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On h-Quasi-Hemi-Slant Riemannian Maps

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Abstract: In the present article, we indroduce and study h-quasi-hemi-slant (in short, h-qhs) Riemannian maps and almost h-qhs Riemannian maps from almost quaternionic Hermitian manifolds to Riemannian manifolds. We investigate some fundamental results mainly on h-qhs Riemannian maps: the integrability of distributions, geometry of foliations, the condition for such maps to be totally geodesic, etc. At the end of this article, we give two non-trivial examples of this notion.

Keywords: Riemannian map; hyperkähler manifold; h-quasi-hemi-slant Riemannian map

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1. Introduction

In Riemannian geometry, there are few appropriate maps among Riemannian manifolds that compare their geometric properties. In this direction, as a generalization of the notions of isometric immersions and Riemannian submersions, Riemannian maps between Riemannian manifolds were initiated by Fischer [1], while isometric immersions and Riemannian submersions were widely studied in [2] and [3], respectively. However, the notion of Riemannian maps is a new research topic for geometers. More precisely, a differentiable map $\pi : (N_1, g_1) \to (N_2, g_2)$ between Riemannian manifolds (N_1, g_1) and (N_2, g_2) is called a Riemannian map ($0 < rank\pi_* < \min\{m, n\}$, where $\dim N_1 = m$ and $\dim N_2 = n$) if it satisfies the following equation:

$$g_2(\pi_* W_1, \pi_* W_2) = g_1(W_1, W_2), \text{ for } W_1, W_2 \in \Gamma(\ker \pi_*)^\perp, \tag{1}$$

where π_* is the differentiable map of π.

Consequently, isometric immersions and Riemannian submersions are particular cases of Riemannian maps with $\ker \pi_* = 0$ and $(range\pi_*)^\perp = 0$, respectively [1].

The other prominent basic map for comparing geometric structures between Riemannian manifolds is Riemannian submersion, and it was studied by O'Neill [4] and Gray [5]. In 1976, Watson [6] studied Riemannian submersion between Riemannian manifolds equipped with differentiable structures. After that, several kinds of Riemannian submersions were introduced and studied, including Riemannian submersion [3], H-anti-invariant submersion [7], H-semi-invariant submersion [8] and H-semi-slant submersion [9].

Currently, one of the most inventive topics in differential geometry is the theory of Riemannian maps between different Riemannian manifolds. It is well known that differentiable maps between Riemannian manifolds have wide applications in differential geometry as well as in physics, such as in Yang–Mills theory [10], Kaluza–Klein theory [11], and supergravity and superstring theories [12].

We also note that quarternionic manifolds have many applications, including for nonlinear σ models with super symmetry [12], in the theory of harmonic differential forms [13] and obtaining estimates for the Betti numbers of the manifold [14,15]. In this paper, we have for the first time investigated h-qhs Riemannian maps from almost quarternionic manifolds to Riemannian manifolds. Here, we mainly focus on the most fundamental and interesting geometric properties on the fibers and distributions of these maps.

Nowadays, Riemannian maps and related topics have been actively studied by many authors, such as invariant and anti-invariant Riemannian maps [16], semi-invariant Riemannian maps [17], slant Riemannian maps [18], semi-slant Riemannian maps [19,20], hemi-slant Riemannian maps [21], quasi-hemi-slant Riemannian maps [22], almost h-semi-slant Riemannian maps [23], V-quasi-bi-slant Riemannian maps [24] and Clairaut semi-invariant Riemannian maps [25]. As a generalization of h-slant Riemannian maps [26], h-semi-slant Riemannian maps [9] and h-hemi-slant Riemannian maps, we define and study h-qhs Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. In the near future, we plan to work on conformal h-qhs submersions, conformal h-qhs submersions, h-qhs semi-Riemannian submersions, etc.

This paper is structured as follows. In Section 2, we recall basic facts about Riemannian maps and almost Hermitian manifolds. In Section 3, we define h-qhs Riemannian maps and study the geometry of leaves of distributions that are involved in the definition of such maps. We give necessary and sufficient conditions for h-qhs Riemannian maps to be totally geodesic. Finally, we provide two concrete examples of h-qhs Riemannian maps.

2. Preliminaries

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and $\pi : (N_1, g_1) \to (N_2, g_2)$ be a C^∞-Riemannian map [1].

We define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ [4] by

$$\mathcal{A}_{F_1} F_2 = \mathcal{H}\nabla_{\mathcal{H}F_1} \mathcal{V}F_2 + \mathcal{V}\nabla_{\mathcal{H}F_1} \mathcal{H}F_2, \tag{2}$$

$$\mathcal{T}_{F_1} F_2 = \mathcal{H}\nabla_{\mathcal{V}F_1} \mathcal{V}F_2 + \mathcal{V}\nabla_{\mathcal{V}F_1} \mathcal{H}F_2, \tag{3}$$

for any vector fields F_1, F_2 on N_1, where ∇ is the Levi-Civita connection of g_1.

From Equations (2) and (3), we have

$$\nabla_{Y_1} Y_2 = \mathcal{T}_{Y_1} Y_2 + \mathcal{V}\nabla_{Y_1} Y_2, \tag{4}$$

$$\nabla_{Y_1} U_1 = \mathcal{T}_{Y_1} U_1 + \mathcal{H}\nabla_{Y_1} U_1, \tag{5}$$

$$\nabla_{U_1} Y_1 = \mathcal{A}_{U_1} Y_1 + \mathcal{V}\nabla_{U_1} Y_1, \tag{6}$$

$$\nabla_{U_1} U_2 = \mathcal{H}\nabla_{U_1} U_2 + \mathcal{A}_{U_1} U_2, \tag{7}$$

for $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ and $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$, where $\mathcal{H}\nabla_{Y_1} U_1 = \mathcal{A}_{U_1} Y_1$ and U_1 is basic.

Let $\pi : (N_1, E, g_1) \to (N_2, g_2)$ be a C^∞ map. The second fundamental form of π is given by

$$(\nabla \pi_*)(V_1, V_2) = \nabla^\pi_{V_1} \pi_*(V_2) - \pi_*(\nabla^{N_1}_{V_1} V_2), \tag{8}$$

for $V_1, V_2 \in \Gamma(TN_1)$, where ∇^π is the pullback connection [27]. The map π is said to be a total geodesic if $(\nabla \pi_*)(V_1, V_2) = 0$ for $V_1, V_2 \in \Gamma(TN_1)$.

Let (N_1, E, g_1) be an almost quaternionic Hermitian manifold, where g_1 is a Riemanian metric on the maniifold N_1 and E is a rank 3 subbundle of $End(TN_1)$ such that for any point $p \in N_1$ within some neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying all $\alpha \in \{1, 2, 3\}$ in which

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{9}$$

$$g_1(J_\alpha X_2, J_\alpha X_1) = g_1(X_1, X_2), \tag{10}$$

for $X_1, X_2 \in \Gamma(TN_1)$, where the indices are taken from $\{1,2,3\}$ modulo 3 and $\{J_1, J_2, J_3\}$ is called the quaternionic Hermitian basis. The structure (N_1, E, g_1) is called a quaternionic kähler manifold if there exist locally defined 1 forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1,2,3\}$, we have

$$\nabla_{X_1} J_\alpha = \omega_{\alpha+2}(X_1) J_{\alpha+1} - \omega_{\alpha+1}(X_1) J_{\alpha+2}, \tag{11}$$

for $X_1 \in \Gamma(TN_1)$, where the indices are taken from $\{1,2,3\}$ modulo 3. If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on N_1, then (N_1, E, g_1) is called a hyperkähler. The structure $\{J_1, J_2, J_3, g_1\}$, where g_1, a hyperkähler metric, is called a hyperkähler structure on N_1.

