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A Probe into a $(2 + 1)$ -Dimensional Combined Cosmological Model in $f(R, T)$ Gravity

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Abstract: This research is an extension of our earlier published $(2 + 1)$ dimensional cosmological models in $f(R, T)$ gravity with $\Lambda(R, T)$ (*IOP Conf. Ser. J. Phys. Conf. Ser.* **2019**, 1258, 012026). A different class of cosmological space model is studied under modified theories of $f(R, T)$ gravity, where the cosmological constant Λ is expressed as a function of the Ricci scalar R and the trace of the stress-energy momentum tensor T . We call such a model as “ $\Lambda(R, T)$ gravity”. Such a specific form of $\Lambda(R, T)$ has been defined in the dust as well as in the perfect fluid case. We intend to search for a combined model that satisfies the equation of state for dark energy matter or quintessence matter or perfect matter fluid. Some geometric and intrinsic physical properties of the model are also described. The energy conditions, pressure and density are discussed both when $\Lambda = \Lambda(r)$ is a function of the radial parameter r , as well as when Λ is zero. We study the effective mass function and also the gravitational redshift function, both of which are found to be positive as per the latest observations. The cosmological model is studied in $f(R, T)$ modified theory of gravity, where the gravitational Lagrangian is expressed both in terms of the Ricci scalar R and the trace of the stress-energy tensor T . The equation of state parameter is discussed in terms of ω corresponding to the three cases mentioned above. The behaviour of the cosmological constant is separately examined in the presence of quintessence matter, dark energy matter and perfect fluid matter.

Keywords: general relativity; $f(R, T)$ gravity; dark energy matter; quintessence matter; perfect matter fluid; $\Lambda(R, T)$.



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1. Introduction

We are acquainted with the fact that the standard theory of gravity is General Relativity. Cosmologists have proposed that a mysterious substance called quintessence can explain why our universe is accelerating. As numerous observations and experiments reshape the field, many cosmologists are exploring the possibility that the vast majority of the energy in the universe is in the form of an undiscovered substance until now, called “quintessence”.

Quintessence has the striking physical characteristic that it causes the expansion of the universe to speed up. Most forms of energy, such as matter or radiation, cause the expansion to slow down due to the attractive force of gravity. For quintessence, however, the gravitational force is repulsive, and this causes the expansion of the universe to accelerate. In this work, we study the quintessence field. The accelerating rate of expansion of the universe is better explained by this quintessence field, which is a scalar field or a hypothetical form of dark energy. Quintessence can be either attractive or repulsive depending on the ratio of its kinetic and potential energy. We understand that the value of the cosmological constant, by definition, does not change. But when we explain dark energy cosmological models, it behaves as a dynamic quantity which changes over time.

Cosmic inflation and the accelerated expansion of the universe is, however, characterized by the equation of the state of dark energy. The predicted amount of vacuum

energy differs from the measured amount, which is known as the cosmological constant problem. If the cosmological constant $\Lambda = 0$, then there is an absence of matter ($T_{\mu\nu} = 0$) which leads to an absence of space-time curvature. On the other hand, if $\Lambda \neq 0$, we have gravity associated with the vacuum. Further, the coincidence problem is that the density of the dark energy and matter evolves differently as the Universe expands, even if they are comparable [1].

Alternative theories of gravity are now playing an interesting role to describe the present day observed Universe. The authors of [2] have shown that dust matter-dark energy combined phases can be achieved by the exact solution which is deduced from the power law $f(R)$ cosmological model. Hence it solves the query by which a dust-dominated decelerated phase is required before the dark-energy accelerated phase to form large scale structures.

To explain the present observed large scale structures and cosmic accelerating expansion, we need to search for the huge amounts of “dark matter” and “dark energy”, but no experimental evidence has yet been found to support such mysterious components. Dark matter and dark energy can however bridge the “shortcomings” of General Relativity and explain the “correct” theory of gravity by matching with the present day observational data [3].

Ever since the pioneering work of Karl Schwarzschild in 1916, many authors have investigated the static and spherically symmetric perfect fluid solutions in classical general relativity. Schwarzschild has solved the field equations for the interior region by considering a perfect fluid of constant energy density, and also for the outside vacuum region, the so-called Schwarzschild solutions. The authors of [4] have examined static perfect fluid spheres in the presence of a cosmological constant and hence generated new classes of exact matter solutions. They obtained a class of solutions that generalizes the Einstein static universe and observed that neither its energy density nor its pressure is constant throughout the spacetime.

It is now important to study gravity in $(2 + 1)$ dimensions. We know that such gravity becomes relevant after the BTZ black hole solution. Bañados et al. [5] have observed that a black hole solution, together with a negative cosmological constant, exists in $(2 + 1)$ dimensions of spacetime, for the standard Einstein-Maxwell equations. It was also observed that the $(2 + 1)$ spacetimes for the standard black hole exhibits thermodynamic properties that resemble $(3 + 1)$ -dimensions of Schwarzschild and Kerr black hole solutions [6]. The simple nature of gravitation in such dimensions have also been studied in [7]. We find in [8] that the authors have considered the Einstein theory in a standard form of $(2 + 1)$ dimensions and studied the geometry of the spinning black holes, having a negative cosmological constant. Kamata et al. [9] considered an anti-self dual equation to obtain an exact solution to the Einstein-Maxwell equations in such dimensions for an electrically charged BTZ black hole. The authors of [10] have considered the relation between quantised Chern-Simons gravity and $(2 + 1)$ dimensional quantum theory, and hence obtained the density of states of the BTZ black hole in $(2 + 1)$ dimensions. Further, Oliva et al. [11] observed that the conformally flat solution containing black holes, can be obtained for any value of the cosmological constant. However, for a negative cosmological constant, the black hole is characterized by the “gravitational hair” parameter as well as the mass. In [12], the authors have studied asymptotically flat black holes with deformed horizons in $(2 + 1)$ dimensions. They obtained solutions for static and rotating black holes, as well as for black holes with radiative gravitons. Koch et al. [13] obtained a scale dependent black hole in $(2 + 1)$ dimensions. They derived the corresponding field equations by imposing the “null energy condition” alongwith stationary spherical symmetry. In the BTZ black hole background, the relativistic scattering as well as propagation, with a nonminimal gravitational coupling, has been analyzed in [14]. A new kind of “soft hairy black holes” in $(2 + 1)$ dimensions, with asymptotic symmetries for locally flat spacetime, have been observed in [15]. The authors have derived the results on flat space and, also, made observations on the entropy. A generalization of the rotating black hole in $(2 + 1)$

dimensions is observed in [16], where the authors have analyzed the horizon structure, asymptotic behavior as well as its thermodynamic properties.

The cosmological solutions in $(2 + 1)$ dimensions have been significantly investigated by different authors. In [17] the author has studied the scalar field cosmology in three-dimensions and hence obtained an analytical solution to the Einstein's equations under three distinct space-times. Homogeneous and isotropic $3-d$ cosmological features, containing perfect-fluid matter, have been reviewed in [18]. Further, spatially flat Einstein-Gauss-Bonnet cosmological models in various dimensions have been studied by Pavluchenko [19]. The author has also investigated the Einstein-Gauss-Bonnet cosmological models in a lower dimension and in the presence of perfect fluid in [20]. The aspects of the polynomial affine model of gravity in $3-d$ have been defined in [21], where the authors reformulated the model and discussed the truncated sectors.

The “cosmological constant” problem is a widely debated issue. Variable cosmological terms were studied almost extensively in this context. The evolution of the scale factor with a variable Λ term has been studied in [22]. The authors regeneralized the investigation to include nonzero pressure and obtained new solutions using a variety of observational criteria. The effect of the cosmological constant Λ on the trajectory of the photons and hence on their bending angle as they arrive at the observer has been discussed in [23]. The author found that Λ contributes at the first-order differential equation level.

Cosmological models in varying scenarios are well-known today. A new type of decaying Λ cosmologies is observed by Bonanno et al. [24], where it was found that the cosmological constant $\Lambda = 0$ as $t \rightarrow 0$, at very late times. Further, to explain the cosmic acceleration from the observational point of view, RG-improved swiss-cheese cosmology has been discussed in [25], where variable G and Λ are used. The cosmological constant problem is solved by strong quantum effects near the infrared [26]. Reuter et al. worked on Quantum Einstein Gravity and derived a critical infrared fixed point, where gravity is found to be the scale invariant. It was also found that cosmological models of gravity are scale-dependent [27]. There is a huge discrepancy between the observed value and theoretical value of the cosmological constant Λ in Einstein's equations. It was earlier thought as vacuum energy density and has been studied by Peracaula [28], where old and new ideas are compared.

In [29] the authors have derived that the cosmological constant Λ varies as R^2 , where R is the Ricci scalar. They obtained this behaviour considering some general assumptions in compliance with quantum cosmology. We intend to make a thorough study of the behaviour of the cosmological constant in presence of dark energy matter, quintessence matter or perfect fluid matter. We look for an unified model that satisfies the equation of state in all these cases. The author in [30] has observed that in a $(2+1)$ -dimensional gravitational model, the cosmological constant behaves as a dynamical variable. We are motivated to investigate this dynamic nature of the cosmological constant in our unified model and also when it behaves as a constant as well.

The authors of [31] have studied modified $f(R, T)$ gravity with scalar field and flat FRW model. A particular form $f(R, T) = R + 2f(T)$ is considered and hence obtained the field with an effective energy momentum tensor expressed in terms of the normal (phantom) scalar field and the matter contribution due to modified gravity.

The structure of this paper is as follows: In Section 2, the solutions are studied under $f(R, T)$ gravity models. We have recapitulated some discussions on models from our previous published research paper. Section 3 deals with the field equations when the cosmological constant Λ is zero, when it is defined in terms of the radial parameter r , as well as when it is defined in terms of the Ricci scalar R . The density, pressure, energy conditions [Null energy condition (NEC), Weak energy condition (WEC), Dominant energy condition (DEC) and Strong energy condition (SEC)], cosmological constant are studied in intricate details in all the three cases for quintessence field, dark energy matter and perfect fluid matter. The effective mass and redshift function for the dark energy matter case are discussed in Section 4. Concluding remarks are given in Section 5.

