas quaternions. We also give some relations between higher-order Jacobsthal and Jacobsthal–Lucas quaternions.

Keywords: Jacobsthal–Lucas quaternions; higher-order Jacobsthal–Lucas quaternions; Binet formula

MSC: 11B39; 11R52; 05A15

1. Introduction

Number sequences have attracted the attention of many researchers over the years. Number sequences have found many applications in nature and science and have been analyzed [1–3]. Many generalizations of these number sequences have been made and analyzed [4–7]. Some of these generalizations are related to Jacobsthal and Jacobsthal–Lucas numbers [8–13].

Quaternions are an expansion of complex numbers in mathematics. Quaternions were first discovered by William Rowan Hamilton in 1843 and applied to mathematics in three-dimensional space. Quaternions are not commutative. Hamilton defined a quaternion as the division of two oriented lines in three-dimensional space, or the division of two equivalent vectors [14].

Quaternions are used in applied mathematics, especially in computer science, physics, differential geometry, quantum physics, engineering, algebra and to calculate rotational motions in three-dimensional space.

Many studies have emerged by associating algebra with quaternions.

Horadam defined Fibonacci quaternions in 1963 and gave a generalization of these numbers [15]. In the studies of [16–20], different applications of quaternions of Fibonacci and Lucas numbers were studied, and their properties were examined.

Jacobsthal and Jacobsthal–Lucas quaternions are presented and given their many identities. Jacobsthal numbers and their generalizations have been given, and the properties of these numbers have been examined [21–23].

Keçilioğlu and Akkuş studied Fibonacci octonions as a generalization of quaternions [24].

In [25], Bilgici et al. defined Fibonacci sedenions and gave some identities of these numbers.

In [26], Çimen et al. introduced Jacobsthal and Jacobsthal–Lucas octonions as a generalization of quaternions.

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One of the studies conducted in this field is [18], where the higher-order Fibonacci quaternions were introduced. Additionally, Kızılateş et al. gave their properties and some identities related to these quaternions [18].

Özkan et al. defined higher-order Jacobsthal numbers as a new study of Jacobsthal numbers. Then, higher-order Jacobsthal quaternions were defined with the help of these numbers. The quaternion properties of these numbers and their properties as a sequence of numbers are examined [27].

In this work, we define higher-order Jacobsthal–Lucas numbers. Then we find the Binet formula and the recursive relation for these numbers. Then, we describe higher-order Jacobsthal–Lucas quaternions by using higher-order Jacobsthal–Lucas numbers. Moreover, we give the basic quaternion properties, such as the norm and conjugate. We also obtain the Binet formula and the generating function, which are important concepts in the number sequences for higher-order Jacobsthal–Lucas quaternions. We also calculate Cassini, Catalan, Vajda and d'Ocagne identities for higher-order Jacobsthal–Lucas quaternions. Finally, we give some relations between higher-order Jacobsthal and Jacobsthal–Lucas quaternions.

2. Definitions

The Jacobsthal numbers J_n are defined by

$$J_{n+2} = J_{n+1} + 2J_n,\ n \geq 0$$

with $J_0 = 0$ and $J_1 = 1$ [21].

Similarly, the Jacobsthal–Lucas numbers j_n are defined by

$$j_{n+2} = j_{n+1} + 2j_n,\ n \geq 0$$

with $j_0 = 2$ and $j_1 = 1$ [21].

Their Binet formulas are given by, respectively,

$$J_n = \frac{a^n - b^n}{a - b} = \frac{2^n - (-1)^n}{3}$$

and

$$j_n = a^n + b^n = 2^n - (-1)^n$$

where a and b are roots of the equation $x^2 - x - 2 = 0$.

Quaternions are defined in the following form. With p being a quaternion, p is written as

$$p = p_0 + p_1\mathbb{i} + p_2\mathbb{j} + p_3\mathbb{k}$$

where $p_0, p_1,\ p_2$ and p_3 are real numbers, and $\mathbb{i}, \mathbb{j}, \mathbb{k}$ are the main quaternions which satisfy rules in Table 1.

Table 1. The main multiplications.

	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
$\mathbb{i}$	-1	$\mathbb{k}$	$-\mathbb{j}$
$\mathbb{j}$	$-\mathbb{k}$	-1	$\mathbb{i}$
$\mathbb{k}$	$\mathbb{j}$	$-\mathbb{i}$	-1

Let p^* and $\|p\|$ show conjugate and norm of the quaternion p, respectively.

$$p^* = p_0 - p_1\mathbb{i} - p_2\mathbb{j} - p_3\mathbb{k},$$

$\|p\| = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}$. Note that $\|p\|^2 = pp^*$.