A map $\pi : (N_1, E_1, g_1) \to (N_2, E_2, g_2)$ is called an (E_1, E_2)-holomorphic map if for any point $p \in N_1$ and $J \in (E_1)_p$, there exists $J' \in (E_2)_{\pi(p)}$ such that

$$\pi_* \circ J = J' \circ \pi_*.$$

A Riemannian submersion between quaternionic kähler manifolds $\pi : (N_1, E_1, g_1) \to (N_2, E_2, g_2)$, which is an (E_1, E_2)-holomorphic map, is known as a quaternionic kähler submersion (or a hyperkähler submersion) [9]:

Definition 1 ([23]). *A Riemannian map π from the almost quaternionic Hermitian manifold (N_1, E, g_1) to the Riemannian manifold (N_2, g_2) is called an h-semi-slant Riemannian map if, given a point $p \in N_1$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, the following is true:*

$$\ker \pi_* = D_1 \oplus D_2, R(D_1) = D_1,$$

in which the angle $\theta_R = \theta_R(Z_1)$ between RZ_1 and the space $(D_2)_q$ is constant for a non-zero $Z_1 \in (D_2)_q$ and $q \in U$, where D_2 is an orthogonal complement of D_1 in $\ker \pi_*$.

Furthermore, assume we have

$$\theta = \theta_I = \theta_J = \theta_K,$$

Then, we call the map $\pi : (N_1, E, g_1) \to (N_2, g_2)$ a strictly h-semi-slant Riemannian map, the basis $\{I, J, K\}$ a strictly h-semi-slant basis and the angle θ a strictly h-semi-slant angle.

3. *h*-Quasi-Hemi-Slant Riemannian Maps

Motivated by the studies given in Section 2, we give the definition of the h-qhs Riemannian map as follows:

Definition 2. *A Riemannian map π from the almost quaternionic Hermitian manifold (N_1, E, g_1) to the Riemannian manifold (N_2, g_2) is called an h-qhs Riemannian map if, given a point $p \in N_1$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $D \subset (\ker \pi_*)$ on U such that*

$$\ker \pi_* = D \oplus D_1 \oplus D_2, R(D) = D, R(D_2) \subset (\ker \pi_*)^\perp,$$

and the angle $\theta_R = \theta_R(Z_1)$ between RZ_1 and the space $(D_1)_q$ is constant for a non-zero $Z_1 \in (D_1)_q$ and $q \in U$, where $\ker \pi_*$ *admits three orthogonal complementary distributions D, D_1 and D_2 such that D is invariant, D_1 is a slant with an angle θ_R and D_2 is anti-invariant.*

We call the basis $\{I, J, K\}$ an h-qhs basis and the angles $\{\theta_I, \theta_J, \theta_K\}$ h-qhs angles.
Furthermore, let us say we have

$$\theta = \theta_I = \theta_J = \theta_K,$$

Then, we call the map $\pi : (N_1, E, g_1) \to (N_2, g_2)$ a strictly h-qhs Riemannian map, the basis $\{I, J, K\}$ a strictly quasi-hemi-slant basis and the angle θ a strictly quasi-hemi-slant angle:

Definition 3. *A Riemannian map π from the almost quaternionic Hermitian manifold (N_1, E, g_1) to the Riemannian manifold (N_2, g_2) is called an almost h-qhs Riemannian map if, given a point $p \in N_1$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in \{I, J, K\}$, there is a distribution $D^R \subset (\ker \pi_*)$ on U such that*

$$\ker \pi_* = D^R \oplus D_1^R \oplus D_2^R, R(D^R) = D^R, R(D_2^R) \subset (\ker \pi_*)^\perp,$$

and the angle $\theta_R = \theta_R(Z_1)$ between RZ_1 and the space $(D_1^R)_q$ is constant for a non-zero $Z_1 \in (D_1^R)_q$ and $q \in U$, where the vertical distribution $\ker \pi_*$ admits three orthogonal complementary distributions D^R, D_1^R and D_2^R such that D^R is invariant, D_1^R is a slant with an angle θ_R and D_2^R is anti-invariant.

We call the basis $\{I, J, K\}$ an almost h-qhs basis and the angles $\{\theta_I, \theta_J, \theta_K\}$ almost h-qhs angles.

Let $\pi : (N_1, E, g_1) \to (N_2, g_2)$ be an almost h-qhs Riemannian map. We can easily observe the following:

(a) If $\dim D^R \neq 0$, $\dim D_1^R \neq 0$, $0 < \theta_R < \frac{\pi}{2}$ and $\dim D_2^R = 0$, then π is an almost proper h-semi-slant Riemannian map with a semi-slant angle θ_R;
(b) If $\dim D^R = 0$, $\dim D_1^R \neq 0$, $0 < \theta_R < \frac{\pi}{2}$ and $\dim D_2^R \neq 0$, then π is an almost h-hemi-slant Riemannian map.

We say that the almost h-qhs Riemannian map $\pi : (N_1, E, g_1) \to (N_2, g_2)$ is proper if $D^R \neq \{0\}$, $D_2^R \neq \{0\}$ and $\theta_R \neq 0, \frac{\pi}{2}$. Thus, one can easily see that the h-hemi-slant Riemannian map, h-semi-invariant Riemannian map and h-semi-slant Riemannian map are examples of h-qhs Riemannian maps.

Thus, we have

$$(\ker \pi_*)^\perp = \omega_R(D_1^R) \oplus R(D_2^R) \oplus \mu_R.$$

Obviously, μ_R is an invariant sub-bundle of $(\ker \pi_*)^\perp$ with respect to the complex structure R.

For $V_1 \in \Gamma(\ker \pi_*)$, we have

$$V_1 = P_R V_1 + Q_R V_1 + S_R V_1, \tag{12}$$

where $P_R V_1 \in \Gamma(D^R)$, $Q_R V_1 \in \Gamma(D_1^R)$, $S_R V_1 \in \Gamma(D_2^R)$ and $R \in \{I, J, K\}$.

For $Z_1 \in \Gamma(\ker \pi_*)$, we obtain

$$RZ_1 = \phi_R Z_1 + \omega_R Z_1, \tag{13}$$

where $\phi_R Z_1 \in \Gamma(\ker \pi_*)$, $\omega_R Z_1 \in \Gamma(\ker \pi_*)^\perp$ and $R \in \{I, J, K\}$.

For $X_1 \in \Gamma(\ker \pi_*)^\perp$, we have

$$RX_1 = B_R X_1 + C_R X_1, \tag{14}$$

where $B_R X_1 \in \Gamma(\ker \pi_*)$, $C_R X_1 \in \Gamma(\mu_R)$ and $R \in \{I, J, K\}$.

We will denote an almost h-qhs Riemannian map from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-qhs basis by π.

The following lemmas can be easily obtained:

Lemma 1. *For $\pi : (N_1, g_1, E_1) \to (N_2, g_2, E_2)$, we get*

$$\phi_R D^R = D^R, \omega_R D^R = 0, \phi_R D_2^R = 0, \omega_R D_2^R \subset (\ker \pi_*)^\perp,$$

where $R \in \{I, J, K\}$.

Lemma 2. *For* $\pi : (N_1, g_1, E_1) \to (N_2, g_2, E_2)$, *we have*

$$\begin{aligned} \phi_R^2 Z_1 + B_R \omega_R Z_1 &= -Z_1, \ \ \omega_R \phi_R Z_1 + C_R \omega_R Z_1 = 0, \\ \phi_R B_R Z_2 + B_R C_R Z_2 &= 0, \ \ \omega_R B_R Z_2 + C_R^2 Z_2 = -Z_2, \end{aligned}$$

for any $Z_1 \in \Gamma(\ker \pi_*)$, $Z_2 \in \Gamma(\ker \pi_*)^\perp$ *and* $R \in \{I, J, K\}$.