2. Exploring New Solutions under $f(R, T)$ Models

The modified gravity action is given by [32]

$$S = \frac{1}{16\pi} \int (\sqrt{-g}f(R, T) + \sqrt{-g}\mathcal{L}_m) d^3x, \quad (1)$$

where we define T as the trace of the stress-energy momentum tensor of matter fields in the space-time and $f(R, T)$ as an arbitrary function of the Ricci scalar R . $T_{\mu\nu}$ is given by $T = g^{\mu\nu}T_{\mu\nu}$ and \mathcal{L}_m stands for the matter Lagrangian density. The matter may be quintessence, dark matter or perfect fluid. $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g}\mathcal{L}_m}{\delta g^{\mu\nu}}. \quad (2)$$

Further, the variation of T with respect to the metric tensor is defined as

$$\frac{\delta g^{\alpha\beta}T_{\alpha\beta}}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu}, \quad (3)$$

where the tensor

$$\Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \quad (4)$$

Depending on the matter source, various models of $f(R, T)$ gravity have been investigated by different authors, which may be elucidated as follows

$$\begin{aligned} (i) f(R, T) &= f_1(R) + f_2(T) & (ii) f(R, T) &= R + 2f(T) \\ (iii) f(R, T) &= f_1(R) + f_2(R)f_3(T) & (iv) f(R, T) &= f(R, T^\phi), \end{aligned} \quad (5)$$

where ϕ is a scalar field.

The cosmological model with $f(R, T) = R + 2f(T)$ has been studied in [33,34].

We consider the case (iii) in Equation (5). We obtain the gravitational field equations in $(2+1)$ -dimensions as a counterpart to the $(3+1)$ -dimensions, which are observed in [32]. We have

$$\begin{aligned} [f'_1(R) + f'_2(R)f_3(T)]R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)[f'_1(R) + f'_2(R)f_3(T)] \\ = 2\pi T_{\mu\nu} - f_2(R)f'_3(T)T_{\mu\nu} - f_2(R)f'_3(T)\Theta_{\mu\nu} + \frac{1}{2}f_2(R)f_3(T)g_{\mu\nu}. \end{aligned} \quad (6)$$

However, for the perfect fluid, they reduce as

$$\begin{aligned} [f'_1(R) + f'_2(R)f_3(T)]R_{\mu\nu} - \frac{1}{2}f_1(R)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)[f'_1(R) + f'_2(R)f_3(T)] \\ = 2\pi T_{\mu\nu} + f_2(R)f'_3(T)T_{\mu\nu} + f_2(R)[f'_3(T)p + \frac{1}{2}f_3(T)]g_{\mu\nu}. \end{aligned} \quad (7)$$

Here, the symbol $\square = \nabla^i\nabla_i$ and the prime denotes the derivative with respect to the argument. We have considered $f_1(R) = \lambda_1 R, f_2(R) = \lambda_2 R, f_3(T) = \lambda_3 T$ together with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.

We consider $(g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)[f'_1(R) + f'_2(R)f_3(T)] = 0$. Moreover, $TR_{\mu\nu} = RT_{\mu\nu}$ in Equation (7). [The term $\lambda^2 TR_{\mu\nu}$ appears on the left side of Equation (7) and the term $\lambda^2 RT_{\mu\nu}$ appears on the right side of Equation (7) after simplification. However, they are same because if we apply the metric $g_{\mu\nu}g^{\mu\nu}$ to $TR_{\mu\nu}$ we obtain $RT_{\mu\nu}$, and the symbols have their usual meaning. Hence Equations (6) and (7) both reduce significantly [35]

$$\lambda[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}] = 2\pi T_{\mu\nu} + (\lambda^2 p + \frac{1}{2}\lambda^2 T)Rg_{\mu\nu}, \quad (8)$$

hence as,

$$G_{\mu\nu} - (p + \frac{1}{2}T)\lambda R g_{\mu\nu} = \frac{2\pi}{\lambda} T_{\mu\nu}. \quad (9)$$

Comparing Equation (9) with Einstein equation [36]

$$G_{\mu\nu} - \Lambda(g_{\mu\nu}) = -2\pi T_{\mu\nu}, \quad (10)$$

we obtain

$$\Lambda(R, T) = \lambda(p + \frac{1}{2}T)R. \quad (11)$$

The arbitrary λ in Equation (9) takes a very small value. The above Equation (11), in the dust case $p = 0$ reduce as

$$\Lambda = \Lambda(R, T) = \frac{1}{2}\lambda RT. \quad (12)$$

In the case of the perfect fluid [37], where $T = \rho - 2p$, the above Equation (11) becomes

$$\Lambda = \frac{1}{2}\lambda\rho R. \quad (13)$$

One may perceive that the universe is not only expanding but also accelerating as a result of the cosmological constant, which is positive in sign. Moreover, if the equation of Einstein general relativity theory is solved with a positive cosmological constant, one would derive the solution for spacetime having a positive Gaussian curvature. So, we may presume to be in a de-Sitter universe. Here, the positivity of the scalar curvature $R > 0$ implies that $\rho > 0$ for the positive cosmological constant. The cosmological constant can be positive, negative or zero as found by different authors. It is, however, considered as $\Lambda \approx 10^{-123}$ (using geometric units) pertinent with the present observational data [38].

3. (2+1)-Dimensional Spacetime Metric

In (2 + 1)-dimensional gravity, the metric is given by [39]

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2d\theta^2, \quad (14)$$

where $A(r)$ and $B(r)$ are two unknown metric functions. The stress-energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (15)$$

Here, $u^\mu = (1, 0, 0)$ is the velocity vector which satisfies $u^\mu u_\mu = -1$ and $u^\mu \nabla_\nu u_\mu = 0$. The energy density and pressure of the fluid are denoted by ρ and p , respectively.

We use the cosmological Equation (6) for the energy momentum tensor defined in Equation (15) along with the metric in Equation (14), to derive the field equations as follows (using geometrized units, such as the gravitational constant G and the speed of light c as unity)

$$\begin{aligned}
\frac{e^{-2B(r)}}{r}B' &= \frac{2\pi}{\lambda}\rho + \Lambda, \dots\dots\dots 16(i), \\
\frac{e^{-2B(r)}}{r}A' &= \frac{2\pi}{\lambda}p - \Lambda, \dots\dots\dots 16(ii), \\
e^{-2B(r)}(A'' + A'^2 - A'B') &= \frac{2\pi}{\lambda}p - \Lambda, \dots\dots\dots 16(iii), \\
R &= -\frac{1}{r}[2e^{-2B(r)}(A' + rA'^2 - B'(1 + rA') + rA'')]\dots\dots\dots 16(iv).
\end{aligned}
\tag{16}$$

Here, a “’” denotes the derivative with respect to the parameter r and R is the Ricci scalar.

The generalized Tolman-Oppenheimer-Volkov (TOV) equation, also known as the conservation equation in (2+1) dimensions is written as

$$(\rho + p)A' + p' = 0. \tag{17}$$

We further consider the equation of state [40] with

$$\omega = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{Q}^2 - V(Q)}{\frac{1}{2}\dot{Q}^2 + V(Q)}. \tag{18}$$

Here, quintessence (Q) denotes the scalar field, $V(Q)$, and the potential energy, and the kinetic energy is given by $\frac{1}{2}\dot{Q}^2$. The cosmological constant is interpreted as arising from a form of energy having negative pressure, equal in magnitude to its (positive) energy density, where $\omega = -1$. Such a cosmological constant is also proposed as a form of dark energy. On the other hand, considering $\omega = -\frac{1}{3}$, we obtain a quintessence field or rather a hypothetical form of dark energy. It is, more precisely, a scalar field that supports the accelerating universe.

In the following, we consider some cases to generate new solutions:

Case (i): Let $\Lambda = 0$.

From Equations 16(i) and (ii) using, $p = \omega\rho$, we obtain

$$A(r) = \omega B(r) + C_1, \dots\dots\dots 19(i).$$

From Equations 16(ii) and (iii) one gets

$$e^{B(r)} = e^{A(r)} \times \frac{A'(r)}{r}, \dots\dots\dots 19(ii).$$

From Equation (17) we have

$$\rho^\omega \times e^{(1+\omega)A(r)} = C_3, \dots\dots\dots 19(iii), \tag{19}$$

where C_1 and C_3 are constants. The following relations are obtained from Equations 19(i) and (ii) as

$$\begin{aligned}
A(r) &= \frac{\omega}{\omega - 1} \ln[(1 - \omega)(C_2 - \frac{e^{-C_1}r^2}{2\omega})] + C_1, \\
B(r) &= \frac{1}{\omega - 1} \ln[(1 - \omega)(C_2 - \frac{e^{-C_1}r^2}{2\omega})],
\end{aligned}
\tag{20}$$

where C_2 is a constant.

Figures 1 and 2 indicate that the metric functions have a finite origin in both the above cases (dark energy matter and quintessence matter, respectively), in the absence of the

cosmological constant term. The metric functions do not, however, exist in the perfect matter fluid case, when $\omega = 1$.

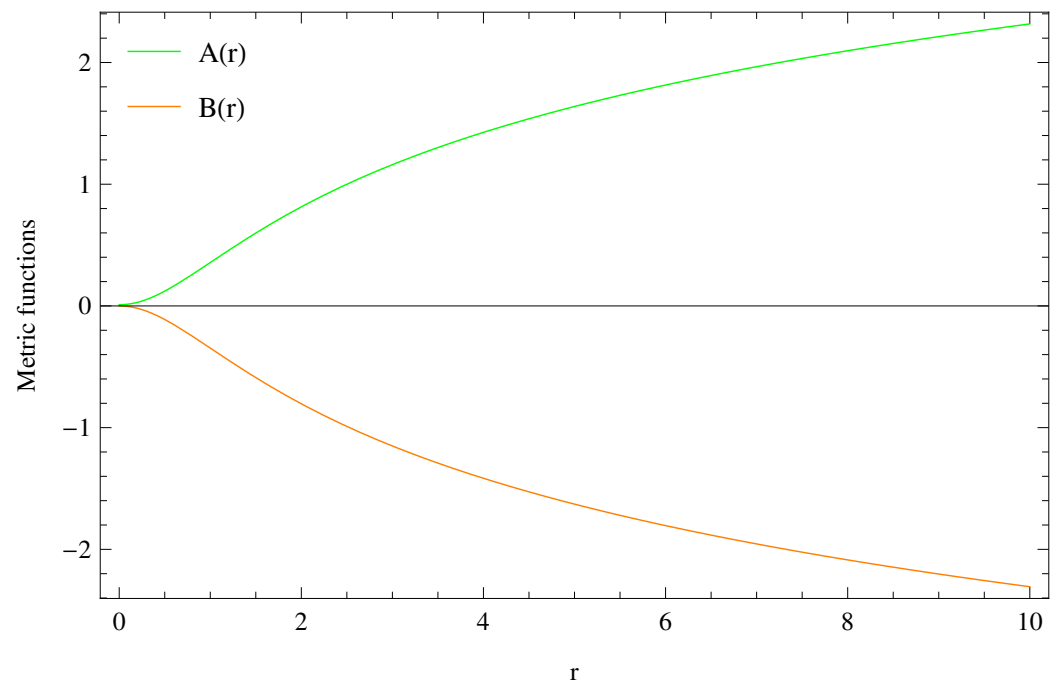


Figure 1. The metric functions for dark energy matter case, $\omega = -1$, are plotted against the radial parameter r , for the values of constants $C_1 = 0.01, C_2 = 0.5$.

