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Existence Solutions for Implicit Fractional Relaxation Differential Equations with Impulsive Delay Boundary Conditions

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Abstract: The aim of this paper is to study the existence and uniqueness of solutions for nonlinear fractional relaxation impulsive implicit delay differential equations with boundary conditions. Some findings are established by applying the Banach contraction mapping principle and the Schauder fixed-point theorem. An example is provided that illustrates the theoretical results.

Keywords: fractional relaxation impulsive implicit differential equations; Riemann–Liouville fractional derivative; Liouville–Caputo fractional derivative; existence; uniqueness; fixed point

MSC: 15B52; 30E25; 34A12; 34D35

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1. Introduction

A branch of mathematics known as fractional calculus looks into the characteristics of non-integer order derivatives and integrals (called fractional derivatives and integrals, briefly differintegrals). The idea and techniques for solving differential equations with fractional derivatives of the unknown function, often known as fractional differential equations, are a focus of this field. Many authors have studied the fractional differential equations (refer to [1–8]). Fractional differential equations provide an excellent tool for describing memory and inherited traits of several materials and processes. The benefits of fractional differential equations become apparent in modeling mechanical and electrical characteristics of actual materials, as well as in the description of rheological effects of rocks, and numerous fields. In [9–11], authors explained the applications of fractional differential equations.

Differential equations including impulse conditions have been widely analyzed in the literature. There has been a lot of interest in the research of these type of problems. The presence of solutions to several classes of fractional order implicit fractional differential equations with impulse conditions has gained a lot of attention recently (see references [12–14]).

Delay differential equations (DDEs) have emerged as a lively research area. It has emerged in applications as a model of equations. In DDEs, the past dictates how the system will evolve at a specific time instant. The life cycle stages, the interval between contagion and the new viruses' emergence, an infectious period, the immunological period, etc. are examples of hidden processes that are related to time delays and time lags in these models. ODEs simultaneously evaluate the unknown state and its derivatives. A differential model becomes substantially more complex when such time delays are added. Therefore, these models' stability and bifurcation analysis are critical for comprehending their qualitative behavior. These models have not undergone adequate sensitivity analysis or parameter identifiability investigations. In [15], Biao Zeng examined the fractional impulsive feedback control systems with finite delay. The authors studied the existence of impulsive neutral fractional differential equations with infinite delay in [16]. We could list certain publications related to fractional differential equations including state-dependent delay [17,18] and discrete delay [19].

Fractional differential equations are important in control, stellar interiors, star clusters, electrochemistry, viscoelasticity, and in optics. The impulsive conditions mount from the real-world problems to describe the dynamics of processes in which sudden, discontinuous jumps occur. In many fields of the life sciences, including population dynamics, immunology, epidemiology, neural networks, and physiology, modeling with DDEs is commonly employed. The entire process is seen in optimal control, electric circuits, neutral network, medicines, and so on. Because of the applications, we have included implicit, impulsive, and delay to our main problem.

In [20], S. Krim and others investigated the existence and uniqueness of solutions for the following Caputo–Hadamard implicit delay fractional differential equations with boundary conditions

$$\begin{cases} u(t) = \zeta(t);\ t \in [1-h, 1], \\ ({}^{HC}D_1^r u)(t) = f(t, u_t, {}^{HC}D_1^r u)(t));\ t \in I := [1, T], \\ u'(T) = u_T, \end{cases}$$

where $r \in (1,2)$, $T > 1$, $h > 0$, $\zeta(t) \in \mathcal{C}$, $u_T \in \mathcal{R}$, $f : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, ${}^{HC}D_1^r$ is the Caputo–Hadamard fractional derivative of order r, and the space of continuous functions $\mathcal{C} := C([1-h, 1], \mathbb{R})$.

In [21], the authors studied the existence and uniqueness of positive solutions of the given non-linear fractional relaxation differential equation

$$\begin{cases} {}^{LC}D^\alpha \varkappa(t) + \lambda \varkappa(t) = f(t, \varkappa(t)), \quad 0 < t \leq 1, \\ \varkappa(0) = \varkappa_0 > 0, \end{cases}$$

where ${}^{LC}D^\alpha$ is the Liouville–Caputo fractional derivative, $\alpha \in (0,1]$. By using the fixed-point theorems and the method of the lower and upper solutions, the existence and uniqueness of solutions have been examined.

In [22], A. Lachouri, A. Djoudi, and A.Ardjouni discussed the existence and uniqueness of solutions for the below fractional relaxation integrodifferential equations with boundary conditions

$$\begin{cases} D^\beta\, {}^{LC}D^\alpha \varkappa(t) + \lambda \varkappa(t) = f(t, \varkappa(t), I^r \varkappa(t)),\ \lambda \in \mathbb{R}, \quad 0 < t < T, \\ {}^{LC}D^\alpha \varkappa(0) = {}^{LC}D^\alpha \varkappa(T) = 0, \quad \varkappa(0) = a \int_0^T \varkappa_s ds + b, \quad a, b \in \mathbb{R}, \end{cases}$$

where ${}^{LC}D^\alpha$ and D^β are Liouville–Caputo (L-C) fractional derivative and the Riemann–Liouville (R-L) fractional derivative of orders α and β, respectively, $\alpha \in (0,1)$, $\beta \in (1,2)$, I^r is the Riemann–Liouville fractional integral of order $r \in (0,1)$, and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function.