Proof. Using Equations (9), (13) and (14), we can find all equations of Lemma 2: □

Lemma 3. *With* $\pi : (N_1, I, J, K, g_1) \to (N_2, g_2)$ *being an almost h-qhs Riemannian map, we then obtain*

$$\mathcal{V}\nabla_{X_1} \phi_R X_2 + \mathcal{T}_{X_1} \omega_R X_2 = B_R \mathcal{T}_{X_1} X_2 + \phi_R \mathcal{V}\nabla_{X_1} X_2, \tag{15}$$

$$\mathcal{T}_{X_1} \phi_R X_2 + \mathcal{H}\nabla_{X_1} \omega_R X_2 = C_R \mathcal{T}_{X_1} X_2 + \omega_R \mathcal{V}\nabla_{X_1} X_2, \tag{16}$$

$$\mathcal{T}_{X_1} B_R Z_1 + \mathcal{H}\nabla_{X_1} C_R Z_1 = C_R \mathcal{H}\nabla_{X_1} Z_1 + \omega_R \mathcal{T}_{X_1} Z_1, \tag{17}$$

$$\mathcal{V}\nabla_{X_1} B_R Z_1 + \mathcal{T}_{X_1} C_R Z_1 = B_R \mathcal{H}\nabla_{X_1} Z_1 + \phi \mathcal{T}_{X_1} Z_1, \tag{18}$$

$$\mathcal{V}\nabla_{Z_1} \phi_R X_1 + \mathcal{A}_{Z_1} \omega_R X_1 = B_R \mathcal{A}_{Z_1} X_1 + \phi_R \mathcal{V}\nabla_{Z_1} X_1, \tag{19}$$

$$\mathcal{A}_{Z_1} \phi_R X_1 + \mathcal{H}\nabla_{Z_1} \omega_R X_1 = C_R \mathcal{A}_{Z_1} X_1 + \omega_R \mathcal{V}\nabla_{Z_1} X_1, \tag{20}$$

$$\mathcal{A}_{Z_1} B_R Z_2 + \mathcal{H}\nabla_{Z_1} C_R Z_2 = C_R \mathcal{H}\nabla_{Z_1} Z_2 + \omega_R \mathcal{A}_{Z_1} Z_2, \tag{21}$$

$$\mathcal{V}\nabla_{Z_1} B_R Z_2 + \mathcal{A}_{Z_1} C_R Z_2 = B_R \mathcal{H}\nabla_{Z_1} Z_2 + \phi_R \mathcal{A}_{Z_1} Z_2, \tag{22}$$

for $X_1, X_2 \in \Gamma(\ker \pi_*)$, $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$ *and* $R \in \{I, J, K\}$.

Proof. Using Equations (4)–(7), (13) and (14), we can easily obtain Equations (15)–(22). □

Now, we define

$$(\nabla_{X_1} \phi_R) X_2 = \mathcal{V}\nabla_{X_1} \phi_R X_2 - \phi_R \mathcal{V}\nabla_{X_1} X_2, \tag{23}$$

$$(\nabla_{X_1} \omega_R) X_2 = \mathcal{H}\nabla_{X_1} \omega_R X_2 - \omega_R \mathcal{V}\nabla_{X_1} X_2, \tag{24}$$

$$(\nabla_{Z_1} B_R) Z_2 = \mathcal{V}\nabla_{Z_1} B_R Z_2 - B_R \mathcal{H}\nabla_{Z_1} Z_2, \tag{25}$$

$$(\nabla_{Z_1} C_R) Z_2 = \mathcal{H}\nabla_{Z_1} C_R Z_2 - C_R \mathcal{H}\nabla_{Z_1} Z_2, \tag{26}$$

for $X_1, X_2 \in \Gamma(\ker \pi_*)$, $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$ and $R \in \{I, J, K\}$.

Lemma 4. *For* $\pi : (N_1, I, J, K, g_1) \to (N_2, g_2)$, *we find*

$$(\nabla_{X_1} \phi_R) X_2 = B_R \mathcal{T}_{X_1} X_2 - \mathcal{T}_{X_1} \omega_R X_2, \ \ (\nabla_{X_1} \omega_R) X_2 = C_R \mathcal{T}_{X_1} X_2 - \mathcal{T}_{X_1} \phi_R X_2,$$

$$(\nabla_{Z_1} C_R) Z_2 = \omega_R \mathcal{A}_{Z_1} Z_2 - \mathcal{A}_{Z_1} B_R Z_2, \ \ (\nabla_{Z_1} B_R) Z_2 = \phi_R \mathcal{A}_{Z_1} Z_2 - \mathcal{A}_{Z_1} C_R Z_2,$$

for all $X_1, X_2 \in \Gamma(\ker \pi_*)$, $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$ *and* $R \in \{I, J, K\}$.

Proof. Using Equations (15) and (16) as well as Equations (21)–(26), Lemma 4 follows. □

If the tensors ϕ_R and ω_R are parallel with respect to the linear connection ∇ on N_1, then

$$B_R \mathcal{T}_{X_1} X_2 = \mathcal{T}_{X_1} \omega_R X_2, C_R \mathcal{T}_{X_1} X_2 = \mathcal{T}_{X_1} \phi_R X_2,$$

for all $X_1, X_2 \in \Gamma(\ker \pi_*)$ and $R \in \{I, J, K\}$:

Lemma 5. *Let $\pi : (N_1, E, g_1) \to (N_2, g_2)$, be an almost h-qhs Riemannian map. Then, we obtain*

$$\phi_R^2 V_1 = -\cos^2 \theta_R V_1, \tag{27}$$

for any non-zero vector field $V_1 \in \Gamma(D_1^R)$ and $R \in \{I, J, K\}$, where $\{I, J, K\}$ is an almost h-qhs basis with the almost h-qhs angles $\{\theta_I, \theta_J, \theta_K\}$.

Proof. For any non-zero vector field $V_1 \in \Gamma(D_1^R)$ and $R \in \{I, J, K\}$, we have

$$\cos \theta_R = \frac{\| \phi_R V_1 \|}{\| RV_1 \|}, \tag{28}$$

and

$$\cos \theta_R = \frac{g_1(RV_1, \phi_R V_1)}{\| \phi_R V_1 \| \| RV_1 \|}, \tag{29}$$

where $\theta_R(V_1)$ is the h-qhs angle.

Using Equations (9) and (13), we obtain

$$\cos \theta_R = -\frac{g_1(V_1, \phi_R{}^2 V_1)}{\| \phi_R V_1 \| \| RV_1 \|}. \tag{30}$$

From Equations (29) and (30), Equation (27) follows. □

Theorem 1. *Let π be an h-qhs Riemannian map from an almost hyperkahler manifold (N_1, I, J, K, g_1) to a Riemannian manifold (N_2, g_2). Then, the following cases are equivalent:*

(a) D^R is integrable;

(b) $g_1(\mathcal{T}_{Z_2} IZ_1 - \mathcal{T}_{Z_1} IZ_2, \omega_I Q_I U_1 + IS_I U_1) = g_1(\mathcal{V}\nabla_{Z_1} IZ_2 - \mathcal{V}\nabla_{Z_2} IZ_1, \phi_I Q_I U_1)$

for $Z_1, Z_2 \in \Gamma(D^I)$ and $U_1 \in \Gamma(D_1^I \oplus D_2^I)$;

(c) $g_1(\mathcal{T}_{Z_2} JZ_1 - \mathcal{T}_{Z_1} JZ_2, \omega_J Q_J U_1 + JS_J U_1) = g_1(\mathcal{V}\nabla_{Z_1} JZ_2 - \mathcal{V}\nabla_{Z_2} JZ_1, \phi_J Q_J U_1)$

for $Z_1, Z_2 \in \Gamma(D^J)$ and $U_1 \in \Gamma(D_1^J \oplus D_2^J)$;

(d) $g_1(\mathcal{T}_{Z_2} KZ_1 - \mathcal{T}_{Z_1} KZ_2, \omega_K Q_K U_1 + KS_K U_1) = g_1(\mathcal{V}\nabla_{Z_1} KZ_2 - \mathcal{V}\nabla_{Z_2} KZ_1, \phi_K Q_K U_1)$

for $Z_1, Z_2 \in \Gamma(D^K)$ and $U_1 \in \Gamma(D_1^K \oplus D_2^K)$.