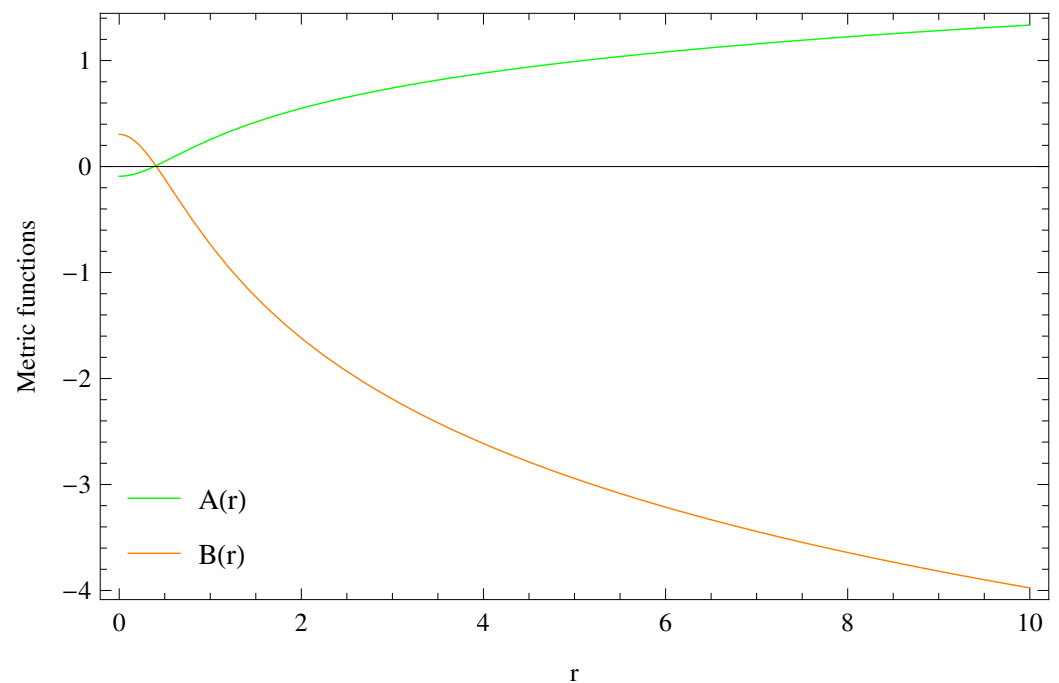


Figure 2. The metric functions for quintessence field, $\omega = -1/3$, are plotted against the radial parameter r , for the values of constants $C_1 = 0.01, C_2 = 0.5$.

The pressure and density are deduced as

$$p(r) = \omega \times \left\{ \frac{C_3}{e^{(1+\omega)(\frac{\omega}{\omega-1} \ln[(1-\omega)(C_2 - \frac{e^{-C_1} r^2}{2\omega})] + C_1)} \right\}^{\frac{1}{\omega}}, \quad (21)$$

$$\rho(r) = \left\{ \frac{C_3}{e^{(1+\omega)(\frac{\omega}{\omega-1} \ln[(1-\omega)(C_2 - \frac{e^{-C_1} r^2}{2\omega})] + C_1)} } \right\}^{\frac{1}{\omega}}. \quad (22)$$

It is observed from Figures 3 and 4 that the pressure is negative, whereas the density is positive in both cases for the dark energy matter and quintessence matter field. Both the pressure and density are non-existent in the perfect fluid matter case. The central density, however, is positive whereas, the central pressure is negative in both existent cases and given by

$$p_0 = \omega \times \left\{ \frac{C_3}{e^{(1+\omega)(\frac{\omega}{\omega-1} \ln[C_2(1-\omega)] + C_1)} } \right\}^{\frac{1}{\omega}}, \quad (23)$$

$$\rho_0 = \left\{ \frac{C_3}{e^{(1+\omega)(\frac{\omega}{\omega-1} \ln[C_2(1-\omega)] + C_1)} } \right\}^{\frac{1}{\omega}}. \quad (24)$$

The Ricci scalar as a function of r from Equation (16) is given as

$$R(r) = \frac{4\pi(\rho - 2p)}{\lambda(1 - 3\rho\lambda)}. \quad (25)$$

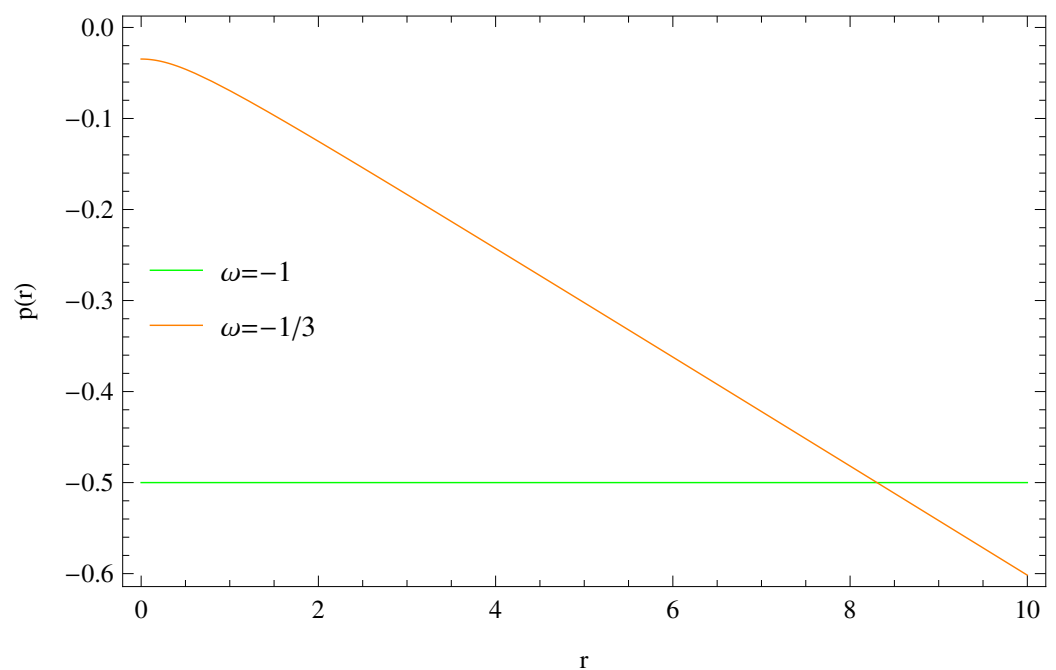


Figure 3. The pressure is plotted against the radial parameter r , for both dark energy matter ($\omega = -1$) and quintessence field ($\omega = -1/3$), for the values of constants $C_1 = 0.01$, $C_2 = 0.5$, $C_3 = 2$.

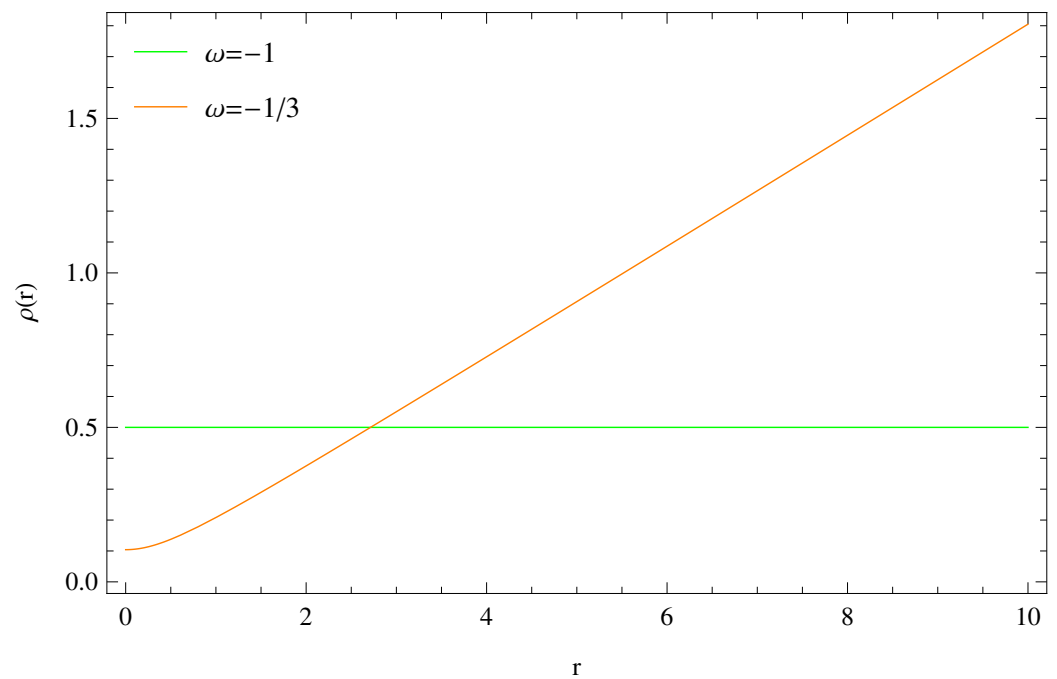


Figure 4. The density is plotted against the radial parameter r , for both dark energy matter ($\omega = -1$) and quintessence field ($\omega = -1/3$), for the values of constants $C_1 = 0.01, C_2 = 0.5, C_3 = 2$.

The radial profile of the Ricci scalar for $\lambda = -0.1$ is plotted in Figures 5 and 6. The Ricci scalar is observed to be negative and constant in Figure 5 (dark energy matter), whereas it is negative and decreasing in Figure 6 (quintessence matter).

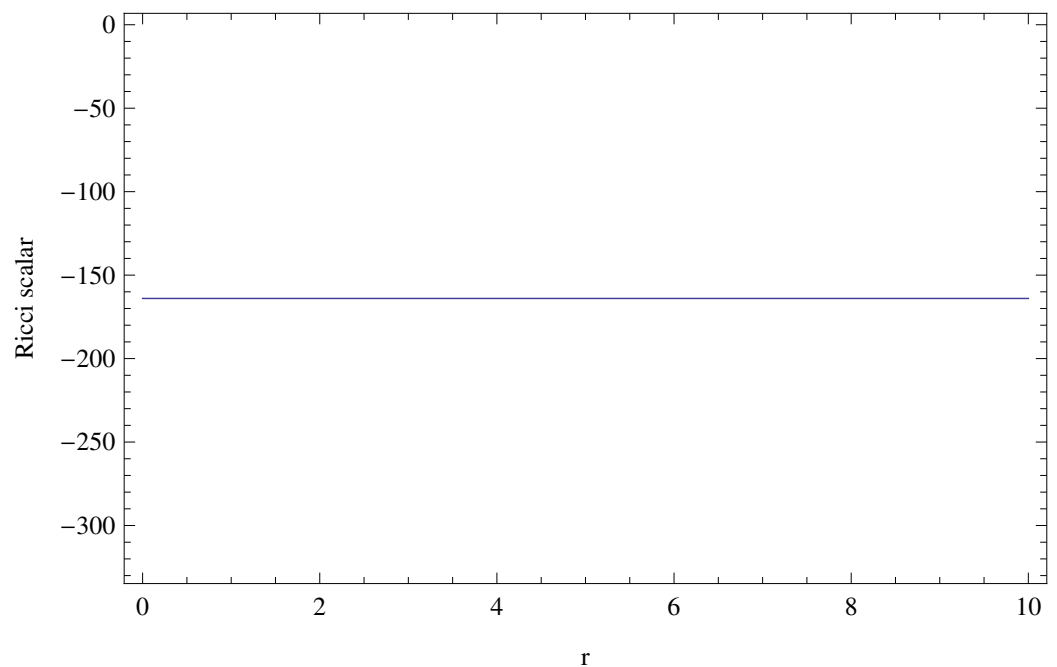


Figure 5. The Ricci scalar is plotted against the radial parameter r , for dark energy matter ($\omega = -1$), for the values of constants $\lambda = -0.1, C_1 = 0.01, C_2 = 0.5, C_3 = 2$.