The higher-order Jacobsthal quaternions, denoted by $OJ_n^{(s)}$, are defined as follows

$$OJ_n^{(s)} = J_n^{(s)} + J_{n+1}^{(s)}\mathbb{i} + J_{n+2}^{(s)}\mathbb{j} + J_{n+3}^{(s)}\mathbb{k},$$

where $\mathbb{i}, \mathbb{j}$ and $\mathbb{k}$ are quaternion units and $J_n^{(s)}$ is a higher-order Jacobsthal number [24].

3. Results

3.1. Higher-Order Jacobsthal–Lucas Numbers

Definition 1. *The higher-order Jacobsthal–Lucas numbers are defined by*

$$j_n^{(s)} = \frac{j_{ns}}{j_s} = \frac{a^{ns} + b^{ns}}{a^s + b^s}. \tag{1}$$

Note that for $s = 1$, higher-order Jacobsthal–Lucas number $j_n^{(1)}$ is the ordinary Jacobsthal–Lucas numbers.

Theorem 1. *The higher-order Jacobsthal–Lucas numbers provide the following equation*

$$j_{n+1}^{(s)} = j_s j_n^{(s)} - (-2)^s j_{n-1}^{(s)}.$$

Proof. By using the Binet formula, we obtain

$$j_s j_n^{(s)} - (-2)^s j_{n-1}^{(s)} = (a^s + b^s)\left(\frac{a^{ns} + b^{ns}}{a^s + b^s}\right) - (-2)^s \frac{a^{(n-1)s} + b^{(n-1)s}}{a^s + b^s}$$

Since $ab = -2,$ we have

$$\begin{aligned} j_s j_n^{(s)} - (-2)^s j_{n-1}^{(s)} &= (a^s + b^s)\left(\tfrac{a^{ns}+b^{ns}}{a^s+b^s}\right) - (ab)^s \tfrac{a^{(n-1)s}+b^{(n-1)s}}{a^s+b^s} \\ &= \tfrac{a^{ns+s}+a^s b^{ns}+b^s a^{ns}+b^{ns+s}-a^{ns}b^s-b^{ns}a^s}{a^s+b^s} \\ &= \tfrac{a^{(n+1)s}+b^{(n+1)s}}{a^s+b^s} = j_{n+1}^{(s)}. \end{aligned}$$

Thus, the desired is obtained. □

Theorem 2. *There are the following equations for $J_n^{(s)}$ and $j_n^{(s)}$.*

$$(i)\ J_n^{(s)} j_n^{(s)} = j_n^{(2s)},$$

$$(\boldsymbol{ii})\ J_n^{(s)} + j_n^{(s)} = \frac{J_{n+1}^{(s)}}{j_s},$$

$$(\boldsymbol{iii})\ J_n^{(s)} - j_n^{(s)} = \frac{2(-2)^s J_{n-1}^{(s)}}{j_s}.$$

Proof. By using the Binet formula, we obtain

$$\begin{aligned} (\boldsymbol{i})\ J_n^{(s)} j_n^{(s)} &= \left(\tfrac{a^{ns}-b^{ns}}{a^s-b^s}\right)\left(\tfrac{a^{ns}+b^{ns}}{a^s+b^s}\right) \\ &= \left(\tfrac{a^{2ns}-b^{2ns}}{a^{2s}-b^{2s}}\right) = \left(\tfrac{(a^n)^{2s}-(b^n)^{2s}}{a^{2s}-b^{2s}}\right) = j_n^{(2s)}. \end{aligned}$$

The proofs of (ii) and (iii) are performed similarly to that of (i). □

3.2. Higher-Order Jacobsthal–Lucas Quaternions

In this section, we define higher-order Jacobsthal–Lucas quaternions and give some of their identities.

Definition 2. *The higher-order Jacobsthal–Lucas quaternions, denoted by* $Oj_n^{(s)}$, *are defined as*

$$Oj_n^{(s)} = j_n^{(s)} + j_{n+1}^{(s)}\mathbb{i} + j_{n+2}^{(s)}\mathbb{j} + j_{n+3}^{(s)}\mathbb{k} \tag{2}$$

where $\mathbb{i}, \mathbb{j}$ *and* $\mathbb{k}$ *are quaternion units and* $j_n^{(s)}$ *is a higher-order Jacobsthal–Lucas number.*

If we take $s = 1$ *in* (2), *then we obtain the Jacobsthal–Lucas quaternions.*

Definition 3. *The real and imaginary parts of the higher-order Jacobsthal–Lucas quaternions are as follows, respectively:*

$$Re\left(Oj_n^{(s)}\right) = j_n^{(s)}$$

and

$$Im\left(Oj_n^{(s)}\right) = j_{n+1}^{(s)}\mathbb{i} + j_{n+2}^{(s)}\mathbb{j} + j_{n+3}^{(s)}\mathbb{k}.$$