Motivated by the aforementioned works, by applying the Schauder and Banach fixed-point theorems, we examine the existence and uniqueness of solutions for the following implicit fractional relaxation differential equation with impulsive delay conditions of the form

$$\begin{cases} D^{\beta}\ {}^{LC}D^{\alpha}\varkappa(\tau) + \lambda\varkappa(\tau) = f(\tau, \varkappa_\tau, D^{\beta}\quad {}^{LC}D^{\alpha}\varkappa(\tau)), \quad \tau \neq \tau_\kappa, \quad \tau \in \mathcal{L} = [0, T], \quad \lambda \in \mathbb{R}, \\ \Delta\varkappa(\tau_\kappa) = I_\kappa(\varkappa(\tau_\kappa^-)), \quad \kappa = 1, 2, ..., m, \\ \varkappa(\tau) = \zeta(\tau),\ \tau \in [-h, 0] \\ {}^{LC}D^{\alpha}\varkappa(0) = {}^{LC}\ D^{\alpha}\varkappa(T) = 0, \quad \varkappa(0) = \mu\int_0^T \varkappa_s ds + \nu, \quad \mu, \nu \in \mathbb{R}, \end{cases} \tag{1}$$

where D^{β} and ${}^{LC}D^{\alpha}$ are the R-L fractional derivative and L-C fractional derivative of orders β and α, respectively, $0 < \alpha < 1$, $\zeta \in C$, $1 < \beta < 2$, and $f : \mathcal{L} \times PC([-h, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function. $I_{kappa} : PC([-h, 0], \mathbb{R}) \to \mathbb{R}$, $\Delta\varkappa(\tau_\kappa) = \varkappa(\tau_\kappa^+) - \varkappa(\tau_\kappa^-)$ indicates the jump of $\varkappa$ at $\tau = \tau_\kappa$, $\varkappa(\tau_\kappa^+)$ and $\varkappa(\tau_\kappa^-)$ represents the right and left limits of $\varkappa(\tau)$ at $\tau = \tau_\kappa$, respectively, and $\kappa = 1, 2, ..., m$. is the space of continuous functions $\mathcal{C} := \mathcal{C}([-h, 0], \mathbb{R})$. For any $\tau \in \mathcal{L}$, we represent $\varkappa_\tau$ by

$$\varkappa_\tau(s) = \varkappa(\tau + s) \text{ and } -h \leq s \leq 0.$$

that is, $\varkappa_\tau(.)$ represents the history of the state from time $\tau - h$ up to time τ.

We utilize the usual fixed-point theorem due to Schauder and Banach, is used to study the existence and uniqueness results. Finally, this work is strengthened by providing examples. The main novelty of this paper is focused with implicit and delay conditions. Basic results are discussed in preliminaries section. New findings are mentioned in the main results and the illustration is presented in the example section.

2. Preliminaries

This section introduces several terminologies, notations, and results related to fractional calculus.

Denote the Banach space of all continuous real functions by $\mathcal{C}(\mathcal{L}) = C(\mathcal{L}, \mathbb{R})$ on $\mathcal{L} := [0, T]$ endowed with the norm

$$\|\varkappa\|_\infty := \sup\{|\varkappa(\tau)| : 0 \leq \tau \leq T\}.$$

Consider the sets of functions

$$\begin{aligned} PC([-h, 0], \mathbb{R}) \quad &= \quad \{\varkappa : [-h, 0] \to \mathbb{R} : \varkappa \in C((\tau_k, \tau_{k+1}], \mathbb{R}), k = 0, ..., l, \text{and there exist} \\ &\qquad \varkappa(\tau_K^-) \text{ and } \varkappa(\tau_K^+), k = 1, ..., l, \text{with} \varkappa(\tau_k^-) = \varkappa(\tau_k^-)\}. \end{aligned}$$

$PC([-h, 0], \mathbb{R})$ is a Banach space with the norm

$$\|\varkappa\|_{PC} = \sup_{\tau \in [-h,0]} |\varkappa(\tau)|.$$

$$\begin{aligned} PC([0, T], \mathbb{R}) \quad &= \quad \{\varkappa : [0, T] \to \mathbb{R} : \varkappa \in C((\tau_k, \tau_{k+1}], \mathbb{R}), k = 0, ..., m, \text{and there exist} \\ &\qquad \varkappa(\tau_K^-) \text{ and } \varkappa(\tau_K^+), k = 1, ..., m, \text{with} \varkappa(\tau_k^-) = \varkappa(\tau_k^-)\}. \end{aligned}$$

$PC([0, T], \mathbb{R})$ is a Banach space with the norm

$$\|\varkappa\|_{PC_1} = \sup_{\tau \in [0,T]} |\varkappa(\tau)|.$$

Moreover, the Banach space $\mathcal{C} := \mathcal{C}([-h, T])$ with the norm

$$\|\varkappa\|_{\mathcal{C}} := \sup\{|\varkappa(\tau)| : -h \leq \tau \leq T]\}.$$

The space of absolutely continuous valued functions $AC(\mathcal{L})$ from $\mathcal{L}$ into $\mathbb{R}$, and set

$$AC^m(\mathcal{L}) = \{\varkappa : \mathcal{L} \to \mathbb{R} : \varkappa, \varkappa^{'}, \varkappa^{''}, \dots, \varkappa^{m-1} \in \mathcal{C} \quad and \quad \varkappa^{m-1} \in AC(\mathcal{L})\}.$$

Now we give out some fractional calculus results and properties.