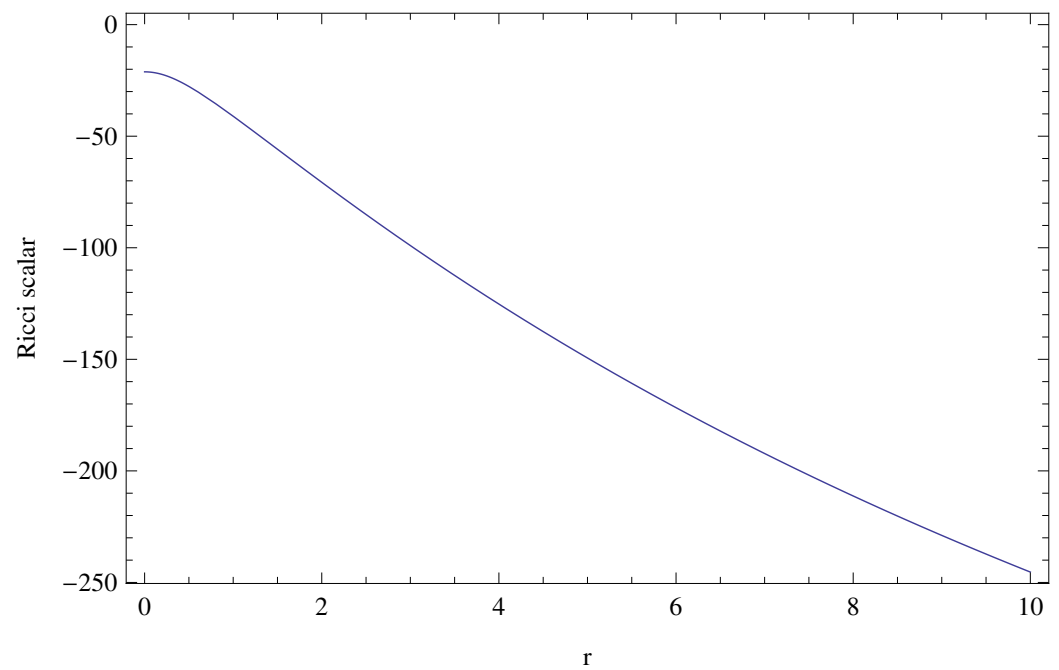


Figure 6. The Ricci scalar is plotted against the radial parameter r , for quintessence matter ($\omega = -\frac{1}{3}$), for the values of constants $\lambda = -0.1, C_1 = 0.01, C_2 = 0.5, C_3 = 2$.

The radial profile of the Ricci scalar for $\lambda = 0.1$ is plotted in Figures 7 and 8 below. The Ricci scalar is observed to be positive and constant in Figure 7 (dark energy matter), whereas it is positive and increasing in Figure 8 (quintessence matter).

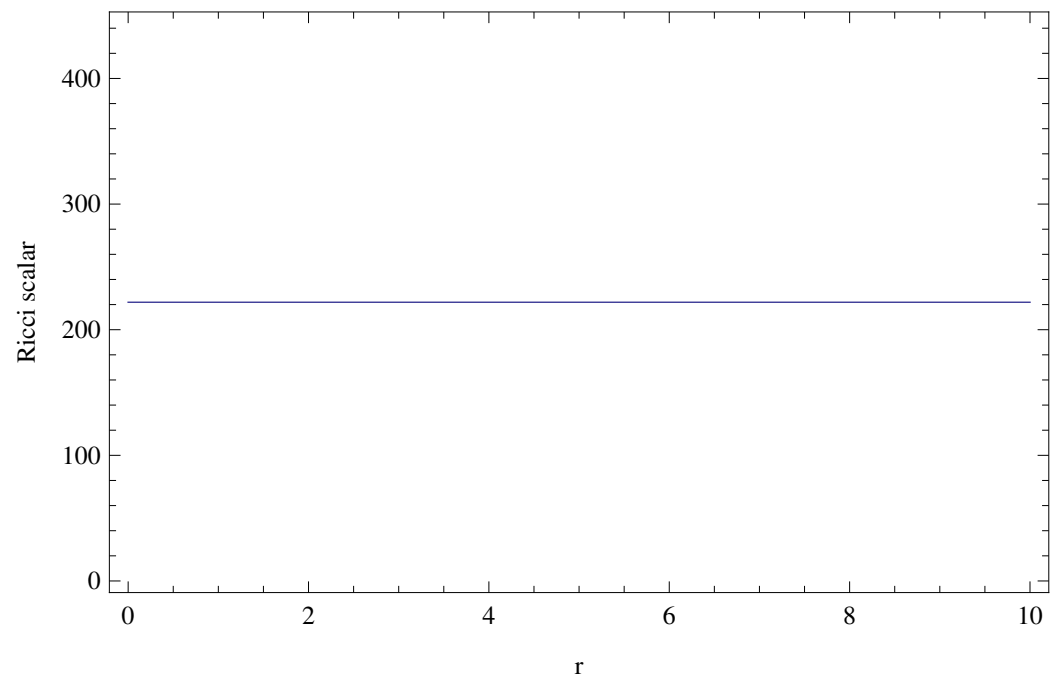


Figure 7. The Ricci scalar is plotted against the radial parameter r , for dark energy matter ($\omega = -1$), for the values of constants $\lambda = 0.1, C_1 = 0.01, C_2 = 0.5, C_3 = 2$.

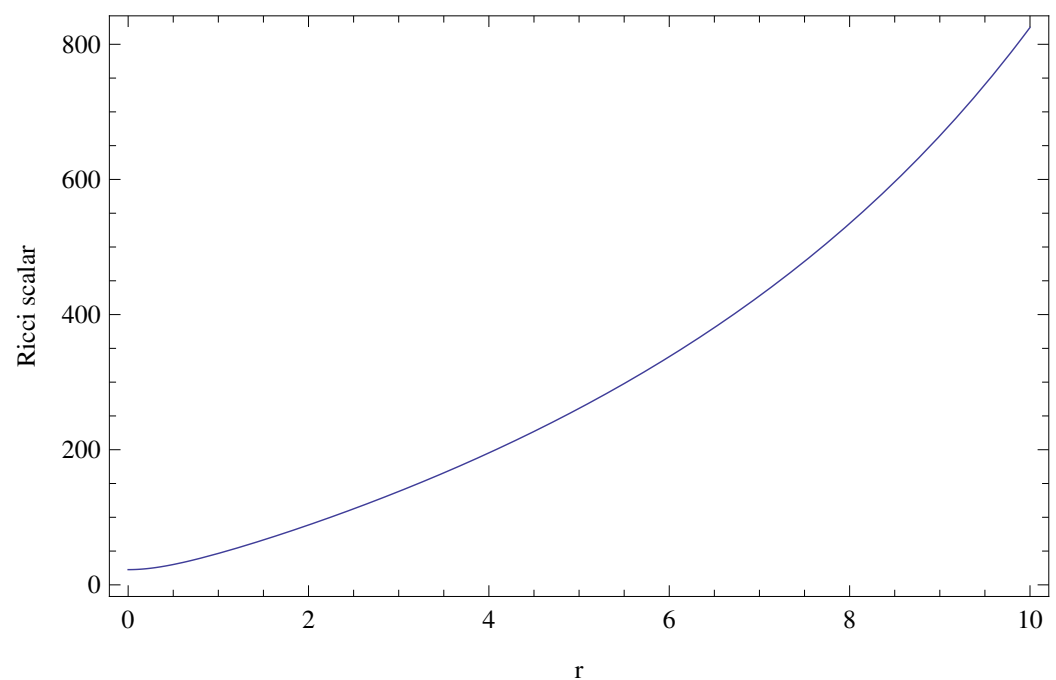


Figure 8. The Ricci scalar is plotted against the radial parameter r , for quintessence matter ($\omega = -\frac{1}{3}$), for the values of constants $\lambda = 0.1, C_1 = 0.01, C_2 = 0.5, C_3 = 2$.

In the Riemannian geometry, the Ricci scalar resembles the simplest curvature invariant of a Riemannian manifold. At each point on a Riemannian manifold, it represents a single real number defined by the intrinsic geometry of the manifold in the vicinity of that point. Moreover, the scalar curvature represents the amount by which the volume of a small geodesic ball in a Riemannian manifold deviates from that of the ball (standard) in an Euclidean space.

If the scalar curvature is positive at a point, the volume of a ball (small) about that point has smaller volume than a ball having the same radius in the Euclidean space. On the other hand, when the scalar curvature is rather negative at a point, the volume of a ball (small) is larger than it would be in the Euclidean space.

Hamilton's Ricci flow on a 3-manifold with a metric of positive scalar curvature has been studied in [41]. In [42], the author observed that positive mean curvature on some time interval is the case of the expanding universe, which confirms with recent cosmological observations. Zero Ricci scalar has been observed in [43] and negative Ricci scalar in [44].

Presently, we intend to search for the energy conditions, which are expressed as

- (i) Null energy condition or NEC : $\rho + p \geq 0$,
- (ii) Weak energy condition or WEC : $\rho + p \geq 0, \rho \geq 0$,
- (iii) Dominant energy condition or DEC : $\rho \geq |p|$,
- (iv) Strong energy condition or SEC : $\rho + p \geq 0, \rho + 2p \geq 0$.

Figure 9 shows that all the energy conditions NEC, WEC, DEC are satisfied. SEC is, however, not satisfied.

From Figure 10 we observe that all the energy conditions NEC, WEC, DEC and SEC are satisfied.

