



Article Wave Patterns inside Transparent Scatterers

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Abstract: It may happen that under a certain wave interrogation, a medium scatterer produces no scattering. In such a case, the scattering field is trapped inside the scatterer and forms a certain interior resonant mode. We are concerned with the behavior of the wave propagation inside a transparent scatterer. It turns out that the study can be boiled down to analyzing the interior transmission eigenvalue problem. For isotropic mediums, it is shown in a series of recent works that the transmission eigenfunctions possess rich patterns. In this paper, we show that those spectral patterns also hold for anisotropic mediums.

Keywords: acoustic scattering; transparency; invisibility; wave pattern; vanishing; boundary localization

MSC: 35P30; 35P10; 65N25

1. Introduction

Achieving "transparency" or "invisibility" has been a fascinating topic. This subject has a huge impact on technological and industrial applications. Regarding the invisibility cloaking, transformation optics [1-5] is one of the most notable methods; see also [6-16] for more related mathematical works. This cloak reduces both back-scattering and forwardscattering with metamaterial technology. It is interesting to note that the cloaking layer can steer the wave propagation so that it slides over the surface of the object being cloaked and returns to its original path after passing through the cloaking layer. Another approach to realize the invisibility cloaking is to utilize plasmonic structures [17–22]. The transparencies/invisibilities mentioned above are "artificial" in the sense that they are achieved by artificially engineered material structures. In this paper, we are concerned with "natural" transparencies/invisibilities, namely, the phenomena that occur for generic natural materials. It turns out that the study can be boiled down to analyzing the so-called interior transmission eigenvalue problem. In fact, if transparency occurs, the scattering wave is trapped inside the scatterer, which together with the interrogating/incident wave forms a pair of transmission eigenfunctions. The corresponding wavenumber is an interior transmission eigenvalue.

The study of the interior transmission eigenvalue problem has a long and colorful history. It is a class of non-elliptic and non-selfadjoint eigenvalue problems that stems from the scattering theory for inhomogeneous media [23–25]. The spectral properties of the transmission eigenvalues have been intensively and extensively studied in the literature. In particular, it is known that for a generic medium scatterer, there exists an infinite and discrete set of eigenvalues satisfying $0 < k_1 \leq k_2 \leq \cdots \leq k_l \leq \cdots \rightarrow +\infty$, with $+\infty$ the only accumulating point [23,25,26]. However, there are few results on the spectral geometry of transmission eigenfunctions, which corresponds to the pattern of the wave propagation inside the scatterer when transparency/invisibility occurs. Very recently, it was revealed in [27–32] in different physical contexts that the transmission eigenfunctions



Citation: He, Y.; Liu, H.; Wang, X. Wave Patterns inside Transparent Scatterers. *Axioms* **2022**, *11*, 661. https://doi.org/10.3390/ axioms11120661

Academic Editor: Mourad Sini

Received: 11 October 2022 Accepted: 16 November 2022 Published: 22 November 2022

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Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). possess rich spectral patterns. These studies show that near a corner/high-curvature point on the boundary of the scatterer, the transmission eigenfunctions must be nearly vanishing. Furthermore, concerning the global property, Chow et al. [30] discovered that under generic scenarios, either the transmission eigenfunction w or v is localized on the boundary surface in \mathbb{R}^3 or the boundary curve in \mathbb{R}^2 . These discoveries about the geometric properties of the transmission eigenfunction also have some interesting applications, e.g., establishing unique identifiability results for a variety of inverse problems [33–35], producing a superresolution imaging scheme for the inverse acoustic scattering problem, and generating the so-called pseudo surface plasmon resonant modes with a potential sensing application [30], as well as electromagnetic mirage [31].

So far, all of the discoveries about the geometric patterns of transmission functions have been made in the setting of isotropic media. For the more challenging case of anisotropic media, we find for the first time that the waves inside the "transparent" scatterers have some regular patterns. On the one hand, the transmission eigenfunctions generically vanish near the corner points in both \mathbb{R}^2 and \mathbb{R}^3 . On the other hand, they tend to localize on the boundary of the scatterer if the wavenumber is large or the refractive index of the medium scatterer is high. Due to the limitation of computing resources, our study cannot be exhaustive. Nevertheless, we consider the representative geometric setup. In fact, a wider range of cases have these kinds of properties. We discuss the related results in Section 3.