Definition 4. *The conjugate of the higher-order Jacobsthal–Lucas quaternion is denoted* by $Oj_n^{(s)^*}$ *and defined as*

$$Oj_n^{(s)^*} = j_n^{(s)} - j_{n+1}^{(s)}\mathbb{i} - j_{n+2}^{(s)}\mathbb{j} - j_{n+3}^{(s)}\mathbb{k}. \tag{3}$$

Definition 5. *The norm of the higher-order Jacobsthal–Lucas quaternion is denoted by* $N\left(Oj_n^{(s)}\right)$ *and defined as*

$$N\left(Oj_n^{(s)}\right) = Oj_n^{(s)}Oj_n^{(s)^*} = \left(j_n^{(s)}\right)^2 + \left(j_{n+1}^{(s)}\right)^2 + \left(j_{n+2}^{(s)}\right)^2 + \left(j_{n+3}^{(s)}\right)^2. \tag{4}$$

Proposition 1: *For the higher-order Jacobsthal–Lucas quaternion, we have*

$$Oj_n^{(s)} + Oj_n^{(s)^*} = 2j_n^{(s)}.$$

Proof. From Definition 3, we obtain

$$Oj_n^{(s)} + Oj_n^{(s)^*} = j_n^{(s)} + j_{n+1}^{(s)}\mathbb{i} + j_{n+2}^{(s)}\mathbb{j} + j_{n+3}^{(s)}\mathbb{k} + j_n^{(s)} - j_{n+1}^{(s)}\mathbb{i} - j_{n+2}^{(s)}\mathbb{j} - j_{n+3}^{(s)}\mathbb{k} = 2j_n^{(s)}.$$

□

Proposition 2. *The higher-order Jacobsthal–Lucas quaternions satisfy the following identity:*

$$\left(Oj_n^{(s)}\right)^2 = Oj_n^{(s)}Oj_n^{(s)^*} + 2j_n^{(s)}Oj_n^{(s)}.$$

Proof. By using (2), we obtain

$$\left(Oj_n^{(s)}\right)^2 = \left(j_n^{(s)} + j_{n+1}^{(s)}\mathbb{i} + j_{n+2}^{(s)}\mathbb{j} + j_{n+3}^{(s)}\mathbb{k}\right)\left(j_n^{(s)} + j_{n+1}^{(s)}\mathbb{i} + j_{n+2}^{(s)}\mathbb{j} + j_{n+3}^{(s)}\mathbb{k}\right)$$

$= -\left(\left(j_n^{(s)}\right)^2 + \left(j_{n+1}^{(s)}\right)^2 + \left(j_{n+2}^{(s)}\right)^2 + \left(j_{n+3}^{(s)}\right)^2\right) + 2j_n^{(s)}\left(j_n^{(s)} + j_{n+1}^{(s)}\mathbb{i} + j_{n+2}^{(s)}\mathbb{j} + j_{n+3}^{(s)}\mathbb{k}\right)$, from (4) $= Oj_n^{(s)}Oj_n^{(s)^*} + 2j_n^{(s)}Oj_n^{(s)}$. □

Theorem 3. ***(Binet formula)*** *The Binet formula of the higher-order Jacobsthal–Lucas quaternions is defined by*

$$Oj_n^{(s)} = \frac{(a^s)^n\hat{a} + (b^s)^n\hat{b}}{a^s + b^s} \tag{5}$$

where $\hat{a} = 1 + a^s\mathbb{i} + a^{2s}\mathbb{j} + a^{3s}\mathbb{k}$ *and* $\hat{b} = 1 + b^s\mathbb{i} + b^{2s}\mathbb{j} + b^{3s}\mathbb{k}$.

Proof. Using (1) and (2), we obtain

$$\begin{aligned} Oj_n^{(s)} &= j_n^{(s)} + j_{n+1}^{(s)}\mathbb{i} + j_{n+2}^{(s)}\mathbb{j} + j_{n+3}^{(s)}\mathbb{k} \\ &= \tfrac{(a^s)^n}{a^s+b^s}\left[1 + a^s\mathbb{i} + a^{2s}\mathbb{j} + a^{3s}\mathbb{k}\right] + \tfrac{(b^s)^n}{a^s+b^s}\left[1 + b^s\mathbb{i} + b^{2s}\mathbb{j} + b^{3s}\mathbb{k}\right] \\ &= \tfrac{(a^s)^n\hat{a}}{a^s+b^s} + \tfrac{(b^s)^n\hat{b}}{a^s+b^s} = \tfrac{(a^s)^n\hat{a}+(b^s)^n\hat{b}}{a^s+b^s}. \end{aligned}$$