Definition 1 ([10]). *Let $\alpha > 0$ be the fractional integral order of a function $h : \mathcal{L} \to \mathbb{R}$ is defined by*

$$I^\alpha h(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} h(s) ds,$$

given that the integral exists.

Definition 2 ([10]). *Let $\alpha > 0$ be the L-C fractional derivative order of a function $h : \mathcal{L} \to \mathbb{R}$ is denoted by*

$${}^{LC}D^\alpha h(\tau) = D^\alpha \left[h(\tau) - \sum_{j=0}^{m_1-1} \frac{h^{(j)}(0)}{j!} \tau^j \right],$$

where

$$m_1 = 1 + [\alpha] \quad for \quad \alpha \notin \mathbb{N}_0, \quad m_1 = \alpha \quad for \quad \alpha \in \mathbb{N}_0, \tag{2}$$

and $D^\alpha_{0^+}$ is a fractional derivative in R-L sense of order α given by

$$D^\alpha h(\tau) = D^{m_1} I^{m_1-\alpha} h(\tau) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{m_1}}{d\tau^{m_1}} \int_0^\tau (\tau - s)^{m_1-\alpha-1} h(s) ds.$$

The Liouville–Caputo fractional derivative ${}^{LC}D^\alpha_{0^+}$ exists for $\varkappa$ belonging to $AC^{m_1}(\mathcal{L})$. In this case, it is defined by

$${}^{LC}D^\alpha h(\tau) = I^{m_1-\alpha} x^{(m_1)}(\tau) = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau - s)^{m_1-\alpha-1} h^{(m)_1}(s) ds.$$

Remark that when $\alpha = m_1$, we get ${}^{LC}D^\alpha h(\tau) = h^{(m_1)}(\tau)$.

Lemma 1 ([10]). *Let $\alpha > 0$ and m be the given by (2). If $h \in AC^m(\mathcal{L}, \mathbb{R})$, then*

$$(I^{\alpha LC} D^\alpha h)(\tau) = h(\tau) - \sum_{j=0}^{m-1} \frac{h^{(j)}(0)}{j!} \tau^j,$$

where $h^{(j)}$ is the usual derivative of h of order j.

Lemma 2 ([10]). *For $\alpha > 0$ and m be given by (2), then the L-C fractional differential equation ${}^{LC}D^\alpha h(\tau) = 0$ has a general solution*

$$h(\tau) = a_0 + a_1\tau + a_2\tau^2 + \dots + a_{m-1}\tau^{m-1},$$

where $a_k \in \mathbb{R}$, $k = 0, 1, 2, \dots, m-1$. Furthermore, the R-L fractional differential equation

$$D^\alpha h(\tau) = 0,$$

has a general solution

$$h(\tau) = a_1\tau^{\alpha-1} + a_2\tau^{\alpha-2} + a_3\tau^{\alpha-3} + \dots + a_m\tau^{\alpha-m}, \quad a_k \in \mathbb{R}, \quad k = 1, 2, \dots, m.$$

Lemma 3 ([10]). *For any $0 \leq \alpha, \beta < \infty$ and , then*

$$\frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\beta-1} s^{\alpha-1} ds = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tau^{\alpha+\beta-1}.$$

Lemma 4. *(Banach fixed-point theorem [7]). Let Ψ be a nonempty closed convex subset of a Banach space $(S^*, \|\cdot\|)$, and then any contraction mapping ϕ of Ψ into itself has a unique fixed point.*

Lemma 5. *(Schauder fixed-point theorem [7]). Let Ψ be a nonempty bounded closed convex subset of a Banach space S^* and $\phi : \Psi \to \Psi$ be a continuous compact operator. Then has a fixed point in Ψ.*

The subsequent lemma is required to get our conclusions.