The paper is organized as follows. Section 2 shows the wave patterns inside a variety of "transparent" scatterers. Among them, the isotropic results are displayed in Section 2.1, and the numerical results for anisotropic media are demonstrated in Section 2.2. The paper concludes in Section 3 with a summary of the most important results.

2. Results

In this section, the main numerical results of the internal patterns of those "transparent" waves are presented. We assume that the scatterer $D \subset \mathbb{R}^d$, d = 2,3 is a bounded connected domain with Lipschitz boundary ∂D . Let p_b be the background pressure field and p_s be the scattered pressure field; then, the total pressure field p can be represented by

$$p(\mathbf{x},t) = p_b(\mathbf{x},t) + p_s(\mathbf{x},t).$$
(1)

Now the wave equation for theacoustic field in the fluid satisfies the following system

$$\rho(\mathbf{x})\frac{\partial \mathbf{v}}{\partial t}(\mathbf{x},t) + \nabla p(\mathbf{x},t) = 0, \quad \kappa(\mathbf{x})\frac{\partial p}{\partial t}(\mathbf{x},t) + \nabla \cdot \mathbf{v}(\mathbf{x},t) = 0, \tag{2}$$

where ρ denotes a symmetric matrix valued function, v denotes the perturbed velocity of the fluid, and κ is the compressibility. Moreover, $\kappa^{-1} = B$ denotes the bulk modulus. By eliminating the velocity field v, the last wave equation can be rewritten as

$$\frac{\partial^2 p}{\partial t^2} - \kappa^{-1} \nabla \cdot (\rho \,\nabla p) = 0. \tag{3}$$

For the time-harmonic case, let $p(x,t) = \Re(u(x)e^{i\omega t})$; then, *u* satisfies the reduced wave equation

$$\nabla \cdot (\boldsymbol{\rho} \,\nabla u(x)) + \omega^2 \kappa u(x) = 0, \tag{4}$$

where $\omega > 0$ denotes the frequency. Correspondingly, the reduced time-harmonic background field u_b satisfying the Helmholtz equation

$$\nabla^2 u_b + \omega^2 u_b = 0. \tag{5}$$

The presence of the scatterer *D* interrupts the propagation of the background field u_b , bringing on the scattered field u_s . Let $u := u_b + u_s$ denote the total wave field. If we denote

$$\begin{cases} \nabla \cdot \left[(I+Q)\nabla u \right] + k^2(1+p)u = 0 & \text{in } \mathbb{R}^d, \\ \Delta u_b + k^2 u_b = 0 & \text{in } \mathbb{R}^d, \\ u = u_b + u_s & \text{in } \mathbb{R}^d, \\ \lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial u_s}{\partial r} - \mathrm{i} k u^s \right) = 0, \end{cases}$$
(6)

where r = |x| and the last limit in (6) characterizes the outgoing nature of the scattered wave field u_s . The well-posedness of the scattering system (6) is known, and in particular, a unique solution $u \in H^2_{loc}(\mathbb{R}^d)$ exists. Furthermore, the scattered field has the following asymptotic expansion:

$$u_{s}(x,\theta,k) = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8k\pi}} \left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} \sqrt{\frac{k}{2\pi}} \right)^{d-2} \frac{\mathrm{e}^{\mathrm{i}kr}}{r^{\frac{d-1}{2}}} \left\{ u^{\infty}(\hat{x},\theta,k) + \mathcal{O}\left(\frac{1}{r}\right) \right\} \quad \text{as } r \to \infty,$$
(7)