Proof. For $Z_1, Z_2 \in \Gamma(D^R)$, $U_1 \in \Gamma(D_1^R \oplus D_2^R)$, $U_2 \in (\ker \pi_*)^\perp$ and $R \in \{I, J, K\}$, since $[Z_1, Z_2] \in (\ker \pi_*)$, we have $g_1([Z_1, Z_2], U_2) = 0$. Thus, D^R is integrable $\Leftrightarrow g_1([Z_1, Z_2], U_1) = 0$. Now, using Equations (4) and (12)–(14), we have

$$\begin{aligned}
& g_1([Z_1, Z_2], U_1) \\
= \; & g_1(R\nabla_{Z_1} Z_2, RU_1) - g_1(R\nabla_{Z_2} Z_1, RU_1), \\
= \; & g_1(\nabla_{Z_1} RZ_2, RU_1) - g_1(\nabla_{Z_2} RZ_1, RU_1), \\
= \; & g_1(\mathcal{T}_{Z_1} RZ_2 - \mathcal{T}_{Z_2} RZ_1, \omega_R Q_R U_1 + JRU_1) \\
& - g_1(\mathcal{V}\nabla_{Z_1} RZ_2 - \mathcal{V}\nabla_{Z_2} RZ_1, \phi_R Q_R U_1).
\end{aligned}$$

Since D^R is R-invariant, we have

$$(a) \Leftrightarrow (b), \;\; (a) \Leftrightarrow (c), \;\; (a) \Leftrightarrow (d).$$

Therefore, we obtain the result. □

Theorem 2. *The following cases are equivalent for the map π defined in Theorem 1:*

(a) D_1^R *is integrable;*

$$(b)\ \ g_1(\mathcal{T}_{Y_1}\omega_I\phi_I Y_2 - \mathcal{T}_{Y_2}\omega_I\phi_I Y_1, V_1) = g_1(\mathcal{T}_{Y_1}\omega_I Y_2 - \mathcal{T}_{Y_2}\omega_I Y_1, \phi_I P_I V_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_I Y_2 - \mathcal{H}\nabla_{Y_2}\omega_I Y_1, \omega_I S_I V_1)$$

for all $Y_1, Y_2 \in \Gamma(D_1^I)$ *and* $V_1 \in \Gamma(D^I \oplus D_2^I)$;

$$(c)\ \ g_1(\mathcal{T}_{Y_1}\omega_J\phi_J Y_2 - \mathcal{T}_{Y_2}\omega_J\phi_J Y_1, V_1) = g_1(\mathcal{T}_{Y_1}\omega_J Y_2 - \mathcal{T}_{Y_2}\omega_J Y_1, \phi_J P_J V_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_J Y_2 - \mathcal{H}\nabla_{Y_2}\omega_J Y_1, \omega_J S_J V_1)$$

for all $Y_1, Y_2 \in \Gamma(D_1^J)$ *and* $V_1 \in \Gamma(D^J \oplus D_2^J)$;

$$(d)\ \ g_1(\mathcal{T}_{Y_1}\omega_K\phi_K Y_2 - \mathcal{T}_{Y_2}\omega_K\phi_K Y_1, V_1) = g_1(\mathcal{T}_{Y_1}\omega_K Y_2 - \mathcal{T}_{Y_2}\omega_K Y_1, \phi_K P_K V_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_K Y_2 - \mathcal{H}\nabla_{Y_2}\omega_K Y_1, \omega_K S_K V_1)$$

for all $Y_1, Y_2 \in \Gamma(D_1^K)$ *and* $V_1 \in \Gamma(D^K \oplus D_2^K)$.

Proof. For $Y_1, Y_2 \in \Gamma(D_1^R)$, $V_1 \in \Gamma(D^R \oplus D_2^R)$, $V_2 \in (\ker F_*)^\perp$ and $R \in \{I, J, K\}$, since $[Y_1, Y_2] \in (\ker \pi_*)$, we have $g_1([Y_1, Y_2], V_2) = 0$. Thus, D_1^R is integrable $\Leftrightarrow g_1([Y_1, Y_2], V_1) = 0$. Using Equations (4), (5), (12) and (13) as well as Lemma 5, we have

$$\begin{aligned}
& g_1([Y_1, Y_2], V_1) \\
= {} & g_1(\nabla_{Y_1} RY_2, RV_1) - g_1(\nabla_{Y_2} RY_1, RV_1), \\
= {} & g_1(\nabla_{Y_1}\phi_R Y_2, RV_1) + g_1(\nabla_{Y_1}\omega_R Y_2, RV_1) - g_1(\nabla_{Y_2}\phi_R Y_1, RV_1) - g_1(\nabla_{Y_2}\omega_R Y_1, RV_1), \\
= {} & \cos^2\theta_R g_1(\nabla_{Y_1} Y_2, V_1) - \cos^2\theta_R g_1(\nabla_{Y_2} Y_1, V_1) - g_1(\mathcal{T}_{Y_1}\omega_R\phi_R Y_2 - \mathcal{T}_{Y_2}\omega_R\phi_R Y_1, V_1) \\
& + g_1(\mathcal{H}\nabla_{Y_1}\omega_R Y_2 + \mathcal{T}_{Y_1}\omega_R Y_2, RP_R V_1 + \omega_R S_R V_1) \\
& - g_1(\mathcal{H}\nabla_{Y_2}\omega_R Y_1 + \mathcal{T}_{Y_2}\omega_R Y_1, RP_R V_1 + \omega_R S_R V_1),
\end{aligned}$$

which gives

$$\begin{aligned}
\sin^2\theta_1 g_1([Y_1, Y_2], V_1) = {} & g_1(\mathcal{T}_{Y_1}\omega_R Y_2 - \mathcal{T}_{Y_2}\omega_R Y_1, RP_R V_1) \\
& + g_1(\mathcal{H}\nabla_{Y_1}\omega_R Y_2 - \mathcal{H}\nabla_{Y_2}\omega_R Y_1, \omega_R S_R V_1) \\
& - g_1(\mathcal{T}_{Y_1}\omega_R\phi_R Y_2 - \mathcal{T}_{Y_2}\omega_R\phi_R Y_1, V_1).
\end{aligned}$$

Since D_1^R is an R-slant distribution, therefore, we obtain

$$(a) \Leftrightarrow (b), \ \ (a) \Leftrightarrow (c), \ \ (a) \Leftrightarrow (d).$$

Therefore, we find the result. □

Theorem 3. *For the h-qhs Riemannian map π defined in Theorem 1, D_2^R is always integrable.*

Proof. We can easily prove the Theorem as hemi-slant case given in [21]. □

Theorem 4. *For the h-qhs Riemannian map π defined in Theorem 1, any one of the following assertions implies the others:*

(a) $(\ker \pi_*)^\perp$ *defines a totally geodesic foliation on* N_1;

$$(b)\ \ g_1(\mathcal{A}_{Z_1} Z_2, P_I W_1 + \cos^2\theta_I Q_I W_1) = g_1(\mathcal{H}\nabla_{Z_1} Z_2, \omega_I\phi_I P_I W_1 + \omega_I\phi_I Q_I W_1) - g_1(\mathcal{A}_{Z_1} B_I Z_2 + \mathcal{H}\nabla_{Z_1} C_I Z_2, \omega_I W_1)$$

for $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$ *and* $W_1 \in \Gamma(\ker \pi_*)$;

$$(c)\ \ g_1(\mathcal{A}_{Z_1}Z_2, P_J W_1 + \cos^2\theta_J Q_J W_1) = g_1(\mathcal{H}\nabla_{Z_1}Z_2, \omega_J\phi_J P_J W_1 + \omega_J\phi_J Q_J W_1) - g_1(\mathcal{A}_{Z_1}B_J Z_2 + \mathcal{H}\nabla_{Z_1}C_J Z_2, \omega_J W_1)$$

for $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$ *and* $W_1 \in \Gamma(\ker \pi_*)$;

$$(d)\ \ g_1(\mathcal{A}_{Z_1}Z_2, P_K W_1 + \cos^2\theta_K Q_K W_1) = g_1(\mathcal{H}\nabla_{Z_1}Z_2, \omega_K\phi_K P_K W_1 + \omega_K\phi_K Q_K W_1) - g_1(\mathcal{A}_{Z_1}B_K Z_2 + \mathcal{H}\nabla_{Z_1}C_K Z_2, \omega_K W_1)$$

for $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$ *and* $W_1 \in \Gamma(\ker \pi_*)$.