Lemma 6 ([22]). *For any $h \in C(\mathcal{L})$, then the problem*

$$\begin{cases} D^{\beta}\ {}^{LC}D^{\alpha} \varkappa(\tau) + \lambda \varkappa(\tau) = h(\tau), \quad \tau \neq \tau_\kappa, \quad 0 \leq \tau \leq T, \quad \lambda \in \mathbb{R}, \\ \Delta \varkappa(\tau_\kappa) = I_\kappa(\varkappa(\tau_\kappa^-)), \quad \kappa = 1, 2, ..., m, \\ \varkappa(\tau) = \zeta(\tau), \ -h \leq \tau \leq 0 \\ {}^{LC}D^{\alpha} \varkappa(0) = {}^{LC} D^{\alpha} \varkappa(T) = 0, \quad \varkappa(0) = \mu \int_0^T \varkappa_s ds + \nu, \quad \mu, \nu \in \mathbb{R}, \end{cases} \tag{3}$$

is identical to the integral equation

$$\begin{cases} \zeta(\tau); \ \tau \in [-h, 0] \\ \dfrac{1}{\Gamma(\alpha+\beta)} \left(\displaystyle\int_0^\tau (\tau - s)^{\alpha+\beta-1} h(s) ds - \lambda \int_0^\tau (\tau - s)^{\alpha+\beta-1} \varkappa_s ds \right) \\ - \dfrac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(\displaystyle\int_0^T (T - s)^{\beta-1} h(s) ds - \lambda \int_0^T (T - s)^{\beta-1} \varkappa_s ds \right) \\ + \mu \displaystyle\int_0^T \varkappa_s ds + \nu \text{ if } \tau \in (0, \tau_1] \\ \vdots \\ \dfrac{1}{\Gamma(\alpha+\beta)} \displaystyle\sum_{i=1}^{\kappa} \left(\int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{\alpha+\beta-1} h(s) ds - \lambda \int_{\tau_{i-1}}^{\tau_i} (\tau_i - s)^{\alpha+\beta-1} \varkappa_s ds \right) \\ + \dfrac{1}{\Gamma(\alpha+\beta)} \left(\displaystyle\int_{\tau_\kappa}^{\tau} (\tau - s)^{\alpha+\beta-1} h(s) ds - \lambda \int_{\tau_\kappa}^{\tau} (\tau - s)^{\alpha+\beta-1} \varkappa_s ds \right) \\ - \dfrac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(\displaystyle\int_0^T (T - s)^{\beta-1} h(s) ds - \lambda \int_0^T (T - s)^{\beta-1} \varkappa_s ds \right) \\ + \mu \displaystyle\int_0^T \varkappa_s ds + \nu + \sum_{\kappa=1}^{m} I_\kappa(\varkappa(\tau_\kappa^-)) \text{ if } \tau \in (\tau_\kappa, \tau_{\kappa+1}]. \end{cases}$$

3. Main Result

The following uses fixed point theorems to prove the problem's existence and uniqueness of the (1). The following hypotheses are required to get our results.

(H1) Take the constants $k > 0, \ 0 < l < 1$ such that

$$|f(\tau, \varkappa_1, \bar{\varkappa}_1) - f(\tau, \varkappa_2, \bar{\varkappa}_2)| \leq k \|\varkappa_1 - \varkappa_2\|_{PC} + l |\bar{\varkappa}_1 - \bar{\varkappa}_2|,$$

for any $\tau \in \mathcal{L}$ and each $\varkappa_i \in PC([-h, 0], \mathbb{R}), \bar{\varkappa}_i \in \mathbb{R}, i = 1, 2$.

(H2) There exists a constants $K > 0$ and $L \in (0, 1)$ such that

$$|f(\tau, \varkappa, \bar{\varkappa})| \leq K \|\varkappa\|_{PC} + L |\bar{\varkappa}|,$$

for any $\varkappa \in PC([-h, 0], \mathbb{R}), \ \bar{\varkappa} \in \mathbb{R}$ and each $\tau \in \mathcal{L}$.

(H3) There exist $\rho > 0$ that says

$$|I_k(\varkappa) - I_k(\bar{\varkappa})| \leq \rho\|\varkappa - \bar{\varkappa}\|_{PC}, \quad \text{for all} \quad \varkappa, \bar{\varkappa} \in PC([-h,0],\mathbb{R}) \quad with \quad k = 1,2,...,m.$$

3.1. Existence and Uniqueness Results via Banach's Fixed-Point Theorem

Theorem 1. *Consider (H1) holds. If*

$$\chi \quad = \quad \left(\frac{(m+2)\beta+\alpha}{\beta(\alpha+\beta)}\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\left(\frac{k}{1-l}+|\lambda|\right)+|\mu|T+m\rho<1,$$

then the solution of (1) is unique on [−h,T].

Proof. Denote the operator $\sigma : \mathcal{C} \to \mathcal{C}$ as

$$(\sigma\varkappa)(\tau) = \begin{cases} \zeta(\tau);\ \tau \in [-h,0] \\ \dfrac{1}{\Gamma(\alpha+\beta)}\displaystyle\sum_{i=1}^{\kappa}\left(\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{\alpha+\beta-1}h(s)ds - \lambda\int_{\tau_{i-1}}^{\tau_i}(\tau_i-s)^{\alpha+\beta-1}\varkappa_s ds\right) \\ +\dfrac{1}{\Gamma(\alpha+\beta)}\left(\displaystyle\int_{\tau_\kappa}^{\tau}(\tau-s)^{\alpha+\beta-1}h(s)ds - \lambda\int_{\tau_\kappa}^{\tau}(\tau-s)^{\alpha+\beta-1}\varkappa_s ds\right) \\ -\dfrac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\left(\displaystyle\int_0^T(T-s)^{\beta-1}h(s)ds - \lambda\int_0^T(T-s)^{\beta-1}\varkappa_s ds\right) \\ +\mu\displaystyle\int_0^T\varkappa_s ds + \nu + \sum_{\kappa=1}^{m}I_\kappa(\varkappa(\tau_\kappa^-)) \text{ if } \tau \in (\tau_\kappa, \tau_{\kappa+1}]. \end{cases}$$