which holds uniformly for all directions $\hat{x} := x/|x| \in \mathbb{S}^{d-1}$. The complex-valued function $u^{\infty}(\hat{x}, \theta, k)$ defined on the unit sphere \mathbb{S}^{d-1} is known as the far-field pattern of u, which encodes the information of the indexes Q and p. If the presence of the object does not cause scattering, invisibility occurs. In this case, $u^{\infty} \equiv 0$, and by Rellich's Theorem, one has $u_s = 0$ in $\mathbb{R}^d \setminus \overline{D}$. Meanwhile, one immediately can verify that $w = u|_D$ and $v = u_b|_D$ fulfill the following system:

$$\begin{cases} \nabla \cdot [(\mathbf{I} + \mathbf{Q})\nabla w] + k^2(1+p)w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ w = v, \ v \cdot (\mathbf{I} + \mathbf{Q})\nabla w = v \cdot \nabla v & \text{on } \partial D, \end{cases}$$
(8)

where, and in what follows, ν is the unit outward normal to ∂D . A value $k \in \mathbb{R}_+$ for which the transmission eigenvalue problem (8) has non-trivial solutions $(w, v) \in H^1(D) \times H^1(D)$ is called a transmission eigenvalue. The corresponding nonzero solutions (w, v) are called transmission eigenfunctions. In the following, we denote by

$$q^* = \sup_{x \in D} \sup_{\xi \in \mathbb{R}^d, |\xi|=1} (\xi \cdot Q(x)\xi), \tag{9}$$

$$q_* = \inf_{x \in D} \inf_{\xi \in \mathbb{R}^d, |\xi|=1} (\xi \cdot Q(x)\xi), \tag{10}$$

$$p^* = \sup_{x \in D} p(x), \quad p_* = \inf_{x \in D} p(x)$$
 (11)

the essential supremum and infimum of Q and p. We suppose that the matrix valued function Q and the function p are such that either $q_* > 0$ and $p^* < 0$, or $q^* < 0$ and $p_* > 0$. Then, an infinite sequence of transmission eigenvalues exists with $+\infty$ as the only accumulation point [25].

We use the continuous finite-element method to solve the system (8). Multiplying the first two equations in (8) by a test function $\varphi \in H_0^1(D)$ and integrating by parts, one has

$$((\mathbf{I} + \mathbf{Q})\nabla w, \nabla \varphi) - k^2((1+p)w, \varphi) = 0, \quad \forall \varphi \in H_0^1(D),$$

$$(\nabla v, \nabla \varphi) - k^2(v, \varphi) = 0, \qquad \forall \varphi \in H_0^1(D).$$
(12)

To enforce the boundary condition $\nu \cdot (I + Q)\nabla w = \nu \cdot \nabla v$ weakly, we multiply it by a test function $\psi \in H^1(D)$ and integrate by parts, thus obtaining

$$((\mathbf{I} + \mathbf{Q})\nabla w, \nabla \psi) - k^2((1+p)w, \psi) = (\nabla v, \nabla \psi) - k^2(v, \psi), \quad \forall \psi \in H^1(D).$$
(13)

Hence, the variational formulation for (8) is to find $(u, v) \in H^1(D) \times H^1(D)$ satisfying (12) and (13), together with the essential boundary condition w = v on ∂D .

Let \mathcal{T}_h be a regular triangular or tetrahedral mesh of D, and let V_h be the finite element subspace of $H^1(D)$ consisting of piecewise linear functions on each element of \mathcal{T}_h . We also define a subspace $V_h^0 = V_h \cap H_0^1(D)$. Let $\{\eta_i\}_{i=1}^m$ detote a basis for V_h^0 and $\{\eta_i\}_{i=1}^m \cup \{\zeta_j\}_{j=1}^n$ denote a basis for V_h , respectively. To enforce the boundary conditions w = v on ∂D , we set

$$w^{h} = \sum_{i=1}^{m} w^{(i)} \eta_{i} + \sum_{j=1}^{n} u^{(j)} \zeta_{j}, \quad v^{h} = \sum_{i=1}^{m} v^{(i)} \eta_{i} + \sum_{j=1}^{n} u^{(j)} \zeta_{j}, \tag{14}$$

to be the finite element approximations of w and v, respectively. Thus, the discrete form of (12) and (13) can be written as