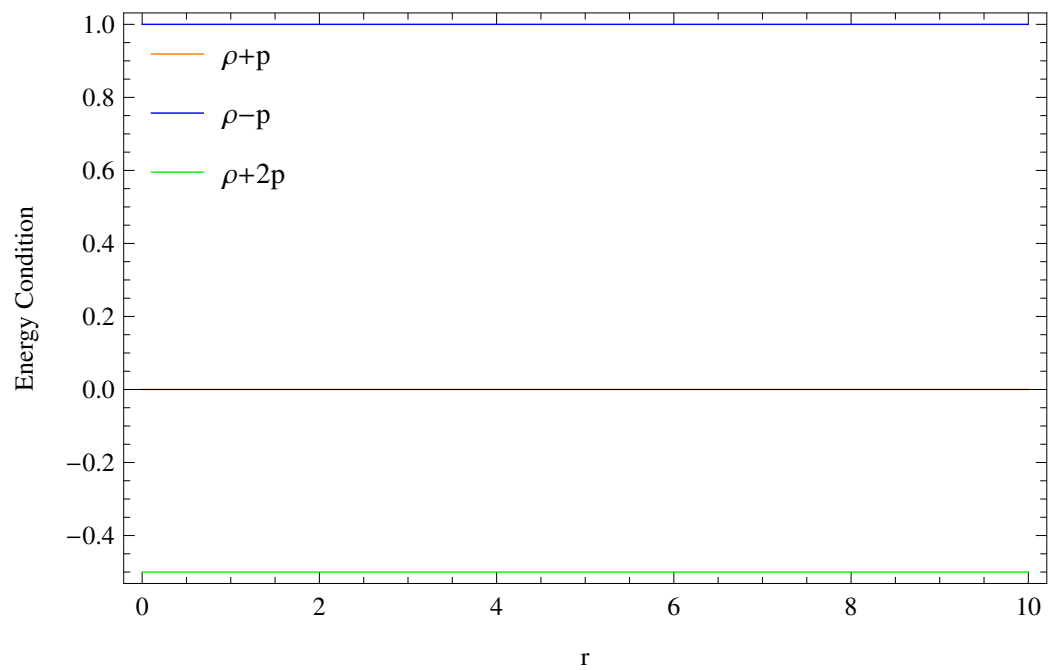


Figure 9. The energy conditions are plotted against the radial parameter r , for dark energy matter ($\omega = -1$), and for the values of constants $C_1 = 0.01, C_2 = 0.5, C_3 = 2$.

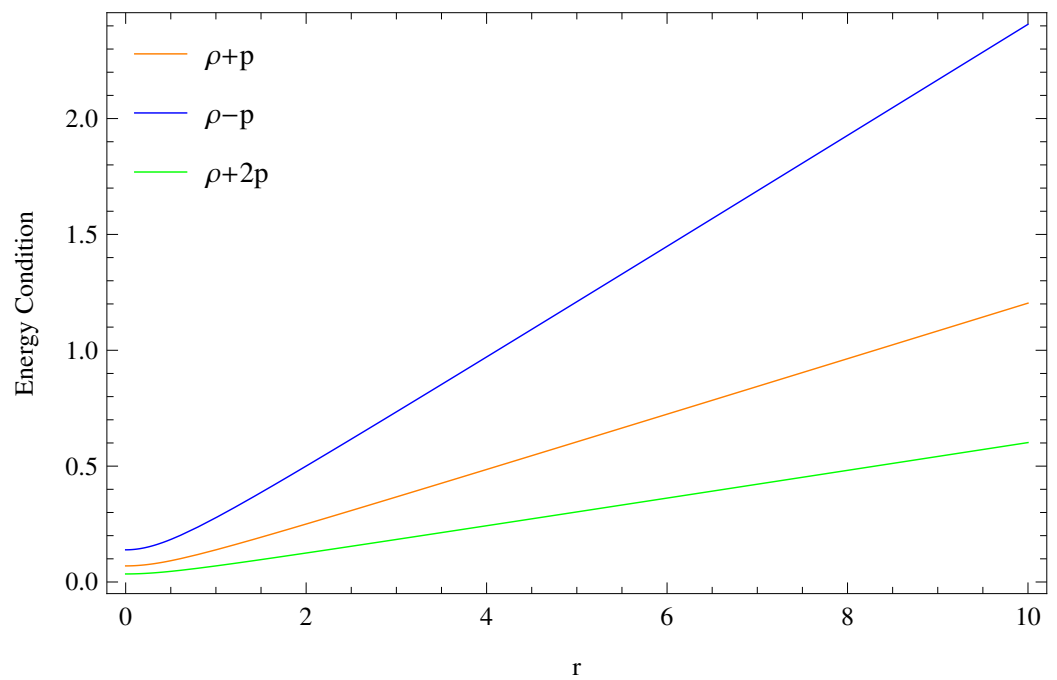


Figure 10. The energy conditions are plotted against the radial parameter r , for quintessence matter ($\omega = -1/3$), and for the values of constants $C_1 = 0.01, C_2 = 0.5, C_3 = 2$.

Case (ii): Let $\Lambda \neq 0, \omega = -1$.
We consider [39]

$$A(r) = C_4 + \frac{1}{2} \ln(r^2 + C_5). \quad (26)$$

The following relations are then relevant from Equations (16) and (17)

$$B(r) = A(r) + \ln A'(r) - \ln r = C_4 - \frac{1}{2} \ln(r^2 + C_5),$$

$$\rho(r) = \left\{ \frac{C_3}{e^{(1+\omega)[C_4 + \frac{1}{2} \ln(r^2 + C_5)]}} \right\}^{\frac{1}{\omega}}, \quad p(r) = \omega \times \left\{ \frac{C_3}{e^{(1+\omega)[C_4 + \frac{1}{2} \ln(r^2 + C_5)]}} \right\}^{\frac{1}{\omega}}. \quad (27)$$

We find that the above solutions are valid only for $\omega = -1$, i.e., for dark matter. Hence the pressure and density reduce as

$$\rho(r) = C_3^{-1}, \quad p(r) = -C_3^{-1}. \quad (28)$$

Figure 11 indicates that the metric functions have a finite origin in presence of the cosmological constant term. Figure 12 reveals that the pressure is negative and constant, whereas Figure 13 shows a positive and constant energy density.

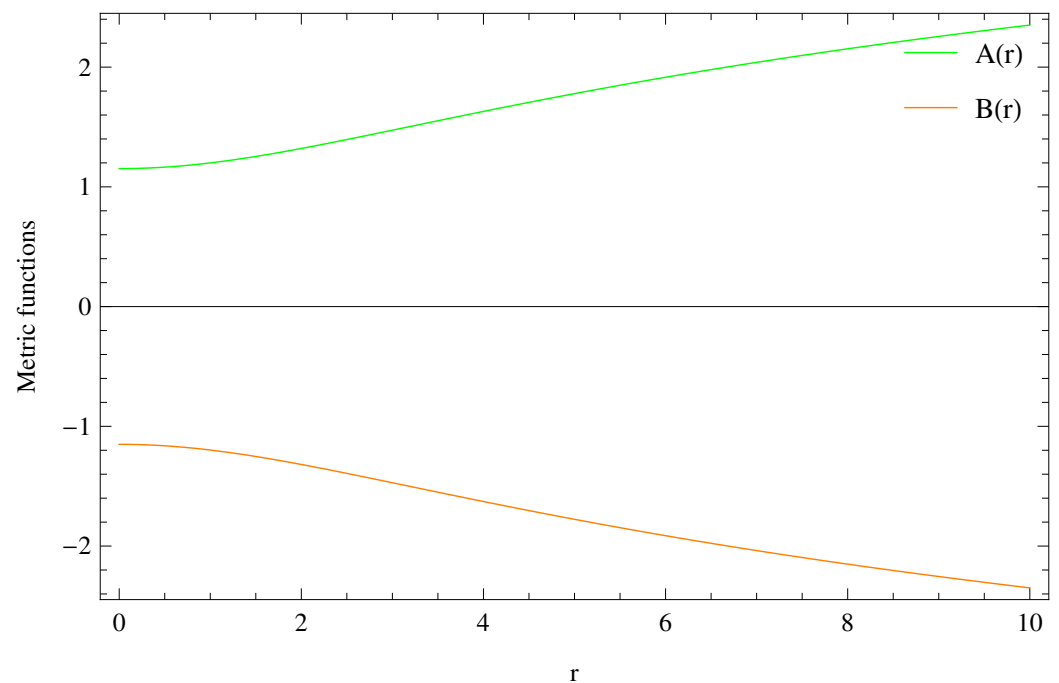


Figure 11. The metric functions are plotted against the radial parameter r for the values of constants $\omega = -1, C_4 = 0.001, C_5 = 10$.

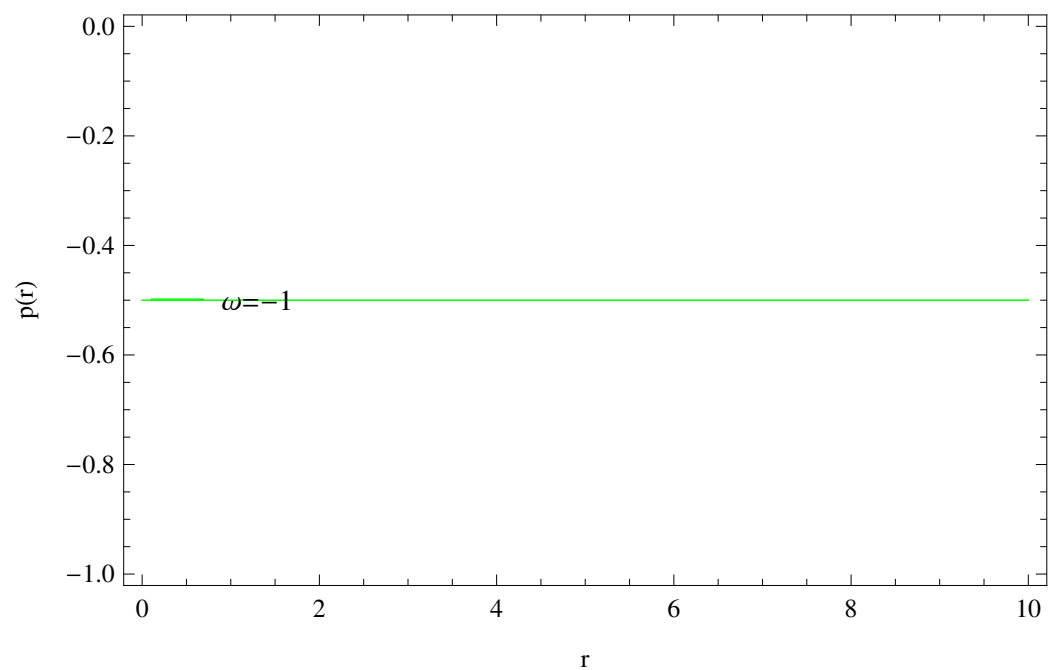


Figure 12. The pressure is plotted against the radial parameter r for the values of constants $C_3 = 2$, $C_4 = 0.001$, $C_5 = 10$.