Proof. For $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp, W_1 \in \Gamma(\ker \pi_*)$ and $R \in \{I, J, K\}$, using Equations (6), (7) and (12)–(14) as well as Lemma 5, we have

$$\begin{aligned}
& g_1(\nabla_{Z_1}Z_2, W_1) \\
= \ & g_1(R\nabla_{Z_1}Z_2, RW_1), \\
= \ & g_1(R\nabla_{Z_1}Z_2, \phi_R P_R W_1 + \phi_R Q_R W_1 + \omega_R Q_R W_1 + \omega_R S_R W_1), \\
= \ & -g_1(\nabla_{Z_1}Z_2, \phi_R^2 P_R W_1 + \omega_R\phi_R P_R W_1 + \omega_R\phi_R Q_R W_1) \\
& + g_1(\nabla_{Z_1}B_R Z_2, \omega_R Q_R W_1 + \omega_R S_R W_1) + g_1(\nabla_{Z_1}C_R Z_2, \omega_R Q_R W_1 + \omega_R S_R W_1), \\
= \ & g_1(\mathcal{A}_{Z_1}Z_2, P_R W_1 + \cos^2\theta_R Q_R W_1) - g_1(\mathcal{H}\nabla_{Z_1}Z_2, \omega_R\phi_R P_R W_1 + \omega_R\phi_R Q_R W_1) \\
& + g_1(\mathcal{A}_{Z_1}B_R Z_2, \omega_R Q_R W_1 + \omega_R S_R W_1) + g_1(\mathcal{H}\nabla_{Z_1}C_R Z_2, \omega_R Q_R W_1 + \omega_R S_R W_1).
\end{aligned}$$

Thus, we obtain

$$(a) \Leftrightarrow (b),\ \ (a) \Leftrightarrow (c),\ \ (a) \Leftrightarrow (d).$$

Therefore, the result follows. □

Theorem 5. *The following conditions are equivalent for the h-qhs Riemannian map* π:

(a) $(\ker \pi_*)$ *defines a totally geodesic foliation on* N_1;

$$\begin{aligned}(b)\ \ g_1(\mathcal{T}_{X_1}P_I X_2 + \cos^2\theta_I \mathcal{T}_{X_1}Q_I X_2, Y_1) = \ & g_1(\mathcal{H}\nabla_{X_1}\omega_I\phi_I P_I X_2 + \mathcal{H}\nabla_{X_1}\omega_I\phi_I Q_I X_2, Y_1) \\ & - g_1(\mathcal{H}\nabla_{X_1}\omega_I Q_I X_2 + \mathcal{H}\nabla_{X_1}\omega_I S_I X_2, C_I Y_1) \\ & - g_1(\mathcal{T}_{X_1}\omega_I Q_I X_2 + \mathcal{T}_{X_1}\omega_I S_I X_2, B_I Y_1)\end{aligned}$$

for $X_1, X_2 \in \Gamma(\ker \pi_*)$ *and* $Y_1 \in \Gamma(\ker \pi_*)^\perp$;

$$\begin{aligned}(c)\ \ g_1(\mathcal{T}_{X_1}P_J X_2 + \cos^2\theta_J \mathcal{T}_{X_1}Q_J X_2, Y_1) = \ & g_1(\mathcal{H}\nabla_{X_1}\omega_J\phi_J P_J X_2 + \mathcal{H}\nabla_{X_1}\omega_J\phi_J Q_J X_2, Y_1) \\ & - g_1(\mathcal{H}\nabla_{X_1}\omega_J Q_J X_2 + \mathcal{H}\nabla_{X_1}\omega_J S_J X_2, C_J Y_1) \\ & - g_1(\mathcal{T}_{X_1}\omega_J Q_J X_2 + \mathcal{T}_{X_1}\omega_J S_J X_2, B_J Y_1)\end{aligned}$$

for $X_1, X_2 \in \Gamma(\ker \pi_*)$ *and* $Y_1 \in \Gamma(\ker \pi_*)^\perp$;

$$\begin{aligned}(d)\ \ g_1(\mathcal{T}_{X_1}P_K X_2 + \cos^2\theta_K \mathcal{T}_{X_1}Q_K X_2, Y_1) = \ & g_1(\mathcal{H}\nabla_{X_1}\omega_K\phi_K P_K X_2 + \mathcal{H}\nabla_{X_1}\omega_K\phi_K Q_K X_2, Y_1) \\ & - g_1(\mathcal{H}\nabla_{X_1}\omega_K Q_K X_2 + \mathcal{H}\nabla_{X_1}\omega_K S_K X_2, C_K Y_1) \\ & - g_1(\mathcal{T}_{X_1}\omega_K Q_K X_2 + \mathcal{T}_{X_1}\omega_K S_K X_2, B_K Y_1)\end{aligned}$$

for $X_1, X_2 \in \Gamma(\ker \pi_*)$ *and* $Y_1 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For $X_1, X_2 \in \Gamma(\ker \pi_*)$, $Y_1 \in \Gamma(\ker \pi_*)^\perp$ and $R \in \{I, J, K\}$, using Equations (4), (5) and (12)–(14) as well as Lemma 5, we have

$$\begin{aligned}
& g_1(\nabla_{X_1} X_2, Y_1) \\
= \; & g_1(R\nabla_{X_1} X_2, RY_1), \\
= \; & g_1(\nabla_{X_1} \phi_R P_R X_2, RY_1) + g_1(\nabla_{X_1} \phi_R Q_R X_2, RY_1) \\
& + g_1(\nabla_{X_1} \omega_R Q_R X_2, RY_1) + g_1(\nabla_{X_1} \omega_R S_R X_2, RY_1), \\
= \; & g_1(\mathcal{T}_{X_1} P_R X_2, Y_1) + \cos^2\theta_R g_1(\mathcal{T}_{X_1} Q_R X_2, Y_1) - g_1(\mathcal{H}\nabla_{X_1} \omega_R \phi_R P_R X_2, Y_1) \\
& - g_1(\mathcal{H}\nabla_{X_1} \omega_R \phi_R Q_R X_2, Y_1) + g_1(\mathcal{H}\nabla_{X_1} \omega_R Q_R X_2 + \mathcal{H}\nabla_{X_1} \omega_R S_R X_2, C_R Y_1) \\
& + g_1(\mathcal{T}_{X_1} \omega_R Q_R X_2 + \mathcal{T}_{X_1} \omega_R S_R X_2, B_R Y_1).
\end{aligned}$$

Thus, we obtain

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, the result follows. □

Theorem 6. *Let π be an h-qhs Riemannian map from an almost hyperkahler manifold (N_1, I, J, K, g_1) to a Riemannian manifold (N_2, g_2). Then, any one of the following assertions implies the others:*

(a) D^R defines a totally geodesic foliation on N_1;

$$(b) \quad \begin{aligned} g_1(\mathcal{T}_{Z_1} IP_I Z_2, \omega_I Q_I Y_1 + \omega_I S_I Y_1) &= -g_1(\mathcal{V}\nabla_{Z_1} IP_I Z_2, \phi_I Y_1), \\ g_1(\mathcal{T}_{Z_1} IP_I Z_2, C_I Y_2) &= -g_1(\mathcal{V}\nabla_{Z_1} IP_I Z_2, B_I Y_2) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D^I)$, $Y_1 \in \Gamma(D_1^I \oplus D_2^I)$ and $Y_2 \in \Gamma(\ker \pi_*)^\perp$;

$$(c) \quad \begin{aligned} g_1(\mathcal{T}_{Z_1} JP_J Z_2, \omega_J Q_J Y_1 + \omega_J S_J Y_1) &= -g_1(\mathcal{V}\nabla_{Z_1} JP_J Z_2, \phi_J Y_1), \\ g_1(\mathcal{T}_{Z_1} JP_J Z_2, C_J Y_2) &= -g_1(\mathcal{V}\nabla_{Z_1} JP_J Z_2, B_J Y_2) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D^J)$, $Y_1 \in \Gamma(D_1^J \oplus D_2^J)$ and $Y_2 \in \Gamma(\ker \pi_*)^\perp$;

$$(d) \quad \begin{aligned} g_1(\mathcal{T}_{Z_1} KP_K Z_2, \omega_K Q_K Y_1 + \omega_K S_K Y_1) &= -g_1(\mathcal{V}\nabla_{Z_1} KP_K Z_2, \phi_K Y_1), \\ g_1(\mathcal{T}_{Z_1} KP_K Z_2, C_K Y_2) &= -g_1(\mathcal{V}\nabla_{Z_1} KP_K Z_2, B_K Y_2) \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D^K)$, $Y_1 \in \Gamma(D_1^K \oplus D_2^K)$ and $Y_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For $Z_1, Z_2 \in \Gamma(D^R)$, $Y_1 \in \Gamma(D_1^R \oplus D_2^R)$, $Y_2 \in \Gamma(\ker \pi_*)^\perp$ and $R \in \{I, J, K\}$, using Equations (4), (12) and (13), we have

$$\begin{aligned}
& g_1(\nabla_{Z_1} Z_2, Y_1) \\
= \; & g_1(\nabla_{Z_1} RZ_2, RY_1), \\
= \; & g_1(\nabla_{Z_1} RP_R Z_2, RQ_R Y_1 + RS_R Y_1), \\
= \; & g_1(\mathcal{T}_{Z_1} \phi_R P_R Z_2, \omega_R Q_R Y_1 + \omega_R S_R Y_1) + g_1(\mathcal{V}\nabla_{Z_1} \phi_R P_R Z_2, \phi_R Q_R Y_1).
\end{aligned}$$