□

Theorem 4. *There is the following recurrence relation for higher-order Jacobsthal–Lucas quaternions*

$$Oj_{n+1}^{(s)} = j_sOj_n^{(s)} - (-2)^sOj_{n-1}^{(s)} \tag{6}$$

Proof. Let us write the right-hand side of the equation according to (5).

$$j_sOj_n^{(s)} - (-2)^sOj_{n-1}^{(s)} = (a^s + b^s)\left(\frac{a^{sn}\hat{a} + b^{sn}\hat{b}}{a^s + b^s}\right) - (-2)^s\left(\frac{a^{sn-s}\hat{a} + b^{sn-s}\hat{b}}{a^s + b^s}\right)$$

Since $ab = -2$, we have

$$\begin{aligned} &j_sOj_n^{(s)} - (-2)^sOj_{n-1}^{(s)} \\ &= \left(a^s\hat{a} + b^s\hat{b}\right)\left(\tfrac{a^{sn}\hat{a}+b^{sn}\hat{b}}{a^s+b^s}\right) - (ab)^s\left(\tfrac{a^{sn-s}\hat{a}+b^{sn-s}\hat{b}}{a^s+b^s}\right) \\ &= \tfrac{a^{sn+s}\hat{a}+a^sb^{sn}\hat{b}+b^sa^{sn}\hat{a}+b^{sn+s}\hat{b}-a^{sn}b^s\hat{a}-a^sb^{sn}\hat{b}}{a^s+b^s} \\ &= \tfrac{a^{sn+s}\hat{a}+b^{sn+s}\hat{b}}{a^s+b^s} = Oj_{n+1}^{(s)}. \end{aligned}$$

Thus, the proof is completed. □

Lemma 1. *We have*

$(\boldsymbol{i})$ $\hat{a} + \hat{b} = 2 + j_s\mathbb{i} + j_{2s}\mathbb{j} + j_{3s}\mathbb{k}$,$(\boldsymbol{ii})$ $\hat{a}b^s + \hat{b}a^s = j_s + (-2)^s(2\mathbb{i} + j_s\mathbb{j} + j_{2s}\mathbb{k})$.

Theorem 5. *If the indicess and n are expanded to negative numbers, then we have*

$$\begin{aligned}(\boldsymbol{i})\ Oj_{-n}^{(s)} &= (-2)^{-sn}\tfrac{(b^s)^n\hat{a}+(a^s)^n\hat{b}}{a^s+b^s},\\ (\boldsymbol{ii})\ Oj_{-n}^{(-s)} &= (-2)^{s}\tfrac{(a^s)^n\hat{a}+(b^s)^n\hat{b}}{a^s+b^s},\\ (\boldsymbol{iii})\ Oj_{n}^{(-s)} &= (-2)^{sn-s}Oj_{-n}^{(s)}.\end{aligned}$$

Proof. By using (5), we obtain

$$\begin{aligned}(\boldsymbol{i})\ Oj_{-n}^{(s)} &= \tfrac{(a^s)^{-n}\hat{a}+(b^s)^{-n}\hat{b}}{a^s+b^s}\\ = \tfrac{\frac{\hat{a}}{a^{sn}}+\frac{\hat{b}}{b^{sn}}}{a^s+b^s} &= \tfrac{(b^s)^n\hat{a}+(a^s)^n\hat{b}}{(ab)^{sn}(a^s+b^s)} \text{ since } ab=-2,\\ &= (-2)^{-sn}\tfrac{(b^s)^n\hat{a}+(a^s)^n\hat{b}}{a^s+b^s}.\end{aligned}$$

Equations (ii) and (iii) are made similarly to that of (i). □

Theorem 6. *The generating function of the higher-order Jacobsthal–Lucas quaternions is given by*

$$G^{(s)}(x) = \frac{2+j_s\mathbb{i}+j_{2s}\mathbb{j}+j_{3s}\mathbb{k}-\left(j_s+(-2)^s(2\mathbb{i}+j_s\mathbb{j}+j_{2s}\mathbb{k})\right)x}{j_s\left(1-j_sx+(-2)^sx^2\right)}.$$

Proof.