Obviously, the solutions of problem (1) are the fixed points of operator σ. By (H1) and (H3), for each $\varkappa, y \in \mathcal{C}$ and $\tau \in \mathcal{L}$, where $h \in \mathcal{C}(\mathcal{L})$ such that $h(\tau) = f(\tau, \varkappa_\tau, h(\tau))$.

Take $\varkappa, y \in \mathcal{C}(\mathcal{L})$, and then for each $\tau \in [-h,0]$, we have

$$\|(\sigma\varkappa)(\tau) - (\sigma y)(\tau)\| = |\zeta(\tau) - \zeta(\tau)| = 0$$

and for each $\tau \in \mathcal{L}$, we have $|(\sigma\varkappa)(\tau) - (\sigma y)(\tau)|$

$$\begin{aligned} \leq\ & \frac{1}{\Gamma(\alpha+\beta)}\sum_{0<\tau_k<\tau}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\alpha+\beta-1}|h(s)-g(s)|ds \\ &+\frac{1}{\Gamma(\alpha+\beta)}\int_{\tau_m}^{\tau}(\tau-s)^{\alpha+\beta-1}|h(s)-g(s)|ds \\ &+\frac{1}{\Gamma(\alpha+\beta)}|\lambda|\sum_{0<\tau_k<\tau}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\alpha+\beta-1}|\varkappa_s-y_s|ds \\ &+\frac{1}{\Gamma(\alpha+\beta)}|\lambda|\int_{\tau_m}^{\tau}(\tau-s)^{\alpha+\beta-1}|\varkappa_s-y_s|ds \\ &+\frac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\int_0^T(T-s)^{\beta-1}|h(s)-g(s)|ds \\ &+\frac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}|\lambda|\int_0^T(T-s)^{\beta-1}|\varkappa(s)-y(s)|ds \\ &+|\mu|\int_0^T|\varkappa(s)-y(s)|ds + \sum_{k=1}^{m}|I_k(\varkappa(\tau_k^-)) - I_k(y(\tau_k^-))| \end{aligned}$$

where $h, g \in \mathcal{C}(\mathcal{L})$ such that

$$h(\tau) = f(\tau, \varkappa_\tau, h(\tau)) \text{ and } g(\tau) = f(\tau, y_\tau, g(\tau)).$$

From (H_1), we get

$$\begin{aligned}|h(\tau)-g(\tau)| &= |f(\tau,\varkappa_\tau,h(\tau))-f(\tau,y_\tau,g(\tau))|\\ &\leq \frac{k}{1-l}\|\varkappa_\tau-y_\tau\|_{PC}.\end{aligned}$$

Thus, for each $\tau\in\mathcal{L}$, we get
$|(\sigma\varkappa)(\tau)-(\sigma y)(\tau)|$

$$\begin{aligned}&\leq \frac{T^{\alpha+\beta}m}{\Gamma(\alpha+\beta+1)}\frac{k}{1-l}\|\varkappa_\tau-y_\tau\|_{PC}+\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\frac{k}{1-l}\|\varkappa_\tau-y_\tau\|_{PC}\\ &\quad+\frac{|\lambda|m}{\Gamma(\alpha+\beta+1)}\|\varkappa_\tau-y_\tau\|_{PC}+\frac{|\lambda|}{\Gamma(\alpha+\beta+1)}\|\varkappa_\tau-y_\tau\|_{PC}\\ &\quad+\frac{T^{\beta+\alpha}}{\beta\Gamma(\beta+\alpha)}\frac{k}{1-l}\|\varkappa_\tau-y_\tau\|_{PC}+\frac{T^{\beta+\alpha}}{\beta\Gamma(\beta+\alpha)}|\lambda|\|\varkappa_\tau-y_\tau\|_{PC}\\ &\quad+|\mu|T\|\varkappa_\tau-y_\tau\|_{PC}+m\rho\|\varkappa_\tau-y_\tau\|_{PC}\\ &\leq \left[\left(\frac{(m+2)\beta+\alpha}{\beta(\alpha+\beta)}\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\left(\frac{k}{1-l}+|\lambda|\right)+|\mu|T+m\rho\right]\|\varkappa_\tau-y_\tau\|_{PC_1}.\\ &\leq \chi\|\varkappa_\tau-y_\tau\|_{PC_1}\\ &\leq \left[\left(\frac{(m+2)\beta+\alpha}{\beta(\alpha+\beta)}\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\left(\frac{k}{1-l}+|\lambda|\right)+|\mu|T+m\rho\right]\|\varkappa-y\|_{\mathcal{C}}.\\ &\leq \chi\|\varkappa-y\|_{\mathcal{C}}.\end{aligned}$$