Moreover, using Equations (4), (12) and (14), we obtain

$$\begin{aligned}
& g_1(\nabla_{Z_1} Z_2, Y_2) \\
= \; & g_1(\nabla_{Z_1} RZ_2, RY_2), \\
= \; & g_1(\nabla_{Z_1} RP_R Z_2, B_R Y_2 + C_R Y_2), \\
= \; & g_1(\mathcal{V}\nabla_{Z_1} RP_R Z_2, B_R Y_2) + g_1(\mathcal{T}_{Z_1} JP_R Z_2, C_R Y_2).
\end{aligned}$$

Hence, we have

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, the result follows. □

Theorem 7. *With $\pi : (N_1, I, J, K, g_1) \to (N_2, g_2)$ being an h-qhs Riemannian map, the following conditions are equivalent:*

(a) D_1^R defines a totally geodesic foliation on N_1;

$$(b)\ \begin{aligned} g_1(\mathcal{T}_{Y_1}\omega_I\phi_I Y_2, Z_1) &= g_1(\mathcal{T}_{Y_1}\omega_I Y_2, \phi_I P_I Z_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_I Y_2, \omega_I S_I Z_1), \\ g_1(\mathcal{H}\nabla_{Y_1}\omega_I\phi_I Y_2, Z_2) &= g_1(\mathcal{H}\nabla_{Y_1}\omega_I Y_2, C_I Z_2) + g_1(\mathcal{T}_{Y_1}\omega_I Y_2, B_I Z_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_1^I)$, $Z_1 \in \Gamma(D^I \oplus D_2^I)$ and $Z_2 \in \Gamma(\ker \pi_*)^\perp$;

$$(c)\ \begin{aligned} g_1(\mathcal{T}_{Y_1}\omega_J\phi_J Y_2, Z_1) &= g_1(\mathcal{T}_{Y_1}\omega_J Y_2, \phi_J P_J Z_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_J Y_2, \omega_J S_J Z_1), \\ g_1(\mathcal{H}\nabla_{Y_1}\omega_J\phi_J Y_2, Z_2) &= g_1(\mathcal{H}\nabla_{Y_1}\omega_J Y_2, C_J Z_2) + g_1(\mathcal{T}_{Y_1}\omega_J Y_2, B_J Z_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_1^J)$, $Z_1 \in \Gamma(D^J \oplus D_2^J)$ and $Z_2 \in \Gamma(\ker \pi_*)^\perp$;

$$(d)\ \begin{aligned} g_1(\mathcal{T}_{Y_1}\omega_K\phi_K Y_2, Z_1) &= g_1(\mathcal{T}_{Y_1}\omega_K Y_2, \phi_K P_K Z_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_K Y_2, \omega_K S_K Z_1), \\ g_1(\mathcal{H}\nabla_{Y_1}\omega_K\phi_K Y_2, Z_2) &= g_1(\mathcal{H}\nabla_{Y_1}\omega_K Y_2, C_K Z_2) + g_1(\mathcal{T}_{Y_1}\omega_K Y_2, B_K Z_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_1^K)$, $Z_1 \in \Gamma(D^K \oplus D_2^K)$ and $Z_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For $Y_1, Y_2 \in \Gamma(D_1^R)$, $Z_1 \in \Gamma(D^R \oplus D_2^R)$, $Z_2 \in \Gamma(\ker \pi_*)^\perp$ and $R \in \{I, J, K\}$, using Equations (5), (12) and (13) as well as Lemma 5, we have

$$\begin{aligned} & g_1(\nabla_{Y_1} Y_2, Z_1) \\ = \ & g_1(\nabla_{Y_1} RY_2, RZ_1), \\ = \ & g_1(\nabla_{Y_1}\phi_R Y_2, RZ_1) + g_1(\nabla_{Y_1}\omega_R Y_2, RZ_1), \\ = \ & \cos^2\theta_R g_1(\nabla_{Y_1} Y_2, Z_1) - g_1(\mathcal{T}_{Y_1}\omega_R\phi_R Y_2, Z_1) \\ & + g_1(\mathcal{T}_{Y_1}\omega_R Y_2, \phi_R P_R Z_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_R Y_2, \omega_R S_R Z_1), \end{aligned}$$

which gives

$$\begin{aligned} & \sin^2\theta_R g_1(\nabla_{Y_1} Y_2, Z_1) \\ = \ & -g_1(\mathcal{T}_{Y_1}\omega_R\phi_R Y_2, Z_1) + g_1(\mathcal{T}_{Y_1}\omega_R Y_2, RP_R Z_1) \\ & + g_1(\mathcal{H}\nabla_{Y_1}\omega_R Y_2, \omega_R S_R Z_1). \end{aligned}$$

Moreover, from Equations (5), (13) and (14) as well as Lemma 5, we have

$$\begin{aligned} & g_1(\nabla_{Y_1} Y_2, Z_2) \\ = \ & g_1(\nabla_{Y_1} RY_2, RZ_2), \\ = \ & g_1(\nabla_{Y_1}\phi_R Y_2, RZ_2) + g_1(\nabla_{Y_1}\omega_R Y_2, RZ_2), \\ = \ & \cos^2\theta_R g_1(\nabla_{Y_1} Y_2, Z_2) - g_1(\mathcal{H}\nabla_{Y_1}\omega_R\phi_R Y_2, Z_2) \\ & + g_1(\mathcal{H}\nabla_{Y_1}\omega_R Y_2, C_R Z_2) + g_1(\mathcal{T}_{Y_1}\omega_R Y_2, B_R Z_2). \end{aligned}$$

Thus, we find that

$$\begin{aligned} & \sin^2\theta_R g_1(\nabla_{Y_1} Y_2, Z_2) \\ = \ & -g_1(\mathcal{H}\nabla_{Y_1}\omega_R\phi_R Y_2, Z_2) + g_1(\mathcal{H}\nabla_{Y_1}\omega_R Y_2, C_R Z_2) + g_1(\mathcal{T}_{Y_1}\omega_R Y_2, B_R Z_2). \end{aligned}$$

Hence, we have

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, the result follows. □

Theorem 8. *For the h-qhs Riemannian map π defined in Theorem 1, any one of the following assertions implies the others:*

(a) D_2^R *defines a totally geodesic foliation on* N_1*;*

$$\begin{aligned} (b)\ \ g_1(\mathcal{H}\nabla_{Y_1}\omega_I Y_2, \omega_I Q_I W_1) &= -g_1(\mathcal{T}_{Y_1}\omega_I S_I Y_2, \phi_I P_I W_1 + \phi_I Q_I W_1), \\ g_1(\mathcal{H}\nabla_{Y_1}\omega_I S_I Y_2, C_I W_2) &= -g_1(\mathcal{T}_{Y_1}\omega_I S_I Y_2, B_I W_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_2^I), W_1 \in \Gamma(D^I \oplus D_1^I)$ *and* $W_2 \in \Gamma(\ker \pi_*)^\perp$*;*

$$\begin{aligned} (c)\ \ g_1(\mathcal{H}\nabla_{Y_1}\omega_J Y_2, \omega_J Q_J W_1) &= -g_1(\mathcal{T}_{Y_1}\omega_J S_J Y_2, \phi_J P_J W_1 + \phi_J Q_J W_1), \\ g_1(\mathcal{H}\nabla_{Y_1}\omega_J S Y_2, C_J W_2) &= -g_1(\mathcal{T}_{Y_1}\omega_J S Y_2, B_J W_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_2^J), W_1 \in \Gamma(D^J \oplus D_1^J)$ *and* $W_2 \in \Gamma(\ker \pi_*)^\perp$*;*