$$\begin{aligned}G^{(s)}(x) &= \sum_{n=0}^{\infty} Oj_n^{(s)}x^n\\ &= \sum_{n=0}^{\infty}\left[\tfrac{(a^n)^s+(b^n)^s}{a^s+b^s}+\tfrac{\left(a^{n+1}\right)^s+\left(b^{n+1}\right)^s}{a^s+b^s}\mathbb{i}+\tfrac{\left(a^{n+2}\right)^s+\left(b^{n+2}\right)^s}{a^s+b^s}\mathbb{j}+\tfrac{\left(a^{n+3}\right)^s+\left(b^{n+3}\right)^s}{a^s+b^s}\mathbb{k}\right]x^n\\ &= \tfrac{1}{a^s+b^s}\sum_{n=0}^{\infty}(a^n)^s\left(1+a^s\mathbb{i}+a^{2s}\mathbb{j}+a^{3s}\mathbb{k}\right)x^n+\tfrac{1}{a^s+b^s}\sum_{n=0}^{\infty}(b^n)^s\left(1+b^s\mathbb{i}+b^{2s}\mathbb{j}+b^{3s}\mathbb{k}\right)x^n\\ &= \tfrac{1}{a^s+b^s}\sum_{n=0}^{\infty}(a^n)^s x^n\hat{a}+\tfrac{1}{a^s+b^s}\sum_{n=0}^{\infty}(b^n)^s x^n\hat{b}\\ &= \tfrac{\hat{a}}{a^s+b^s}\sum_{n=0}^{\infty}(a^s x)^n+\tfrac{\hat{b}}{a^s+b^s}\sum_{n=0}^{\infty}(b^s x)^n\\ &= \left(\tfrac{\hat{a}}{a^s+b^s}\right)\left(\tfrac{1}{1-a^s x}\right)+\left(\tfrac{\hat{b}}{a^s+b^s}\right)\left(\tfrac{1}{1-b^s x}\right)\\ &= \tfrac{\hat{a}+\hat{b}-\left(\hat{a}b^s+\hat{b}a^s\right)x}{(a^s+b^s)\left(1-(a^s+b^s)x+(-2)^s x^2\right)}.\end{aligned}$$

From Lemma 1, we have

$$G^{(s)}(x) = \frac{2+j_s\mathbb{i}+j_{2s}\mathbb{j}+j_{3s}\mathbb{k}-\left(j_s+(-2)^s(2\mathbb{i}+j_s\mathbb{j}+j_{2s}\mathbb{k})\right)x}{j_s\left(1-j_sx+(-2)^sx^2\right)}.$$

Thus, the proof is obtained. □

Theorem 7. *The sum of the higher-order Jacobsthal–Lucas quaternion is*

$$\begin{aligned}&SOj_n^{(s)}\\ &= \sum_{n=0}^{\infty} Oj_n^{(s)} = \tfrac{2-j_s+\left(j_s+(-2)^{s+1}\right)\mathbb{i}+\left(j_{2s}-(-2)^s j_s\right)\mathbb{j}+\left(j_{3s}-(-2)^s j_{2s}\right)\mathbb{k}}{j_s\left(1-j_s+(-2)^s\right)}.\end{aligned}$$

Proof. If we take for $x = 1$ in Theorem 6, the proof is finished. □

Theorem 8. *For $n, m \in \mathbb{Z}$, we have*

$$\sum_{n=0}^{\infty} Oj_{n+m}^{(s)} x^n = \frac{Oj_n^{(s)} + (-2)^s Oj_{m-1}^{(s)} x}{1 + j_s x + (-2)^s x^2}.$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} Oj_{n+m}^{(s)} x^n = \sum_{n=0}^{\infty} \left(\frac{(a^s)^{n+m}\hat{a} + (b^s)^{n+m}\hat{b}}{a^s+b^s} \right) x^n \\
&= \sum_{n=0}^{\infty} \frac{(a^s)^{n+m}\hat{a}}{a^s+b^s} x^n + \sum_{n=0}^{\infty} \frac{(b^s)^{n+m}\hat{b}}{a^s+b^s} x^n = \frac{\hat{a}a^{sm}}{a^s+b^s} \sum_{n=0}^{\infty} a^{sn} x^n + \frac{\hat{b}b^{sm}}{a^s+b^s} \sum_{n=0}^{\infty} b^{sn} x^n \\
&= \left(\frac{\hat{a}a^{sm}}{a^s+b^s} \right) \left(\frac{1}{1-a^s x} \right) + \left(\frac{\hat{b}b^{sm}}{a^s+b^s} \right) \left(\frac{1}{1-b^s x} \right) \\
&= \left(\frac{1}{a^s+b^s} \right) \left[\frac{\hat{a}a^{sm} - \hat{a}a^{sm}b^s x + \hat{b}b^{sm} - \hat{b}b^{sm}a^s x}{1-(b^s+a^s)x+(ab)^s x^2} \right] \\
&= \left(\frac{1}{a^s+b^s} \right) \left[\frac{\hat{a}(a^s)^m + \hat{b}(b^s)^m}{1-j_s x+(-2)^s x^2} - \frac{a^s b^s \left(\hat{a}(a^s)^{m-1} + \hat{b}(b^s)^{m-1} \right) x}{1-j_s x+(-2)^s x^2} \right] \\
&= \left[\frac{Oj_m^{(s)}}{1-j_s x+(-2)^s x^2} + \frac{(-2)^s Oj_{m-1}^{(s)} x}{1-j_s x+(-2)^s x^2} \right] \\
&= \frac{Oj_n^{(s)} + (-2)^s Oj_{m-1}^{(s)} x}{1+j_s x+(-2)^s x^2}.
\end{aligned}$$