$$AX = k^2 BX, (15)$$

where

$$X = [w^{(1)}, \cdots, w^{(m)}, v^{(1)}, \cdots, v^{(m)}, u^{(1)}, \cdots, u^{(n)}] \in \mathbb{C}^{2m+n},$$
(16)

and

$$A = \begin{bmatrix} S_{II} & 0 & S_{IB} \\ 0 & S_{II} & S_{IB} \\ S_{IB}^{\top} & -S_{IB}^{\top} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} M_{2,II} & 0 & M_{2,IB} \\ 0 & M_{1,II} & M_{1,IB} \\ M_{2,IB}^{\top} & -M_{1,IB}^{\top} & M_{2,BB} - M_{1,BB} \end{bmatrix}.$$
(17)

Here, the stiffness and mass matrices are given by

$$S_{II} = (\nabla \eta_i, (\mathbf{I} + \mathbf{Q}) \nabla \eta_j), \quad S_{IB} = (\nabla \eta_i, (\mathbf{I} + \mathbf{Q}) \nabla \zeta_j),$$

$$M_{2,II} = (\eta_i, (1 + p)\eta_j), \qquad M_{2,IB} = (\eta_i, (1 + p)\zeta_j),$$

$$M_{1,II} = (\eta_i, \eta_j), \qquad M_{1,IB} = (\eta_i, \zeta_j),$$

$$M_{2,BB} = (\zeta_i, (1 + p)\zeta_j), \qquad M_{1,BB} = (\zeta_i, \zeta_j).$$
(18)

Here, we use an open-source PDE solver FreeFEM++ to implement the assembly of the above matrices. The eigenvalue and eigenvector for the non-symmetric eigenvalue problem (15) are computed by ARPACK in FreeFEM++, which is based on the Arnoldi algorithm.

2.1. Isotropic Media

As a special case of anisotropic media, we first show the results of isotropic media. Here, $Q \equiv 0$.

2.1.1. Local Geometrical Structures of Transmission Eigenfunctions

For any point on the scatterer boundary, we can quantitatively describe the properties of a function around it. Let $P \in \partial D$ be a point and $B_r(P)$ be a ball of radius $r \in \mathbb{R}_+$ centered at *P*. Define $D_r(P) := B_r(P) \cap D$. Consider a function $w \in L^2(D)$. Then, we say *w* is vanishing near *P* if

$$\lim_{r \to +0} \frac{1}{\sqrt{|D_r(P)|}} \|w(x)\|_{L^2(D_r(P))} = 0,$$
(19)

where $|D_r(P)|$ signifies the area or volume of the region $D_r(P)$ in two or three dimensions, respectively.

Two-Dimensional Example

Here, we consider the transmission eigenvalue problem in a regular hexagon with vertices at $P_1 = (-2, 0)$, $P_2 = (-1, -\sqrt{3})$, $P_3 = (1, -\sqrt{3})$, $P_4 = (2, 0)$, $P_5 = (1, \sqrt{3})$, and $P_6 = (-1, \sqrt{3})$. Let the scalar real valued function p = 3. The vanishing properties of both transmission eigenfunctions w's and v's around each corner can be seen in Figure 1.



Figure 1. The magnitude of transmission eigenfunctions for a regular hexagon with different eigenvalues. (**a**–**c**): eigenfunctions w's; and (**d**–**f**): eigenfunctions v's.

Three-Dimensional Example

Next, we numerically investigate the vanishing property of transmission eigenfunctions in the three-dimensional case. Firstly, we give the definition of vertex points in \mathbb{R}^3 [36]. Let $D \in \mathbb{R}^3$ be a bounded open set. A point $P \in \partial D$ is called a vertex if a neighborhood V of P, a diffeomorphism Ψ of class C^2 , and a polyhedral cone Λ with the vertex at O exist such that

$$\nabla \psi(P) = I_{3\times 3} \in \mathbb{R}^3, \Psi(P) = O, \tag{20}$$

and Ψ maps $V \cap \overline{D}$ onto a neighborhood of O in $\overline{\Lambda}$. Here, let D be a pyramid with a base of length 1 and a top of length 0.5 and the index p = 15. From Figure 2, we can see that both transmission eigenfunctions w's and v's vanish at all vertices of the domain D.