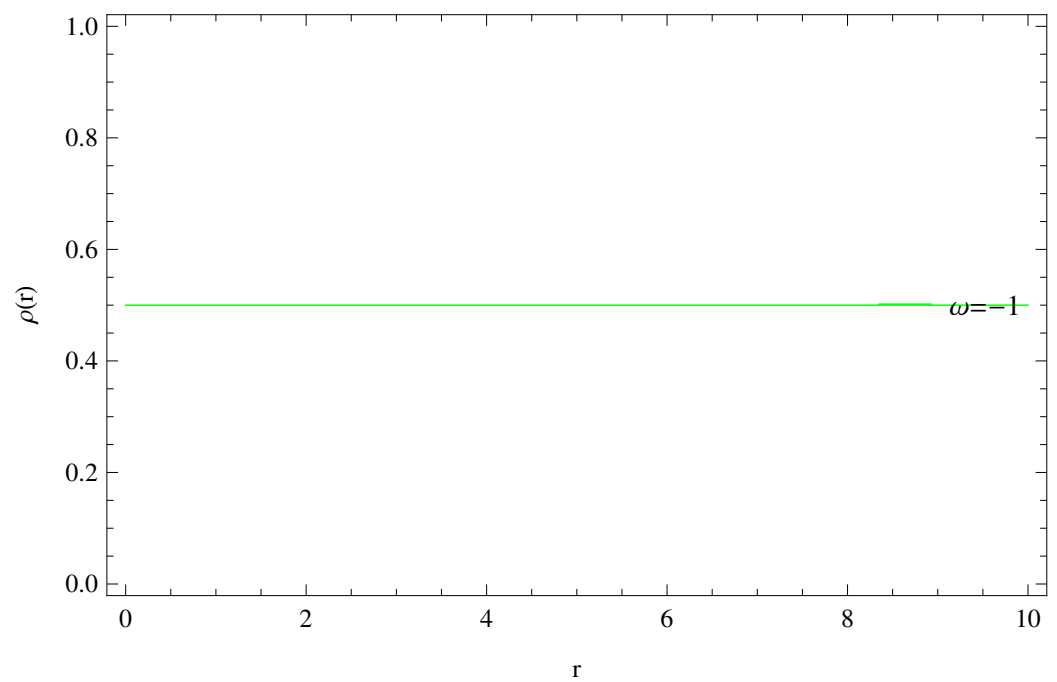


Figure 13. The density is plotted against the radial parameter r with the values of arbitrary constants $C_3 = 2$, $C_4 = 0.001$, $C_5 = 10$.

Figure 14 shows that the strong energy condition (SEC) is not satisfied in presence of the cosmological constant term.

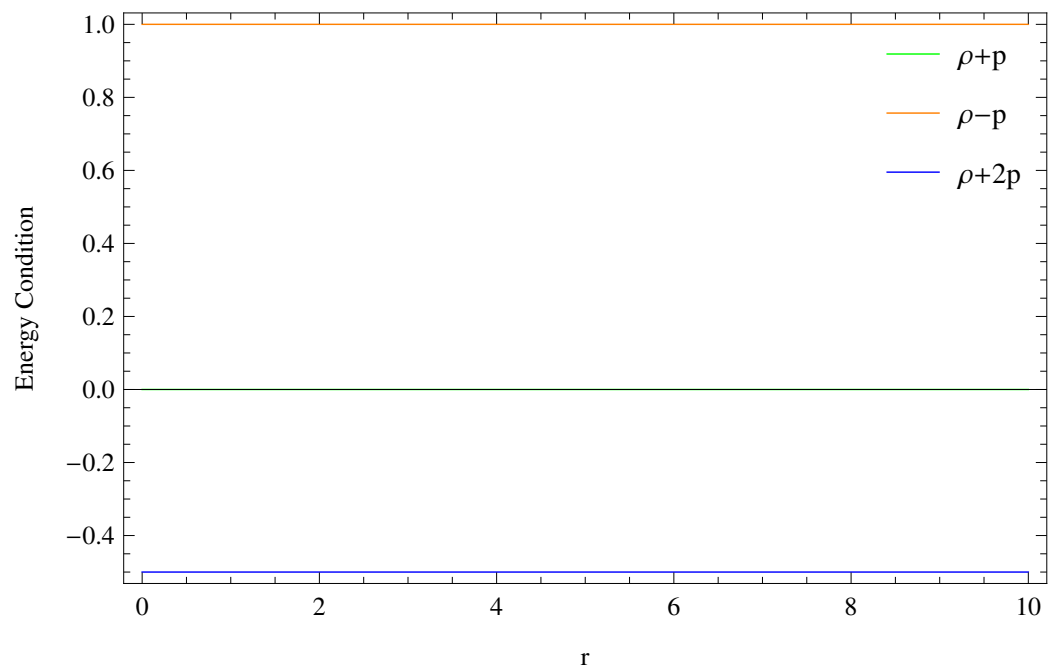


Figure 14. The energy conditions for dark energy matter source are shown against the radial parameter r with the values of arbitrary constants $C_3 = 2, C_4 = 0.001, C_5 = 10$.

The cosmological constant, expressed as a function of r , i.e., $\Lambda = \Lambda(r)$, is deduced from Equations (16) and (17) as (shown graphically in Figure 15).

$$\Lambda(r) = -\frac{e^{-2[c_4 - \frac{1}{2}\ln(r^2 + c_5)]}}{(r^2 + C_5)} - \frac{2\pi}{\lambda} \left\{ \frac{C_3}{e^{(1+\omega)[C_4 + \frac{1}{2}\ln(r^2 + C_5)]}} \right\}^{\frac{1}{\omega}}. \quad (29)$$

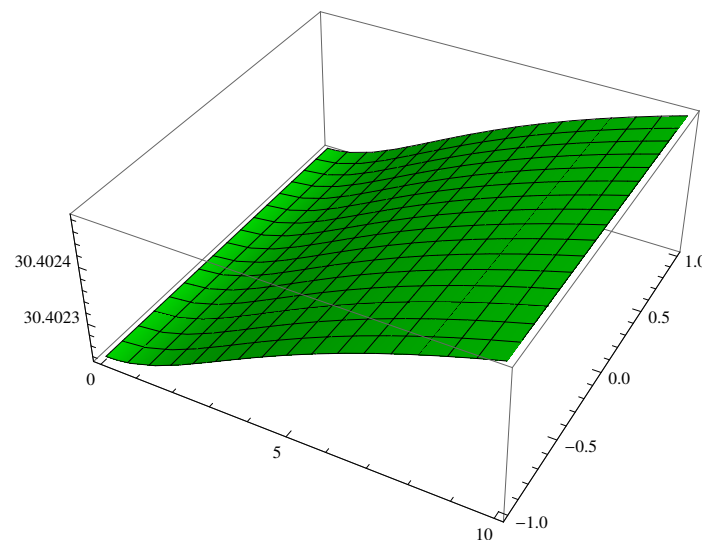


Figure 15. The behaviour of the cosmological constant in presence of dark energy matter source is plotted against r with the values of the arbitrary constants $\lambda = -0.1, C_3 = 2, C_4 = 0.001, C_5 = 10$.

Case (iii): Let $\Lambda \neq 0, \omega \neq -1$.

We consider

$$A(r) = C_4 + \frac{1}{2}\ln(r^2 + C_5). \quad (30)$$

The following relations are then relevant from Equations (16) and (17)

$$B(r) = \frac{(1+\omega)C_4}{2\omega} - \frac{1}{2} \ln \left[\frac{4C_3^{\frac{1}{\omega}} \pi(r^2 + C_5)}{\lambda\omega} \times \left\{ \omega^2(r^2 + c_5)^{-\frac{(1+\omega)}{2\omega}} + 2(1-\omega^2)C_6 \right\} \right], \quad (31)$$

where C_6 is an arbitrary constant. We have

$$B'(r) = (1-\omega)r \left\{ 4C_6(r^2 + C_5)^{\frac{1+\omega}{2\omega}} + \omega(4C_6(r^2 + C_5)^{\frac{1+\omega}{2\omega}} - 1) \right\} \times [2(r^2 + C_5) \{ \omega^2(2C_6(r^2 + C_6)^{\frac{1+\omega}{2\omega}} - 1) - 2C_6(r^2 + C_6)^{\frac{1+\omega}{2\omega}} \}]^{-1}. \quad (32)$$

Figure 16 indicates that the metric functions have a finite origin. Both theory and observations confirm that in the very early universe, there was an inflationary phase where the metric changed very rapidly. The metric function $B(r)$ is constant for $\lambda = 1$ and does not exist for $\lambda = -1$.

The pressure and density for $\omega = -1/3$, is given below as

$$\rho(r) = \left\{ \frac{C_3}{e^{\frac{2}{3}[C_4 + \frac{1}{2}\ln(r^2 + C_5)]}} \right\}^{-3}, \quad p(r) = -\frac{1}{3} \times \left\{ \frac{C_3}{e^{\frac{2}{3}[C_4 + \frac{1}{2}\ln(r^2 + C_5)]}} \right\}^{-3}. \quad (33)$$

Other relation for perfect fluid, $\omega = 1$, follows significantly and is shown in the figures below.

Figure 17 also indicates that the metric functions have a finite origin. We observe from the Figures 12 and 13 and Figures 18 and 19 that the pressure is negative and energy density is positive throughout as $r \rightarrow \infty$ in all the three cases for dark energy matter ($\omega = -1$), quintessence field ($\omega = -\frac{1}{3}$) and perfect matter fluid source ($\omega = 1$).