Thus,

$$\|\sigma\varkappa-\sigma y\|\leq\chi\|\varkappa-y\|.$$

From (4), σ is a contraction. By the Banach contraction mapping theorem, σ has a fixed point that is the unique solution of the problem (1) on $[-h,T]$. This finishes the proof. □

3.2. Existence Results via Schauder's Fixed-Point Theorem

Theorem 2. *Let the hypotheses (H1) and (H2) be fulfilled. If*

$$\left(\frac{(m+2)\beta+\alpha}{\beta(\alpha+\beta)}\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\left(\frac{k}{1-l}+|\lambda|\right)\leq 1,$$

then (1) *has at least one solution on [$-h$,T].*

Proof. Take the operator $\sigma:\mathcal{C}\to\mathcal{C}$. Let $\mathcal{P}>0$ such that

$$\mathcal{P}\geq\max\left\{\|\zeta\|_{\mathcal{C}([-h,0],\mathcal{R})}\frac{|a|T+|b|+m\rho}{1-\left[\left(\frac{(m+2)\beta+\alpha}{\beta(\alpha+\beta)}\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\left(\frac{k}{1-l}+|\lambda|\right)\right]}\right\}.$$

Denote the ball

$$\mathcal{B}_{\mathcal{P}}=\{\varkappa\in\mathcal{C}(\mathcal{L},\mathcal{R}),\|\varkappa\|_C\leq\mathcal{P}\}.$$

Consider that the operator $\sigma:\mathcal{B}_{\mathcal{P}}\to\mathcal{B}_{\mathcal{P}}$ fulfills all conditions of Lemma 3 . Three steps would be taken to present the proof.

Step 1: σ is continuous.

Consider the sequence $\varkappa_n$ such that $\varkappa_n\to\varkappa$ in $\mathcal{B}_{\mathcal{P}}$. For each $\tau\in[-h,0]$, we get

$$\|(\sigma\varkappa_n)(\tau)-(\sigma\varkappa)(\tau)\|=|\zeta(\tau)-\zeta(\tau)|=0$$

and for each $\tau \in \mathcal{L}$, we have
$\|(\sigma\varkappa_n)(\tau) - (\sigma\varkappa)(\tau)\|$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha+\beta)} \sum_{0<\tau_k<\tau} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{\alpha+\beta-1} |h_n(s) - h(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_{\tau_m}^{\tau} (\tau - s)^{\alpha+\beta-1} |h_n(s) - h(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} |\lambda| \sum_{0<\tau_k<\tau} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{\alpha+\beta-1} |\varkappa_{ns} - \varkappa_s| ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} |\lambda| \int_{\tau_m}^{\tau} (\tau - s)^{\alpha+\beta-1} |\varkappa_{ns} - \varkappa_s| ds \\
&\quad + \frac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \int_0^T (T-s)^{\beta-1} |h_n(s) - h(s)| ds \\
&\quad + \frac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} |\lambda| \int_0^T (T-s)^{\beta-1} |\varkappa_{ns} - \varkappa_s| ds \\
&\quad + |a| \int_0^T |\varkappa_{ns} - \varkappa_s| ds + \sum_{k=1}^{m} |I_k(\varkappa_n(\tau_k^-)) - I_k(\varkappa(\tau_k^-))|
\end{aligned}$$

where $h_n, h \in \mathcal{C}(\mathcal{L}, \mathcal{R})$ such that

$$h_n(\tau) = f(\tau, \varkappa_{n\tau}, h_n(\tau)) \text{ and } h(\tau) = f(\tau, \varkappa_\tau, h(\tau)).$$

Here, $f, h,$ and h_n are continuous and $\|\varkappa_n - \varkappa\|_{\mathcal{C}} \to 0$ as $n \to \infty$ then by the Lebesgue-dominated convergence theorem

$$\|\sigma(\varkappa_n) - \sigma(\varkappa)\|_{\mathcal{C}} \to 0 \text{ as } n \to \infty.$$

Hence, σ is continuous.

Step 2: $\sigma(\mathcal{B}_\mathcal{P}) \subset \mathcal{B}_\mathcal{P}$.
Consider $\varkappa \in \mathcal{B}_\mathcal{P}$, If $-h \leq \tau \leq 0$, and then

$$\|(\sigma\varkappa)(\tau)\| \leq \|\zeta\| \leq \mathcal{P}$$

for each $\tau \in \mathcal{L}$, and from (H_2), we get

$$\begin{aligned}
|h(\tau)| &\leq |f(\tau, \varkappa_\tau, h(\tau))| \\
&\leq K\|\varkappa_\tau\|_{[-h,0]} + L|h(\tau)| \\
&\leq K\|\varkappa_\tau\|_{\mathcal{C}} + L\|h\|_\infty \\
&\leq K\mathcal{P} + L\|h\|_\infty.
\end{aligned}$$