From the above isotropic examples, we can see that for both two-dimensional and three-dimensional cases, the transmission eigenfunctions vanish at all vertices of *D*.

2.1.2. Global Geometrical Structures of Transmission Eigenfunctions

The vanishing properties of transmission eigenfunctions discussed in the previous section are of a local nature. In this section, we are going to show the global geometric structures of transmission eigenfunctions numerically in two and three dimensions. Recall that the transmission eigenfunctions associated with the Helmholtz equation are localized on the boundary of D under generic scenarios. For completeness and self-containedness, in this part, we implement some different examples. Let's first review the definition of the localizing of transmission eigenfunctions here, referring to [30]. Consider a function $u \in L^2(D)$. It is said to be surface-localized if a sufficiently small $\epsilon_0 \in \mathbb{R}_+$ exists such that

$$\frac{\|u\|_{L^2(\mathcal{N}_{\epsilon_0}(\partial D))}}{\|u\|_{L^2(D)}} = 1 - \mathcal{O}(\epsilon_0), \tag{21}$$

where

$$\mathcal{N}_{\epsilon_0}(\partial D) := \{ x \in D; \, \operatorname{dist}(x, \partial D) < \epsilon_0 \}.$$
(22)

Two-Dimensional Example

Let *D* be an ellipse with major axis 4 and minor axis 3 and p = 25. We can see from Figure 3 that in this setting, the transmission eigenfunctions v's are surface-localized, while w's are not for the same eigenvalues.



Figure 2. The magnitude of transmission eigenfunctions for a pyramid with different *k*'s. (**a**-**c**): surface of *w*'s and *v*'s; (**d**-**f**): eigenfunctions *w*'s; (**g**-**i**): eigenfunctions *v*'s; and (**j**-**l**): w - v's.



Figure 3. The magnitude of transmission eigenfunctions for an ellipse. (**a**,**b**): k = 1.983; (**c**,**d**): k = 2.347.

Three-Dimensional Example

Next, we consider the domain *D* a triaxial ellipsoid with the principal semi-axes a = 2, b = 1.5, and c = 1, respectively. From Figure 4, it is clear that there is a family of transmission eigenfunctions *v*'s that are localized on the boundary ∂D , while *w*'s are not.

2.2. Anisotropic Media

In this part, we consider the numerical results for a range of anisotropic media, i.e., cases where $\mathbf{Q} \neq \mathbf{0}$.

2.2.1. Local Geometrical Structures of Transmission Eigenfunctions

In this section, we provide several anisotropic numerical examples to verify the vanishing property in a certain geometric setup including corner singularities.

Two-Dimensional Examples

In the first example, we let *D* be an equilateral triangle with vertices at $P_1 = (-1, 0)$, $P_2 = (1, 0)$, and $P_3 = (0, \sqrt{3})$. Let

$$\mathbf{Q} = \begin{pmatrix} -0.5 & 0.2\\ 0.2 & -0.5 \end{pmatrix}, \quad p = 4.$$
(23)

In Figure 5, we calculate the transmission eigenfunctions with respect to several different transmission eigenvalues. This example illustrates that the transmission eigenfunctions w's and v's vanish near every corner point of D.

0. 0 -0 -1 2 0. 0 -0. -1 1 0.5 -0.5 -1 -1.5 -2 0.5 0 -0.5 2 2 1 -2 0 0 -1 -1 -1 -2 -2 -2 (a) k = 1.680 (b) k = 2.140(c) k = 2.215 0 0 -1 3 2 1 0 -1 4 1 0.5 -0. -1 0. 0 -0. -1 0 2 0 0 -2 -2 2 2 2 -4 -6 -2 -3 1 1 1 -4 0 0 0 -1 -1 -1 -6 -2 -2 -2 (d) k = 1.680(e) k = 2.140 (**f**) k = 2.215 1.5 1 0.5 0 -0.5 1 0. -0. -1 1 0. 0 -0 0. 0 -0. -1 0.5 0 -0.5 2 -1 -1.5 2 2 1 1 1 -2 0 0 0 -1 -1 -1 -1 -2 -2 -2 (g) k = 1.680(**h**) k = 2.140 (i) k = 2.215

Figure 4. The magnitude of transmission eigenfunctions for an ellipsoid with different k's. (**a**-**c**): surface of w's and v's; (**d**-**f**): eigenfunctions w's; and (**g**-**i**): eigenfunctions v's.