So, the proof is done. □

Theorem 9. *The exponential generating function of $Oj_n^{(s)}$ is given by*

$$\sum_{n=0}^{\infty} Oj_n^{(s)} \frac{x^n}{n!} = \frac{\hat{a}e^{a^s x} + \hat{b}e^{b^s x}}{a^s + b^s}.$$

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} Oj_n^{(s)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{(a^s)^n \hat{a} + (b^s)^n \hat{b}}{a^s+b^s} \right) \frac{x^n}{n!} \\
&= \frac{1}{a^s+b^s} \sum_{n=0}^{\infty} \frac{(a^s)^n \hat{a} x^n}{n!} + \frac{1}{a^s+b^s} \sum_{n=0}^{\infty} \frac{(b^s)^n \hat{b} x^n}{n!} \\
&= \frac{\hat{a}}{a^s+b^s} \sum_{n=0}^{\infty} \frac{(a^s x)^n}{n!} + \frac{\hat{b}}{a^s+b^s} \sum_{n=0}^{\infty} \frac{(b^s x)^n}{n!} \\
&= \frac{\hat{a}e^{a^s x} + \hat{b}e^{b^s x}}{a^s+b^s}.
\end{aligned}$$

So, the proof is completed. □

3.3. Some Identities of Higher-Order Jacobsthal–Lucas Quaternions

In this section, we give some identities of higher-order Jacobsthal–Lucas quaternions.

Lemma 2. *There are the following equations*

$$\hat{a}\hat{b} = k - \rho l \tag{7}$$

and

$$\hat{b}\hat{a} = k + \rho l \tag{8}$$

where $k = 1 - (-2)^s - (-2)^{2s} - (-2)^{3s} + j_s \mathbb{i} + j_{2s} \mathbb{j} + j_{3s} \mathbb{k}$, $l = (-2)^{2s} \mathbb{i} - (-2)^s j_s \mathbb{j} + (-2)^s \mathbb{k}$ and $\rho = a^s - b^s$.

Proof.

$$\begin{aligned}
&\hat{a}\hat{b} = \left(1 + a^s\mathbb{i} + a^{2s}\mathbb{j} + a^{3s}\mathbb{k}\right)\left(1 + b^s\mathbb{i} + b^{2s}\mathbb{j} + b^{3s}\mathbb{k}\right)\\
&= 1 + b^s\mathbb{i} + b^{2s}\mathbb{j} + b^{3s}\mathbb{k} + a^s\mathbb{i} - a^s b^s + a^s b^{2s}\mathbb{k} - a^s b^{3s}\mathbb{j} + a^{2s}\mathbb{j} - a^{2s}b^s\mathbb{k} - a^{2s}b^{2s}\\
&+ a^{2s}b^{3s}\mathbb{i} + a^{3s}\mathbb{k} + a^{3s}b^s\mathbb{j} - a^{3s}b^{2s}\mathbb{i} - a^{3s}b^{3s}\\
&= 1 + b^s\mathbb{i} + b^{2s}\mathbb{j} + b^{3s}\mathbb{k} + a^s\mathbb{i} - (-2)^s + a^s b^{2s}\mathbb{k} - a^s b^{3s}\mathbb{j} + a^{2s}\mathbb{j} - a^{2s}b^s\mathbb{k} - (-2)^{2s}\\
&+ a^{2s}b^{3s}\mathbb{i} + a^{3s}\mathbb{k} + a^{3s}b^s\mathbb{j} - a^{3s}b^{2s}\mathbb{i} - (-2)^{3s}\\
&= \left(1 - (-2)^s - (-2)^{2s} - (-2)^{3s}\right) + \left(a^s + b^s + a^{2s}b^{3s} - a^{3s}b^{2s}\right)\mathbb{i}\\
&+ \left(a^{2s} + b^{2s} + a^{3s}b^s - a^s b^{3s}\right)\mathbb{j} + \left(a^{3s} + b^{3s} + a^s b^{2s} - a^{2s}b^s\right)\mathbb{k}\\
&= \left(1 - (-2)^s - (-2)^{2s} - (-2)^{3s} + j_s\mathbb{i} + j_{2s}\mathbb{j} + j_{3s}\mathbb{k}\right) - (-2)^{2s}(a^s - b^s)\mathbb{i}\\
&+ (-2)^s\left(a^{2s} - b^{2s}\right)\mathbb{j} - (-2)^s(a^s - b^s)\mathbb{k}\\
&= \left(1 - (-2)^s - (-2)^{2s} - (-2)^{3s} + j_s\mathbb{i} + j_{2s}\mathbb{j} + j_{3s}\mathbb{k}\right) - (-2)^s(a^s - b^s)\left((-2)^s\mathbb{i} - j_s\mathbb{j} + \mathbb{k}\right)\\
&= k - \rho l.
\end{aligned}$$