Then

$$\|h\|_\infty \leq \frac{\mathcal{P}K}{1-L}.$$

Thus,

$$\begin{aligned}
|(\sigma\varkappa)(\tau)| &\leq \frac{1}{\Gamma(\alpha+\beta)} \sum_{0<\tau_k<\tau} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{\alpha+\beta-1} |h(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} |\lambda| \sum_{0<\tau_k<\tau} \int_{\tau_{k-1}}^{\tau_k} (\tau_k - s)^{\alpha+\beta-1} |\varkappa_s| ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_{\tau_m}^{\tau} (\tau - s)^{\alpha+\beta-1} |h(s)| ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha+\beta)}|\lambda|\int_{\tau_m}^{\tau}(\tau-s)^{\alpha+\beta-1}|\varkappa_s|ds \\
&+\frac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\int_0^T(T-s)^{\beta-1}|h(s)|ds \\
&+\frac{\tau^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}|\lambda|\int_0^T(T-s)^{\beta-1}|\varkappa_s|ds \\
&+|\mu|\int_0^T|\varkappa_s|ds+\sum_{k=1}^{m}|I_k(\varkappa(\tau_k^-))| \\
&\leq\left[\left(\frac{(m+2)\beta+\alpha}{\beta(\alpha+\beta)}\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\left(\frac{k}{1-l}+|\lambda|\right)+|\mu|T+m\rho+|\nu|\right] \\
&\leq\mathcal{P}.
\end{aligned}
$$

Hence,

$$\|\sigma(\varkappa)\|_{\mathcal{C}}\leq\mathcal{C}.$$

Consequently, $\sigma(\mathcal{B}_{\mathcal{P}})\subset\mathcal{B}_{\mathcal{P}}$.

Step 3: $\sigma(\mathcal{B}_{\mathcal{P}})$ is equicontinuous.
For $0\leq\tau_{k-1}\leq\tau_k\leq T$ and $\varkappa\in\mathcal{B}_{\mathcal{P}}$, we have
$|(\sigma\varkappa)(\tau_k)-(\sigma\varkappa)(\tau_{k-1})|$

$$
\begin{aligned}
\leq\;&\frac{1}{\Gamma(\alpha+\beta)}\sum_{0<\tau_k<\tau}\int_0^{\tau_{k-1}}\Big((\tau_k-s)^{\alpha+\beta-1}-(\tau_{k-1}-s)^{\alpha+\beta-1}\Big)|h(s)|ds \\
&+\frac{1}{\Gamma(\alpha+\beta)}\sum_{0<\tau_k<\tau}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\alpha+\beta-1}|h(s)|ds \\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_0^{\tau_{k-1}}\Big((\tau_k-s)^{\alpha+\beta-1}-(\tau_{k-1}-s)^{\alpha+\beta-1}\Big)|h(s)|ds \\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\alpha+\beta-1}|h(s)|ds \\
&+\frac{1}{\Gamma(\alpha+\beta)}|\lambda|\sum_{0<\tau_k<\tau}\Big(\int_0^{\tau_{k-1}}(\tau_k-s)^{\alpha+\beta-1}-(\tau_{k-1}-s)^{\alpha+\beta-1}|\varkappa_s|ds \\
&+\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\alpha+\beta-1}|\varkappa_s|ds\Big) \\
&+\frac{1}{\Gamma(\alpha+\beta)}|\lambda|\Big(\int_0^{\tau_{k-1}}(\tau_k-s)^{\alpha+\beta-1}-(\tau_{k-1}-s)^{\alpha+\beta-1}|\varkappa_s|ds \\
&+\int_{\tau_{k-1}}^{\tau_k}(\tau_k-s)^{\alpha+\beta-1}|\varkappa_s|ds\Big) \\
&+\frac{\tau_2^{\beta+\alpha-1}-\tau_1^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\left(\int_0^T(T-s)^{\beta-1}|h(s)|ds+|\lambda|\int_0^T(T-s)^{\beta-1}|\varkappa_s|ds\right) \\
&+\sum_{k=1}^{m}|I_k(\varkappa(\tau_k^-))-I_k(\varkappa(\tau_{k-1}^-))| \\
\leq\;&\frac{(m+1)}{\Gamma(\alpha+\beta+1)}\left(|\lambda|+\frac{\mathcal{P}K}{1-L}\right)\left(\tau_k^{\alpha+\beta}-\tau_{k-1}^{\alpha+\beta}\right)+\frac{\left(\tau_k^{\beta+\alpha-1}-\tau_{k-1}^{\beta+\alpha-1}\right)}{\beta\Gamma(\beta+\alpha)}\left(|\lambda|+\frac{\mathcal{P}K}{1-L}\right) \\
&+|\mu|(\tau_k-\tau_{k-1})+\rho\|(\varkappa(\tau_k))-(\varkappa(\tau_{k-1}))\|.
\end{aligned}
$$

As $\tau_{k-1}\to\tau_k$, we see that the right side of the above inequality tends to zero, and the convergence is not dependent on $\varkappa$ in $\mathcal{B}_{\mathcal{P}}$. This implies $\sigma(\mathcal{B}_{\mathcal{P}})$ is equicontinuous. By the Arzela–Ascoli theorem, σ is compact. Thus, by the Lemma 5, we prove that σ has at least one fixed point $\varkappa\in\mathcal{B}_{\mathcal{P}}$, which is a solution of the problem (1) on $[-h,T]$. $\square$