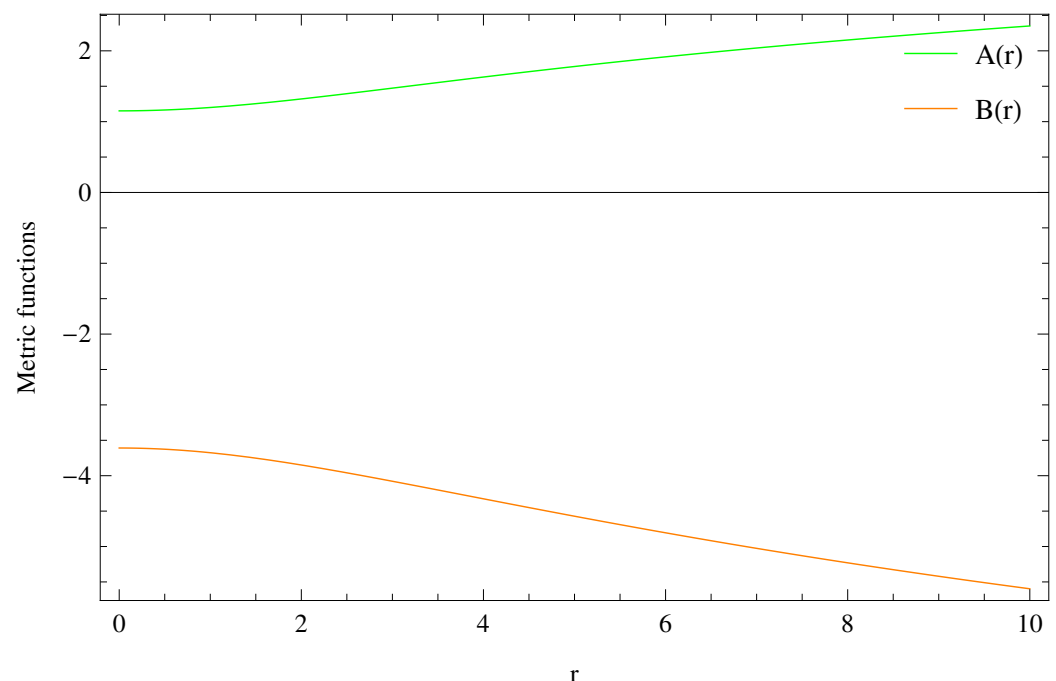


Figure 16. The metric functions are plotted against the radial parameter r with the values of the arbitrary constants $\omega = -1/3, \lambda = -0.1, C_4 = 0.001, C_5 = 10, C_6 = 1$.

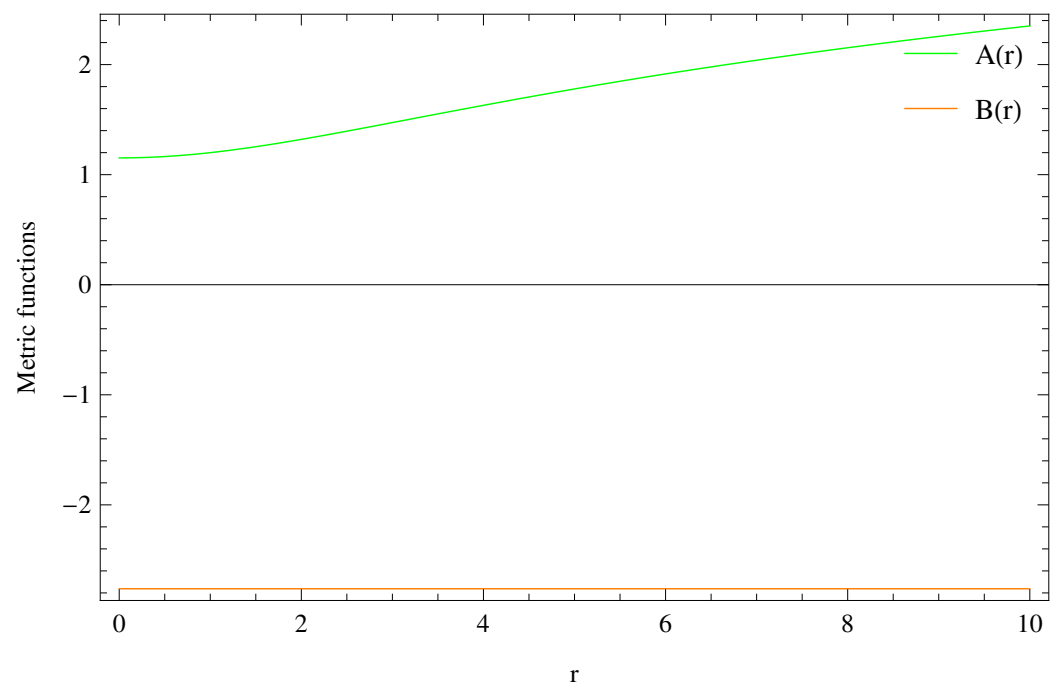


Figure 17. The metric functions are plotted against the radial parameter r with the values of the arbitrary constants $\omega = 1, \lambda = 0.1, C_4 = 0.001, C_5 = 10, C_6 = 1$.

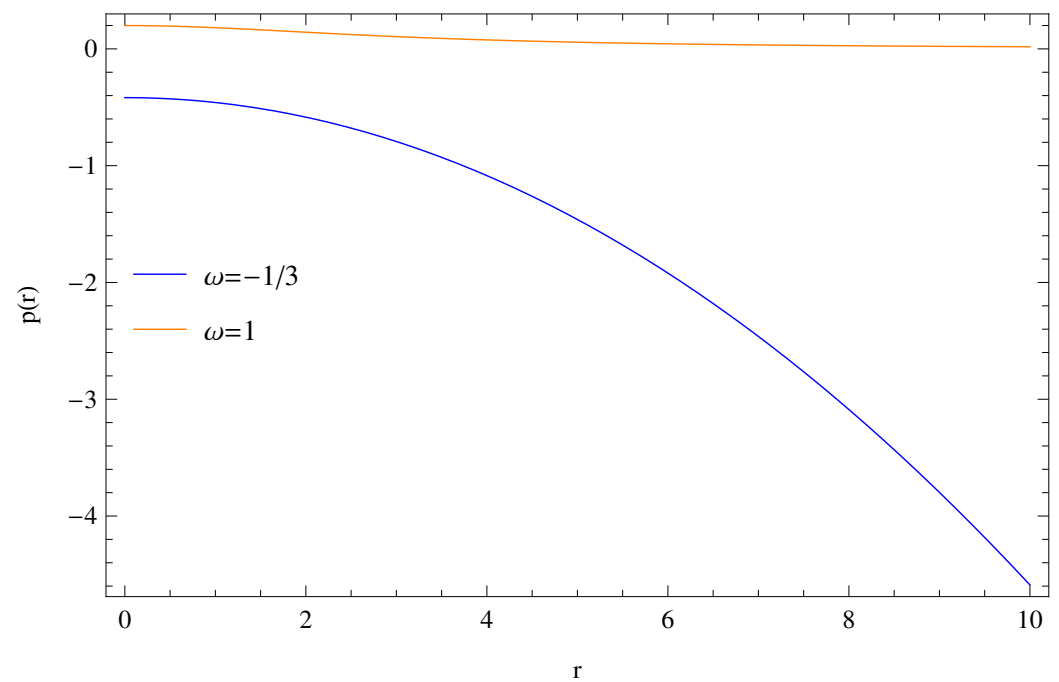


Figure 18. The pressure is plotted against the radial parameter r with the values of arbitrary constants $C_3 = 2, C_4 = 0.001, C_5 = 10$.

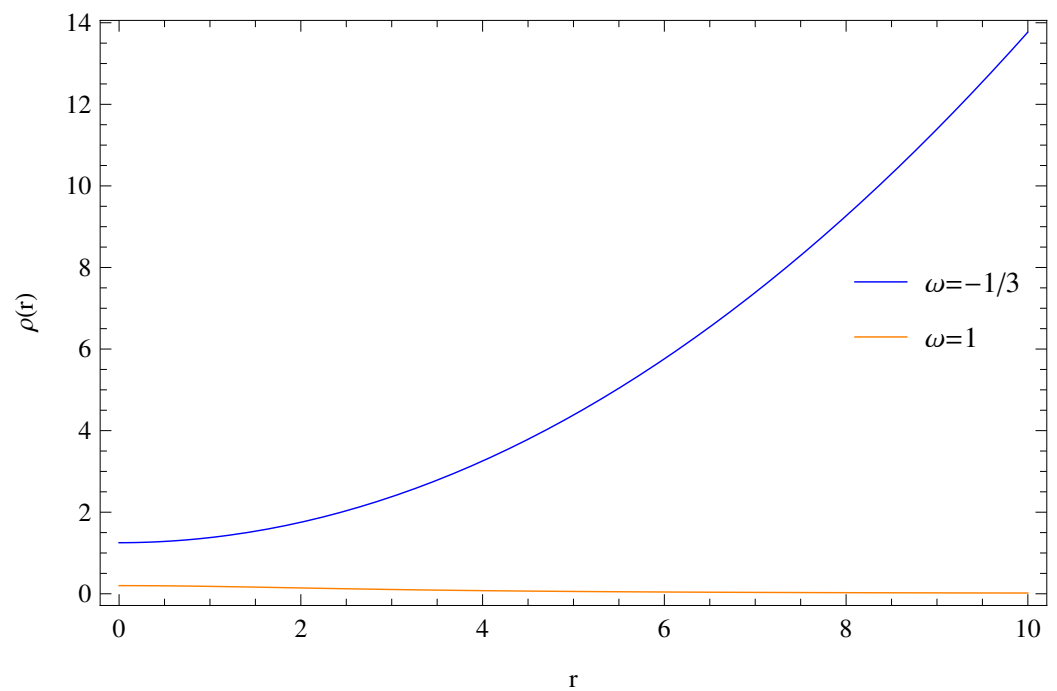


Figure 19. The density is plotted against the radial parameter r with the values of arbitrary constants $C_3 = 2, C_4 = 0.001, C_5 = 10$.

The energy conditions are shown below.

The energy conditions plotted in Figure 14 indicate that only the NEC, DEC and WEC are satisfied for the dark energy matter source. However, Figures 20 and 21 indicate that all the energy conditions NEC, WEC, DEC and SEC are satisfied for the quintessence field and perfect matter fluid source, respectively.

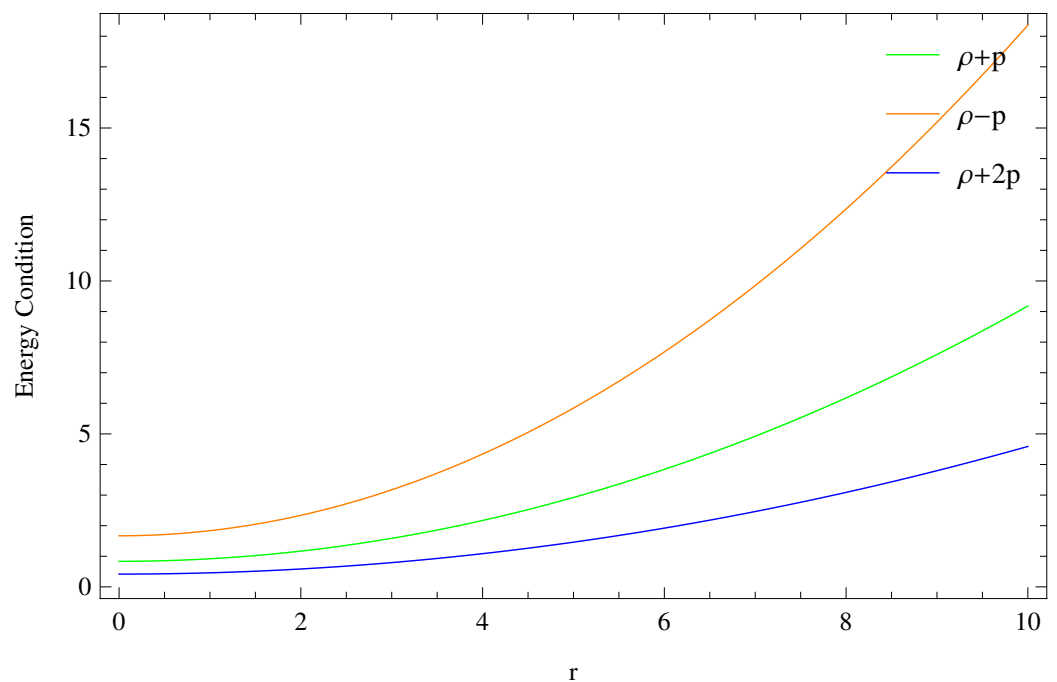


Figure 20. The energy conditions for quintessence field are shown against the radial parameter r with the values of the arbitrary constants $C_3 = 2, C_4 = 0.001, C_5 = 10$.