Figure 5. The magnitude of transmission eigenfunctions for an equilateral triangle with different eigenvalues. (**a**–**c**): eigenfunctions w's; and (**d**–**f**): eigenfunctions v's.

In the next example, we consider the domain *D* to be a square with a side length 2. Let

$$\mathbf{Q} = \begin{pmatrix} 7 & 1\\ 1 & 7 \end{pmatrix}, \quad p = -0.3. \tag{24}$$



The vanishing properties of both transmission eigenfunctions w's and v's near the corners can be seen in Figure 6.

Figure 6. The magnitude of transmission eigenfunctions for a square with different k's. (**a**–**c**): eigenfunctions w's; (**d**–**f**): eigenfunctions v's.

The two examples above show that near the corner points of the domain *D*, either $q^* < 0$, $p_* > 0$ or $q_* > 0$, $p^* < 0$, the transmission eigenfunctions *w*'s and *v*'s both vanish.

Three-Dimensional Examples

In the following examples, we let *D* be some different three-dimensional domains. In the first example, we let *D* be a cube with side length 1. Let

$$\mathbf{Q} = \begin{pmatrix} 49 & 2 & 0\\ 2 & 49 & 0\\ 0 & 0 & 36 \end{pmatrix}, \quad p = -0.4.$$
(25)

From Figure 7, we can see that both transmission eigenfunctions w's and v's vanish at all vertices of D.

Next, we let *D* be a cone with the bottom radius 1 and the height 1.5. Let

$$Q = \begin{pmatrix} 36 & 2 & 0 \\ 2 & 36 & 0 \\ 0 & 0 & 25 \end{pmatrix}, \quad p = -0.5.$$
 (26)

We can see from Figure 8 that the transmission eigenfunctions w's and v's vanish near the top point (0, 0, 1.5), which is the only vertex on D.

2.2.2. Global Geometrical Structures of Transmission Eigenfunctions

In Section 2.1.2, we reviewed the surface-localized property of transmission functions in isotropic media. In this section, we numerically illustrate that the transmission eigenfunctions for anisotropic media also possess the surface-localized property. **Two-Dimensional Examples**

We consider several different two-dimensional configurations in this part. In Figure 9, we calculate the transmission eigenfunctions w's and v's for D being a unit disk. The corresponding

$$Q = \begin{pmatrix} -0.5 & 0.3 \\ 0.3 & -0.5 \end{pmatrix}, \quad p = 8.$$
 (27)

It is clearly seen that for the same transmission eigenvalue, the transmission eigenfunction v is surface-localized, while w is not.



Figure 7. The magnitude of transmission eigenfunctions for a cube with different k's. (**a**–**c**): surface of w's and v's; (**d**–**f**): eigenfunctions w's; and (**g**–**i**): eigenfunctions v's.

From Figure 10, we can see that when *D* is an ellipse, and

$$\mathbf{Q} = \begin{pmatrix} 25 & 2\\ 2 & 25 \end{pmatrix}, \quad p = -0.6,$$
 (28)

the transmission eigenfunctions w's are surface-localized, while v's are not for some certain eigenvalues.