Equation (8) can be similarly proved. □

Theorem 10. ***(Vajda identity)*** *For any $n, m, r \in \mathbb{Z}$, we have*

$$Oj_{n+m}^{(s)} Oj_{n+r}^{(s)} - Oj_n^{(s)} Oj_{n+m+r}^{(s)} = -(-2)^{sn}\rho^2 J_m^{(s)}(j_s)^{-2}\left[kJ_r^{(s)} + lj_{sr}\right].$$

Proof.

$$\begin{aligned}
&Oj_{n+m}^{(s)} Oj_{n+r}^{(s)} - Oj_n^{(s)} Oj_{n+m+r}^{(s)}\\
&\qquad = \left(\frac{(a^s)^{n+m}\hat{a} + (b^s)^{n+m}\hat{b}}{a^s + b^s}\right)\left(\frac{(a^s)^{n+r}\hat{a} + (b^s)^{n+r}\hat{b}}{a^s + b^s}\right)\\
&\qquad - \left(\frac{(a^s)^{n}\hat{a} + (b^s)^{n}\hat{b}}{a^s + b^s}\right)\left(\frac{(a^s)^{n+m+r}\hat{a} + (b^s)^{n+m+r}\hat{b}}{a^s + b^s}\right)\\
&= \left(\frac{1}{(a^s + b^s)^2}\right)\left((a^s)^{n+m}\hat{a}(b^s)^{n+r}\hat{b} + (b^s)^{n+m}\hat{b}(a^s)^{n+r}\hat{a} - (a^s)^n\hat{a}(b^s)^{n+m+r}\hat{b} - (b^s)^n\hat{b}(a^s)^{n+m+r}\hat{a}\right)\\
&= \frac{1}{(a^s + b^s)^2}\left(\hat{a}\hat{b}a^{ns}b^{ns+rs}\left((a^s)^m - (b^s)^m\right) + \hat{b}\hat{a}b^{sn}a^{ns+rs}\left((b^s)^m - (a^s)^m\right)\right)\\
&= \frac{1}{(a^s + b^s)^2}\left(\hat{a}\hat{b}(-2)^{ns}b^{rs}\left((a^s)^m - (b^s)^m\right) - \hat{b}\hat{a}(-2)^{ns}a^{rs}\left((a^s)^m\right) - (b^s)^m\right)\\
&= \frac{1}{(a^s + b^s)^2}\left((-2)^{ns}\left((a^s)^m - (b^s)^m\right)\left(\hat{a}\hat{b}b^{rs} - \hat{b}\hat{a}a^{rs}\right)\right)\text{from Lemma 3.1,}\\
&= \frac{(-2)^{ns}\left((a^s)^m - (b^s)^m\right)}{(a^s + b^s)^2}\left[kb^{rs} - \rho l b^{rs} - ka^{rs} - \rho l a^{rs}\right]\\
&= \frac{(-2)^{ns}\left((a^s)^m - (b^s)^m\right)}{(a^s + b^s)^2}\left[-k(a^{rs} - b^{rs}) - \rho l(a^{rs} + b^{rs})\right]\\
&= \frac{(-2)^{ns}\left((a^s)^m - (b^s)^m\right)}{(a^s + b^s)^2}\left[-k\rho J_r^{(s)} - \rho l j_{sr}\right]\\
&= -(-2)^{sn}\rho^2 J_m^{(s)}(j_s)^{-2}\left[kJ_r^{(s)} + lj_{sr}\right].
\end{aligned}$$

So, the desired is obtained. □

Corollary 1. ***(Catalan identity)*** *For $n, r \in \mathbb{Z}$, we obtain*

$$Oj_{n-r}^{(s)} Oj_{n+r}^{(s)} - \left(Oj_n^{(s)}\right)^2 = -(-2)^{sn}\rho^2 J_{-r}^{(s)}(j_s)^{-2}\left[kJ_r^{(s)} + lj_{sr}\right].$$