$$\begin{aligned} (d)\ \ g_1(\mathcal{H}\nabla_{Y_1}\omega_K Y_2, \omega_K Q_K W_1) &= -g_1(\mathcal{T}_{Y_1}\omega_K S Y_2, \phi_K P_K W_1 + \phi_K Q_K W_1), \\ g_1(\mathcal{H}\nabla_{Y_1}\omega_K S Y_2, C_K W_2) &= -g_1(\mathcal{T}_{Y_1}\omega_K S_K Y_2, B_K W_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_2^K), W_1 \in \Gamma(D^K \oplus D_1^K)$ *and* $W_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. For $Y_1, Y_2 \in \Gamma(D_2^R)$, $W_1 \in \Gamma(D^R \oplus D_1^R)$, $W_2 \in \Gamma(\ker \pi_*)^\perp$ and $R \in \{I, J, K\}$, using Equations (5), (12) and (13), we have

$$\begin{aligned} & g_1(\nabla_{Y_1} Y_2, W_1) \\ = \ & g_1(\nabla_{Y_1} R Y_2, R W_1) \\ = \ & g_1(\nabla_{Y_1}\omega_R S_R Y_2, \phi_R P_R W_1 + \phi_R Q_R W_1 + \omega_R Q_R W_1), \\ = \ & g_1(\mathcal{T}_{Y_1}\omega_R S_R Y_2, \phi_R P_R W_1 + \phi_R Q_R W_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_R S_R Y_2, \omega_R Q_R W_1). \end{aligned}$$

Again, using Equations (5), (13) and (14), we have

$$\begin{aligned} g_1(\nabla_{Y_1} Y_2, W_2) &= g_1(\nabla_{Y_1} R Y_2, R W_2) \\ &= g_1(\nabla_{Y_1}\omega_R S_R Y_2, B_R W_2 + C_R W_2), \\ &= g_1(\mathcal{T}_{Y_1}\omega_R S_R Y_2, B_R W_2) + g_1(\ \mathcal{H}\nabla_{Y_1}\omega_R R Y_2, C_R W_2). \end{aligned}$$

Hence, we have

$$(a) \Leftrightarrow (b), \ \ (a) \Leftrightarrow (c), \ \ (a) \Leftrightarrow (d).$$

Therefore, the result follows. □

Theorem 9. *Let π be an h-qhs Riemannian map from an almost hyperkahler manifold (N_1, I, J, K, g_1) to a Riemannian manifold (N_2, g_2). Then, the following conditions are equivalent:*

(a) π *is a totally geodesic map;*

$$\begin{aligned} (b)\ \ & g_1(\mathcal{T}_{Y_1} P_I Y_2 + \cos^2\theta_I \mathcal{T}_{Y_1} Q_I Y_2 - \mathcal{H}\nabla_{Y_1}\omega_I \phi_I P_I Y_2 - \mathcal{H}\nabla_{Y_1}\omega_I \phi_I Q_I Y_2, W_1) \\ & = g_1(\mathcal{T}_{Y_1}\omega_I Q_I Y_2 + \mathcal{T}_{Y_1}\omega_I S_I Y_2, B_I W_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_I \phi_I Q_I Y_2 + \mathcal{H}\nabla_{Y_1}\omega_I \phi_I S_I Y_2, W_1), \end{aligned}$$

$$\begin{aligned} & g_1(\mathcal{A}_{W_1} P_I Y_1 + \cos^2\theta_I \mathcal{A}_{W_1} Q_I Y_1 - \mathcal{H}\nabla_{W_1}\omega_I \phi_I P_I Y_1 - \mathcal{H}\nabla_{W_1}\omega_I \phi_I Q_I Y_1, W_2) \\ & = g_1(\mathcal{A}_{W_1}\omega_I Q_I Y_1 + \mathcal{A}_{W_1}\omega_I S_I Y_1, B_I W_2) + g_1(\mathcal{H}\nabla_{W_1}\omega_I Q_I Y_1 + \mathcal{H}\nabla_{W_1}\omega_I S_I Y_1, C_I W_2) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ *and* $W_1, W_2 \in \Gamma(\ker \pi_*)^\perp$*;*

$$\begin{aligned} (c)\ \ & g_1(\mathcal{T}_{Y_1} P_J Y_2 + \cos^2\theta_J \mathcal{T}_{Y_1} Q_J Y_2 - \mathcal{H}\nabla_{Y_1}\omega_J \phi_J P_J Y_2 - \mathcal{H}\nabla_{Y_1}\omega_J \phi_J Q_J Y_2, W_1) \\ & = g_1(\mathcal{T}_{Y_1}\omega_J Q_J Y_2 + \mathcal{T}_{Y_1}\omega_J S_J Y_2, B_J W_1) + g_1(\mathcal{H}\nabla_{Y_1}\omega_J \phi_J Q_J J Y_2 + \mathcal{H}\nabla_{Y_1}\omega_J \phi_J S_J Y_2, W_1), \end{aligned}$$

$$g_1(\mathcal{A}_{W_1}P_JY_1+\cos^2\theta_J\mathcal{A}_{W_1}Q_JY_1-\mathcal{H}\nabla_{W_1}\omega_J\phi_JP_JY_1-\mathcal{H}\nabla_{W_1}\omega_J\phi_JQ_JY_1,W_2)$$
$$=g_1(\mathcal{A}_{W_1}\omega_JQ_JY_1+\mathcal{A}_{W_1}\omega_JS_JY_1,B_JW_2)+g_1(\mathcal{H}\nabla_{W_1}\omega_JQ_JY_1+\mathcal{H}\nabla_{W_1}\omega_JS_JY_1,C_JW_2)$$

for $Y_1,Y_2\in\Gamma(\ker\pi_*)$ *and* $W_1,W_2\in\Gamma(\ker\pi_*)^\perp$;

(d) $$g_1(\mathcal{T}_{Y_1}P_KY_2+\cos^2\theta_K\mathcal{T}_{Y_1}Q_KY_2-\mathcal{H}\nabla_{Y_1}\omega_K\phi_KP_KY_2-\mathcal{H}\nabla_{Y_1}\omega_K\phi_KQ_KY_2,W_1)$$
$$=g_1(\mathcal{T}_{Y_1}\omega_KQ_KY_2+\mathcal{T}_{Y_1}\omega_KS_KY_2,B_KW_1)+g_1(\mathcal{H}\nabla_{Y_1}\omega_K\phi_KQ_KY_2+\mathcal{H}\nabla_{Y_1}\omega_K\phi_KS_KY_2,W_1),$$

$$g_1(\mathcal{A}_{W_1}P_KY_1+\cos^2\theta_K\mathcal{A}_{W_1}Q_KY_1-\mathcal{H}\nabla_{W_1}\omega_K\phi_KP_KY_1-\mathcal{H}\nabla_{W_1}\omega_K\phi_KQ_KY_1,W_2)$$
$$=g_1(\mathcal{A}_{W_1}\omega_KQ_KY_1+\mathcal{A}_{W_1}\omega_KS_KY_1,B_KW_2)+g_1(\mathcal{H}\nabla_{W_1}\omega_KQ_KY_1+\mathcal{H}\nabla_{W_1}\omega_KS_KY_1,C_KW_2)$$

for $Y_1,Y_2\in\Gamma(\ker\pi_*)$ *and* $W_1,W_2\in\Gamma(\ker\pi_*)^\perp$.

Proof. Since π is a Riemannian map, therefore, we have

$$(\nabla\pi_*)(W_1,W_2)=0,$$

for $W_1,W_2\in\Gamma(\ker\pi_*)^\perp$.