-1 -1 -1 (**a**) k = 6.930 (b) k = 8.370 (c) k = 9.193 0.25 0.2 0.15 0.2 0.15 0.1 0.05 0 -0.05 -0.1 -0.15 -0.2 0.1 0.08 0.06 0.04 0.02 0 -0.02 -0.04 -0.06 -0.08 -1.s ٦t. 1 1 1 0.05 0.5 0.5 0.5 0.05 0 0 0 -0.15 -1 -1 (d) k = 6.930(e) k = 8.370 (**f**) k = 9.193 0.8 0.6 0.2 0 -0.2 -0.4 -0.6 -0.8 4 3 2 1 -1 -2 -3 -4 -5 1.5 ₂ -1.s 1 1 1 0 0.5 0.5 -1 0 0 -2 -1 -1 -1 (h) k = 8.370 (i) k = 9.193 (g) k = 6.930

0.2

1.s

1

0.5

0

0.15

0.13 0.05 0 -0.05 -0.1 -0.15 -0.2

0.3

0.1

0

-0.1 0

-0.2

-0.3

-1.5 0.2

1

0.5

Figure 8. The magnitude of transmission eigenfunctions for a cone with different k's. (a-c): surface of *w*'s and *v*'s; (**d**–**f**): eigenfunctions *w*'s; and (**g**–**i**): eigenfunctions *v*'s.



Figure 9. The magnitude of transmission eigenfunctions for a unit disk. (**a**,**b**): k = 2.576; (**c**,**d**): k = 3.301.

0.3 0.2

0.1

-0.3

0

it.s

1

0 -0.2

0.5 -0.1



Figure 10. The magnitude of transmission eigenfunctions for an ellipse with the semi-major radius 1.5 and the semi-minor radius 1. (**a**,**b**): k = 8.736; (**c**,**d**): k = 9.621.

Three-Dimensional Examples

Next, we conduct several numerical examples in three dimensions. We first consider a unit sphere with

$$Q = \begin{pmatrix} 25 & 1 & 0 \\ 1 & 25 & 0 \\ 0 & 0 & 17 \end{pmatrix}, \quad p = -0.5.$$
⁽²⁹⁾

From Figure 11, it is clear that there is a family of transmission eigenfunctions w's that is localized on the boundary ∂D , while v's are not.

Figure 12 plots the transmission eigenfunctions with three different eigenvalues. The domain D considered here is a torus, and

$$\mathbf{Q} = \begin{pmatrix} 36 & 2 & 0\\ 2 & 36 & 0\\ 0 & 0 & 25 \end{pmatrix}, \quad p = -0.4.$$
(30)

For these different transmission eigenvalues, the transmission eigenfunctions w's are surface-localized, while v's are not.



Figure 11. The magnitude of transmission eigenfunctions for a unit sphere with different *k*'s. (**a**–**c**): surface of *w*'s and *v*'s; (**d**–**f**): eigenfunctions *w*'s; and (**g**–**i**): eigenfunctions *v*'s; (**j**–**l**): w - v's.

From the above examples, we can see that when $q^* < 0$ and $p_* > 0$, the transmission eigenfunction v is surface-localized, while w is not. If $q_* > 0$ and $p^* < 0$, the transmission eigenfunction w is surface-localized, while v is not. This result agrees to some extent with the result for the isotropic situation [30]. Recall the isotropic cases; if p > 0, the transmission eigenfunction v tends to localized on the boundary of the scatterer if the wavenumber is large. If -1 , the same global geometric property holds for the transmission eigenfunction <math>w. Practically, we cannot compute very large eigenvalues, so we take the relatively large indexes Q or p to illustrate the localization property. As the numerical experiments cannot be complete, further theoretical analysis is needed on how the localization property depends on the indexes.

For summary and comparison purposes, we list the necessary parameters of all of the numerical results in Table 1.

0.1 0.1 0.05 0.05 0.05 0.5 0 0 -0.5 -0.5 -0.05 -0.05 0.05 -0.1 -0.1 (a) k = 7.674(**b**) k = 8.374 (c) k = 9.0871 0 0 0 0.1 0.1 0.05 0.05 0.05 0.5 0.5 0 0 0 -0.5 0.5 -0.05 -0.05 -0.05 -0.1 -0.1 (d) k = 7.674(e) k = 8.374 (f) k = 9.0871 1 1 1 0 0 0 3 2 1 -1 -2 -3 3 2 1 0 -1 -2 -3 0 -1 -2 -3 -0.5 -0.5 (g) k = 7.674(**h**) k = 8.374 (i) k = 9.087

Figure 12. The magnitude of transmission eigenfunctions for a torus with different *k*'s. (**a**–**c**): surface of *w*'s and *v*'s; (**d**–**f**): eigenfunctions *w*'s; and (**g**–**i**): eigenfunctions *v*'s.