4. Example

Consider the following fractional relaxation impulsive integro-differential equation

$$\begin{cases} D^{\frac{3}{2}}\ {}^{LC}D^{\frac{1}{2}}\varkappa(\tau) + \frac{1}{4}\varkappa(\tau) = f\Big(\tau, \varkappa(\tau), D^{\frac{3}{2}}\ {}^{LC}D^{\frac{1}{2}}\varkappa(\tau)\Big), \quad \tau \neq \tau_\kappa \quad \tau \in [0,1], \\ \Delta\varkappa(\tau_\kappa) = I_\kappa(\varkappa(\tau_\kappa^-)), \quad \kappa = 1, 2, ..., m, \\ \varkappa(\tau) = 1 + \tau^2 : \ \tau \in [-1, 0] \\ {}^{LC}D^{\frac{1}{2}}\varkappa(0) = {}^{LC}\ D^{\frac{1}{2}}\varkappa(1) = 0, \quad \varkappa(0) = \frac{1}{10}\int_0^1 \varkappa(s)ds + 2. \end{cases} \tag{4}$$

Here $\alpha = \frac{1}{2}, \beta = \frac{3}{2}, \lambda = \frac{1}{4}, \mu = \frac{1}{10}$, and $\nu = 2$. Set

$$f\Big(\tau, \varkappa(\tau), D^{\frac{3}{2}}\ {}^{LC}D^{\frac{1}{2}}\varkappa(\tau)\Big) = \frac{sin(\tau)}{\exp(\tau^2) + 7}\left(\frac{1}{90(|\varkappa(\tau)| + 1)} + \frac{1}{1 + |D^{\frac{3}{2}}\ {}^{LC}D^{\frac{1}{2}}\varkappa(\tau)|}\right).$$

For $\varkappa_i, y_i \in \mathbb{R}, i = 1, 2$, we have
$|f(\tau, \varkappa_1, \varkappa_2) - f(\tau, y_1, y_2)|$

$$\begin{aligned} &= \left|\frac{sin(\tau)}{\exp(\tau^2) + 7}\left(\left(\frac{1}{90(|\varkappa_1| + 1)} - \frac{1}{90(|y_1| + 1)}\right)\right.\right. \\ &+ \left.\left.\left(\frac{1}{1 + |D^{\frac{3}{2}}\ {}^{LC}D^{\frac{1}{2}}\varkappa(\tau)|} - \frac{1}{1 + |D^{\frac{3}{2}}\ {}^{LC}D^{\frac{1}{2}}y(\tau)|}\right)\right)\right| \\ &\leq \frac{1}{\exp(\tau^2) + 7}\left(\frac{\|\varkappa_1 - y_1\|_{[-h,0]}}{90(1 + |\varkappa_1|)(1 + |y_1|)} + \frac{|\varkappa_2 - y_2|}{30(1 + |\varkappa_2|)(1 + |y_2|)}\right) \\ &\leq \frac{1}{8}(\frac{1}{90}\|\varkappa_1 - y_1\|_{[-h,0]} + \frac{1}{30}|\varkappa_2 - y_2|), \end{aligned}$$

and thus, the assumption (H1) is satisfied with $k = \frac{1}{720}, l = \frac{1}{240}, T = 1, \ \rho = \frac{1}{2}$ & $m = 1$. We will evaluate that condition (4) is satisfied. Indeed,

$$\begin{aligned} \sigma &= \left(\frac{(m+2)\beta + \alpha}{\beta(\alpha + \beta)}\right)\frac{T^{\alpha+\beta}}{\Gamma(\beta + \alpha)}\left(\frac{k}{1 - l} + |\lambda|\right) + |\mu|T + m\rho \\ &= \left(\frac{(1+2)\frac{3}{2} + \frac{1}{2}}{\frac{3}{2}(\frac{1}{2} + \frac{3}{2})}\right)\left(\frac{\frac{1}{720}}{1 - \frac{1}{240}} + \frac{1}{4}\right) + \frac{1}{10} + \frac{1}{3} \\ &\simeq 0.7651 < 1. \end{aligned}$$

The conditions of Theorem 1 are fulfilled.

$$\begin{aligned} &\left(\frac{(m+2)\beta + \alpha}{\beta(\alpha + \beta)}\right)\frac{T^{\alpha+\beta}}{\Gamma(\alpha + \beta)}\left(\frac{k}{1 - l} + |\lambda|\right) \\ &= \left(\frac{(1+2)\frac{3}{2} + \frac{1}{2}}{\frac{3}{2}(\frac{1}{2} + \frac{3}{2})}\right)\left(\frac{\frac{1}{720}}{1 - \frac{1}{240}} + \frac{1}{4}\right) \\ &= 0.3351 < 1. \end{aligned}$$

The conditions of Theorem 2 are fulfilled.

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