For $Y_1,Y_2\in\Gamma(\ker\pi_*)$, $W_1,W_2\in\Gamma(\ker\pi_*)^\perp$ and $R\in\{I,J,K\}$, using Equations (4), (5) and (12)–(14) as well as Lemma 5, we have

$$\begin{aligned}
&g_2((\nabla\pi_*)(Y_1,Y_2),\pi_*(W_1))\\
=\;&-g_1(\nabla_{Y_1}Y_2,W_1)\\
=\;&-g_1(\nabla_{Y_1}RY_2,RW_1)\\
=\;&-g_1(\nabla_{Y_1}RP_RY_2,RW_1)-g_1(\nabla_{Y_1}RQ_RY_2,RW_1)-g_1(\nabla_{Y_1}RS_RY_2,RW_1),\\
=\;&-g_1(\nabla_{Y_1}\phi_RP_RY_2,RW_1)-g_1(\nabla_{Y_1}\phi_RQ_RY_2,RW_1)\\
&-g_1(\nabla_{Y_1}\omega_RQ_RY_2,RW_1)-g_1(\nabla_{Y_1}\omega_RS_RY_2,RW_1),\\
=\;&-g_1(\mathcal{T}_{Y_1}P_RY_2+\cos^2\theta_R\mathcal{T}_{Y_1}Q_RY_2-\mathcal{H}\nabla_{Y_1}\omega_R\phi_RP_RY_2-\mathcal{H}\nabla_{Y_1}\omega_R\phi_RQ_RY_2,W_1)\\
&-g_1(\mathcal{T}_{Y_1}\omega_RQ_RY_2+\mathcal{T}_{Y_1}\omega_RS_RY_2,B_RW_1)\\
&-g_1(\mathcal{H}\nabla_{Y_1}\omega_R\phi_RQ_RY_2+\mathcal{H}\nabla_{Y_1}\omega_R\phi_RS_RY_2,W_1).
\end{aligned}$$

Moreover, using Equations (4), (5) and (12)–(14) as well as Lemma 5, we have

$$\begin{aligned}
&g_2((\nabla\pi_*)(W_1,Y_1),\pi_*(W_2))\\
=\;&-g_1(\nabla_{W_1}Y_1,W_2),\\
=\;&-g_1(\nabla_{W_1}RY_1,RW_2),\\
=\;&-g_1(\nabla_{W_1}RP_RY_1,RW_2)-g_1(\nabla_{W_1}RQ_RY_1,RW_2)-g_1(\nabla_{W_1}RS_RY_1,RW_2),\\
=\;&-g_1(\nabla_{W_1}\phi_RP_RY_1,RW_2)-g_1(\nabla_{W_1}\phi_RQ_RY_1,RW_2)\\
&-g_1(\nabla_{W_1}\omega_RQ_RY_1,RW_2)-g_1(\nabla_{W_1}\omega_RS_RY_1,RW_2),\\
=\;&-g_1(\mathcal{A}_{W_1}P_RY_1+\cos^2\theta_R\mathcal{A}_{W_1}Q_RY_1-\mathcal{H}\nabla_{W_1}\omega_R\phi_RP_RY_1-\mathcal{H}\nabla_{W_1}\omega_R\phi_RQ_RY_1,W_2)\\
&-g_1(\mathcal{A}_{W_1}\omega_RQ_RY_1+\mathcal{A}_{W_1}\omega_RS_RY_1,B_RW_2)\\
&-g_1(\mathcal{H}\nabla_{W_1}\omega_RQ_RY_1+\mathcal{H}\nabla_{W_1}\omega_RS_RY_1,C_RW_2).
\end{aligned}$$

Hence, we obtain

$$(a)\Leftrightarrow(b),(a)\Leftrightarrow(c),(a)\Leftrightarrow(d).$$

Thus, the theorem is proven. □

4. Example

Note that given a Euclidean space $\mathbb{R}^{4n}$ with coordinates $(x_1, x_2, \ldots.., x_{4n})$, we can canonically choose complex structures I, J and K on $\mathbb{R}^{4n}$ as follows:

$$
\begin{aligned}
I(\frac{\partial}{\partial x_{4s+1}}) &= \frac{\partial}{\partial x_{4s+2}}, I(\frac{\partial}{\partial x_{4s+2}}) = -\frac{\partial}{\partial x_{4s+1}}, I(\frac{\partial}{\partial x_{4s+3}}) = \frac{\partial}{\partial x_{4s+4}}, \\
I(\frac{\partial}{\partial x_{4s+4}}) &= -\frac{\partial}{\partial x_{4s+3}}, J(\frac{\partial}{\partial x_{4s+1}}) = \frac{\partial}{\partial x_{4s+3}}, J(\frac{\partial}{\partial x_{4s+2}}) = -\frac{\partial}{\partial x_{4s+4}}, \\
J(\frac{\partial}{\partial x_{4s+3}}) &= -\frac{\partial}{\partial x_{4s+1}}, J(\frac{\partial}{\partial x_{4s+4}}) = \frac{\partial}{\partial x_{4s+2}}, K(\frac{\partial}{\partial x_{4s+1}}) = \frac{\partial}{\partial x_{4s+4}}, \\
K(\frac{\partial}{\partial x_{4s+2}}) &= \frac{\partial}{\partial x_{4s+3}}, K(\frac{\partial}{\partial x_{4s+3}}) = -\frac{\partial}{\partial x_{4s+2}}, K(\frac{\partial}{\partial x_{4s+4}}) = -\frac{\partial}{\partial x_{4s+1}},
\end{aligned}
$$

for $s \in \{0, 1, 2, \ldots., \ldots, n-1\}$.

Then, we can easily check that $(I, J, K, \langle , \rangle)$ is a hyperkähler structure on $\mathbb{R}^{4n}$, where $\langle , \rangle$ denotes the Euclidean metric on $\mathbb{R}^{4n}$. Throughout this section, we will use these notations.

Example 1. *Define a map* $\pi : \mathbb{R}^{12} \to \mathbb{R}^6$ *by*

$$\pi(x_1, x_2, \ldots\ldots\ldots, x_{12}) = (2020, x_2, x_6, \frac{x_8 - x_9}{\sqrt{2}}, x_{10}, 2022).$$

Then, the map π is an almost h-qhs Riemannian map such that

$$\ker \pi_* = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \right\rangle,$$

$$(\ker \pi_*)^\perp = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_6}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_8} - \frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_{10}} \right\rangle,$$

$$D^I = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \right\rangle, D_1^I = \left\langle \frac{\partial}{\partial x_7}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_9}) \right\rangle, D_2^I = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_5} \right\rangle,$$

$$D^J = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} \right\rangle, D_1^J = \left\langle \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_{11}} \right\rangle, D_2^J = \left\langle \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{12}} \right\rangle,$$

$$D^K = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4} \right\rangle, D_1^K = \left\langle \frac{\partial}{\partial x_5}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_{12}} \right\rangle, D_2^K = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_{11}} \right\rangle,$$

with the almost h-qhs angles $\{\theta_I = \theta_J = \theta_K = \frac{\pi}{4}\}$.

Example 2. *Define a map* $\pi : \mathbb{R}^{16} \to \mathbb{R}^8$ *by*

$$\pi(x_1, x_2, \ldots\ldots\ldots, x_{16}) = (101, \frac{\sqrt{3}x_5 - x_9}{2}, x_6, x_8, x_{11}, x_{14}, 202, x_{15}).$$

Then, the map π is an almost h-qhs Riemannian map such that

$$\ker \pi_* = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{1}{2}(\frac{\partial}{\partial x_5} + \sqrt{3}\frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{16}} \right\rangle,$$

$$(\ker \pi_*)^\perp = \left\langle \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{15}}, \right\rangle,$$

$$D^I = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle, D_1^I = \left\langle \frac{1}{2}(\frac{\partial}{\partial x_5} + \sqrt{3}\frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_{10}} \right\rangle, D_2^I = \left\langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{16}} \right\rangle,$$

$$D^J = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{12}} \right\rangle, D_1^J = \left\langle \frac{1}{2}(\frac{\partial}{\partial x_5} + \sqrt{3}\frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_7} \right\rangle, D_2^J = \left\langle \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{16}} \right\rangle,$$

$$D^K = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{13}}, \frac{\partial}{\partial x_{16}} \right\rangle, D_1^K = \left\langle \frac{1}{2}(\frac{\partial}{\partial x_5} + \sqrt{3}\frac{\partial}{\partial x_9}), \frac{\partial}{\partial x_{12}} \right\rangle, D_2^K = \left\langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_{10}} \right\rangle,$$

with the almost h-qhs angles $\{\theta_I = \frac{\pi}{6}, \theta_J = \frac{\pi}{3}, \theta_K = \frac{\pi}{6}\}$.

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