No.	Shape	Medium	Q	р
Figure 1	hexagon	isotropic	0	3
Figure 2	pyramid	isotropic	0	15
Figure 3	ellipse	isotropic	0	25
Figure 4	triaxial ellipsoid	isotropic	0	16
Figure 5	equilateral triangle	anisotropic	$\left(\begin{array}{cc}-0.5&0.2\\0.2&-0.5\end{array}\right)$	4
Figure 6	square	anisotropic	$\left(\begin{array}{cc}7&1\\1&7\end{array}\right)$	-0.3
Figure 7	cube	anisotropic	$ \left(\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrr$	-0.4
Figure 8	cone	anisotropic	$\left(\begin{array}{rrrr} 36 & 2 & 0 \\ 2 & 36 & 0 \\ 0 & 0 & 25 \end{array}\right)$	-0.5
Figure 9	disk	anisotropic	$\left(\begin{array}{cc}-0.5&0.3\\0.3&-0.5\end{array}\right)$	8
Figure 10	ellipse	anisotropic	$\left(\begin{array}{rrr} 25 & 2\\ 2 & 25 \end{array}\right)$	-0.6
Figure 11	sphere	anisotropic	$\left(\begin{array}{rrrr} 25 & 1 & 0 \\ 1 & 25 & 0 \\ 0 & 0 & 17 \end{array}\right)$	-0.5
Figure 12	torus	anisotropic	$ \left(\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrr$	-0.4

 Table 1. Summary of numerical results.

3. Summary and Conclusions

In this paper, we numerically investigate the intrinsic geometric structures of transmission eigenfunctions for isotropic media and anisotropic media, especially for anisotropic media. Extensive numerical results show that these eigenfunctions have similar geometric properties to the transmission eigenfunctions associated with the Helmholtz system for isotropic media. Locally, the transmission eigenfunctions vanish near the corner points of the scatterer. Globally, they tend to localize on the boundary of the scatterer if the wavenumber is large or the refractive index is high. The results also show the dependence of the localization property on the wavenumber and the refractive index of the medium scatterer. Both two- and three-dimensional experiments qualitatively confirm our results.

It is noted that the transmission eigenfunction is closely related to the invisibility cloaking. The phenomenon that the transmission eigenfunction is localized on the boundary can be regarded as the cloaking device forcing the field to enclose the hidden object, that is, the "artificial transparency/invisibility" is a natural extension of the "natural transparency/invisibility". On the other hand, it is physically justifiable that the transmission eigenfunctions vanish at the corners. If the field enters the place near a corner, it will be trapped there. Then, it is impossible for the incident field to pass through the obstacle without scattering.

Although our numerical study cannot be exhaustive, we consider the representative geometric setup in both two and three dimensions. This work not only enriches the spectral theory for the interior transmission eigenvalue problem but also provides direction for future theoretical research. Moreover, this study can be applied in many practical projects.

Author Contributions: Conceptualization, Y.H., H.L. and X.W.; software, Y.H. and X.W.; writing original draft preparation, Y.H.; and writing—review and editing, H.L. and X.W. All authors have read and agreed to the published version of the manuscript.

Funding: The work of H Liu was supported by the Hong Kong RGC General Research Funds (projects 12302919, 12301420 and 11300821), the NSFC/RGC Joint Research Fund (project N_CityU101/21), and the France-Hong Kong ANR/RGC Joint Research Grant, A-HKBU203/19. This research of X. Wang was supported by the NSFC grants (12001140 and 11971133) and the Fundamental Research Funds for the Central Universities (FRFCU5710050422 and 2022FRFK060026).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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