Proof. The proof is obtained from the special case of Vajda identity.
For $m = -r$, we get

$$Oj_{n-r}^{(s)} Oj_{n+r}^{(s)} - \left(Oj_n^{(s)}\right)^2 = -(-2)^{sn}\rho^2 J_{-r}^{(s)}(j_s)^{-2}\left[kJ_r^{(s)} + lj_{sr}\right].$$

□

Corollary 2. ***(Cassini identity)*** *For $n \in \mathbb{Z}$, we obtain*

$$Oj_{n-1}^{(s)} Oj_{n+1}^{(s)} - \left(Oj_n^{(s)}\right)^2 = (-2)^{s(n-1)} \rho^2 (j_s)^{-2} [k + l j_s].$$

Proof. For r = 1 and m = −1 in Vajda identity, we have

$$\begin{aligned} Oj_{n-1}^{(s)} Oj_{n+1}^{(s)} - \left(Oj_n^{(s)}\right)^2 &= -(-2)^{sn} \rho^2 J_{-1}^{(s)} (j_s)^{-2} \left[k J_1^{(s)} + l j_s\right] \\ &= (-2)^{s(n-1)} \rho^2 (j_s)^{-2} [k + l j_s]. \end{aligned}$$

□

Corollary 3. ***(d'Ocagne identity)*** *We have*

$$Oj_k^{(s)} Oj_{n+1}^{(s)} - Oj_n^{(s)} Oj_{k+1}^{(s)} = -(-2)^{sn} \rho^2 J_{k-n}^{(s)} (j_s)^{-2} [k + l j_s].$$

Proof: If we take $m + n = k$ and $r = 1$ in Vajda identity, the following is obtained.

$$Oj_k^{(s)} Oj_{n+1}^{(s)} - Oj_n^{(s)} Oj_{k+1}^{(s)} = -(-2)^{sn} \rho^2 J_{k-n}^{(s)} (j_s)^{-2} [k + l j_s].$$

□

Now, we give some identities between higher-order Jacobsthal and Jacobsthal–Lucas quaternions.

Theorem 11. *We have*

$$(\boldsymbol{i})\ OJ_n^{(s)} + Oj_n^{(s)} = \frac{OJ_{n+1}^{(s)}}{j_s},$$
$$(\boldsymbol{ii})\ OJ_n^{(s)} - Oj_n^{(s)} = \frac{2(-2)^s OJ_{n-1}^{(s)}}{j_s}.$$

Proof. We use Theorem 2 for the proof.

$$\begin{aligned} &(\boldsymbol{i})\ OJ_n^{(s)} + Oj_n^{(s)} = J_n^{(s)} + J_{n+1}^{(s)} \mathbb{i} + J_{n+2}^{(s)} \mathbb{j} + J_{n+3}^{(s)} \mathbb{k} + j_n^{(s)} + j_{n+1}^{(s)} \mathbb{i} + j_{n+2}^{(s)} \mathbb{j} + j_{n+3}^{(s)} \mathbb{k} \\ &= \left(J_n^{(s)} + j_n^{(s)}\right) + \left(J_{n+1}^{(s)} + j_{n+1}^{(s)}\right) \mathbb{i} + \left(J_{n+2}^{(s)} \mathbb{j} + j_{n+2}^{(s)}\right) \mathbb{j} + \left(J_{n+3}^{(s)} \mathbb{k} + j_{n+3}^{(s)}\right) \mathbb{k} \\ &= \frac{J_{n+1}^{(s)}}{j_s} + \frac{J_{n+2}^{(s)}}{j_s} \mathbb{i} + \frac{J_{n+3}^{(s)}}{j_s} \mathbb{j} + \frac{J_{n+4}^{(s)}}{j_s} \mathbb{k} \\ &= \frac{OJ_{n+1}^{(s)}}{j_s}. \end{aligned}$$

The proof of (ii) is performed similarly to that of (i). □

4. Discussion

Based on this study, as an application of these numbers, hyper complex numbers whose parts are higher-order Jacobsthal–Lucas numbers can be defined.

5. Conclusions

In this paper, we studied higher-order Jacobsthal–Lucas quaternions. We defined the higher-order Jacobsthal–Lucas numbers and gave the recurrence relation. Using higher-order Jacobsthal numbers, we introduced higher-order Jacobsthal–Lucas numbers. Then we gave concepts of the norm and conjugate for these numbers in terms of the quaternion. Additionally, we gave the recurrence relation, the Binet formula, the generating function, and the sum formula for these numbers. We obtained Cassini, Catalan, Vajda and d'Ocagne identities, which are important in number sequences. We gave some identities between higher-order Jacobsthal and Jacobsthal–Lucas quaternions.

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