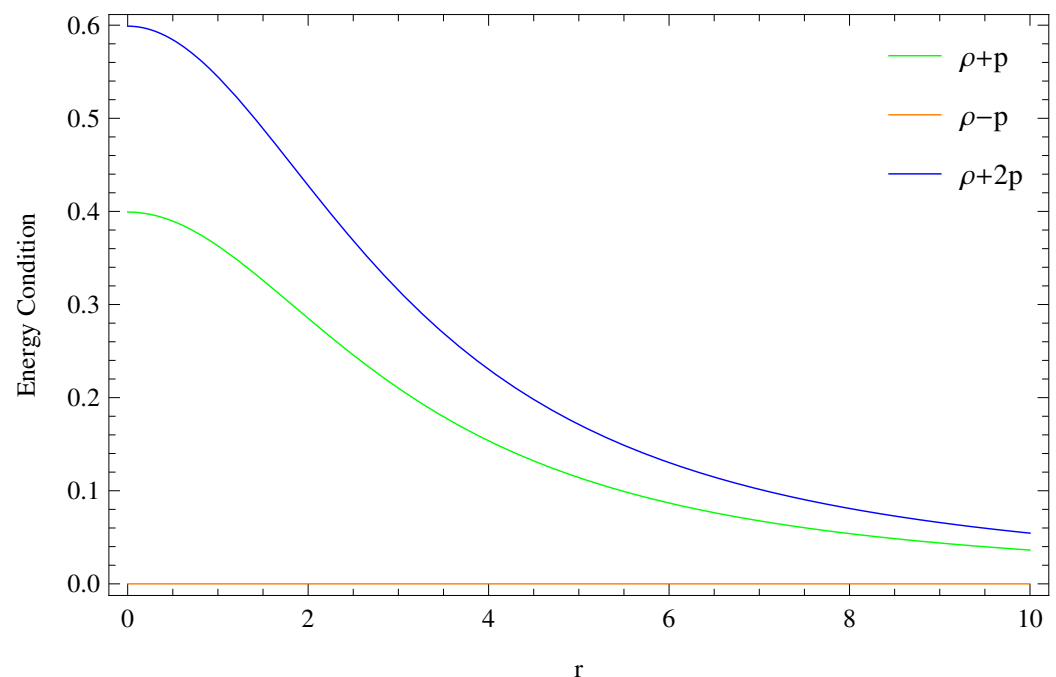


Figure 21. The energy conditions for perfect matter fluid source are plotted against the radial parameter r with the values of the arbitrary constants $C_3 = 2, C_4 = 0.001, C_5 = 10$.

The cosmological parameter results from Equations (16) and (17) as

$$\Lambda(r) = \frac{1}{r} e^{-2\frac{(1+\omega)C_4}{2\omega}} \times \left[\frac{4C_3^{\frac{1}{\omega}} \pi(r^2 + C_5)}{\lambda\omega} \times \left\{ \omega^2(r^2 + C_5)^{-\frac{(1+\omega)}{2\omega}} + 2(1 - \omega^2)C_6 \right\} \right]^{-\frac{1}{2}} \\ \times (1 - \omega)r \left\{ 4C_6(r^2 + C_5)^{\frac{1+\omega}{2\omega}} + \omega(4C_6(r^2 + C_5)^{\frac{1+\omega}{2\omega}} - 1) \right\} \times [2(r^2 + C_5) \{ \omega^2(2C_6(r^2 + C_6)^{\frac{1+\omega}{2\omega}} - 1) - 2C_6(r^2 + C_6)^{\frac{1+\omega}{2\omega}} \}]^{-1} - \frac{2\pi}{\lambda} \left\{ \frac{C_3}{e^{(1+\omega)[C_4 + \frac{1}{2}\ln(r^2 + C_5)]}} \right\}^{\frac{1}{\omega}} \quad (34)$$

and it is shown graphically below.

Figure 22 shows that the cosmological constant is negative and decreasing, in the presence of the quintessence field or perfect fluid matter.

We observe that the actual cosmological constant Λ is almost static, with a fixed energy density and when $\omega = -1$ (shown in Figure 15), whereas in presence of the quintessence field or perfect matter fluid it does not appear to be constant. It is found that the cosmological constant behaves as a dynamical quantity, which is a positive and decreasing function in the perfect fluid matter case $\omega = 1$ (Figure 23) and negative and decreasing function in the presence of the quintessence field $\omega = -\frac{1}{3}$ (Figure 24). Thus, we observe here that the cosmological constant can vary with respect to the radial parameter r in the presence of the perfect fluid or quintessence field.

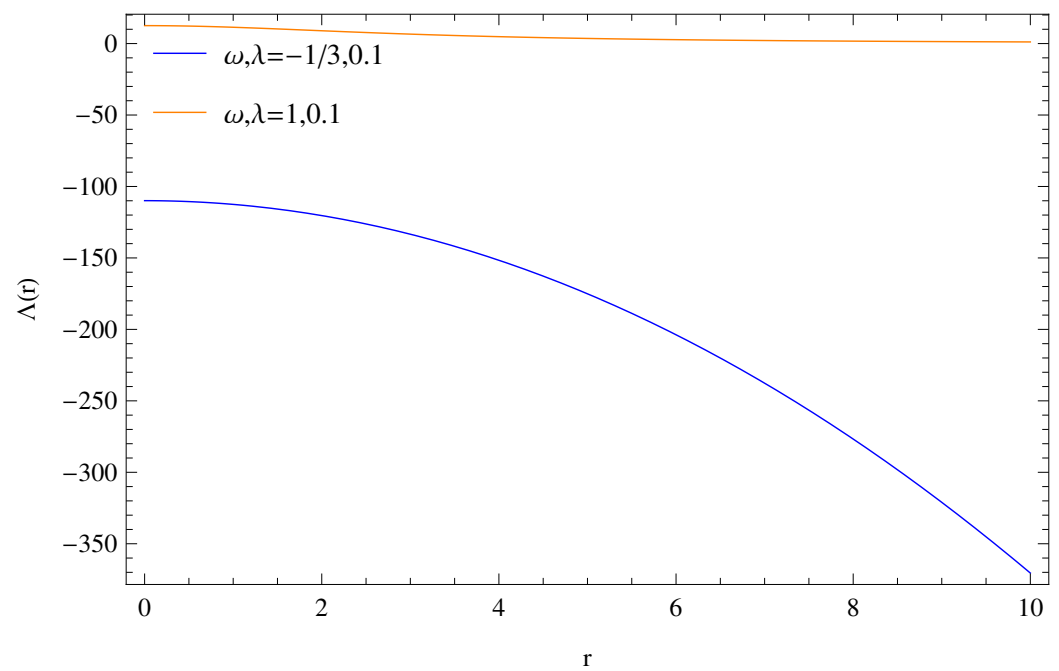


Figure 22. Variation of cosmological constant in presence of quintessence field ($\omega = -\frac{1}{3}$) and perfect fluid matter ($\omega = 1$) is plotted against r with the values of the arbitrary constants $\lambda = -0.1$, $C_3 = 2$, $C_4 = 0.001$, $C_5 = 10$, $C_6 = 1$.

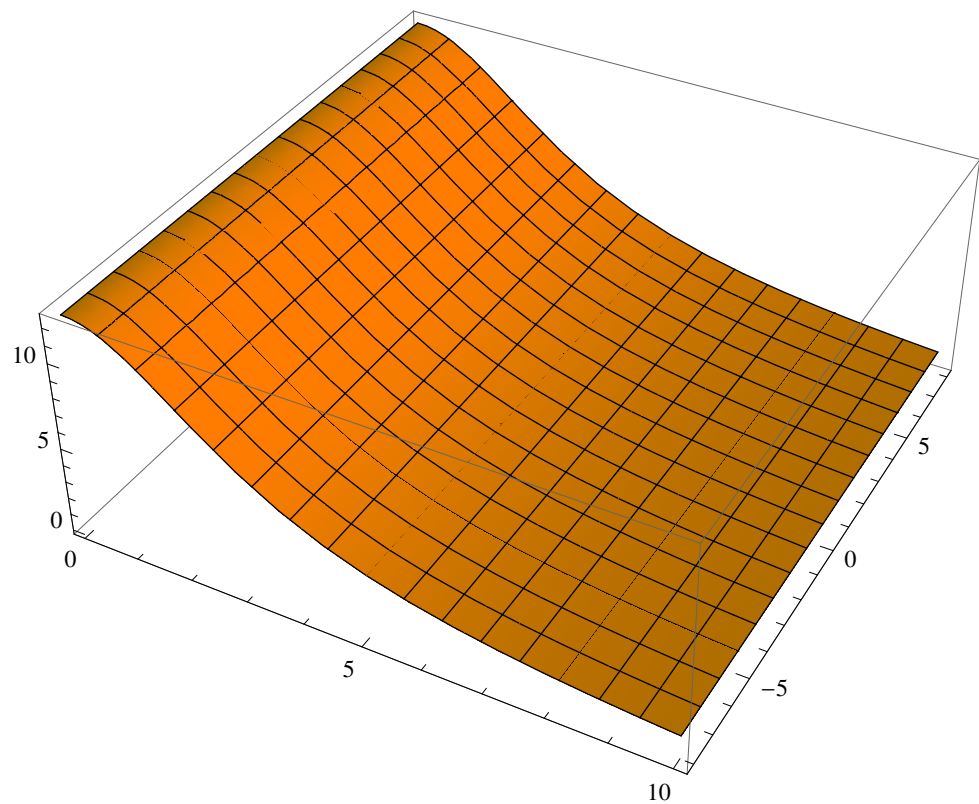


Figure 23. Variation of cosmological constant in presence of perfect fluid matter ($\omega = 1$) is plotted against r with the values of the arbitrary constants $\lambda = 0.1$, $C_3 = 2$, $C_4 = 0.001$, $C_5 = 10$, $C_6 = 1$.

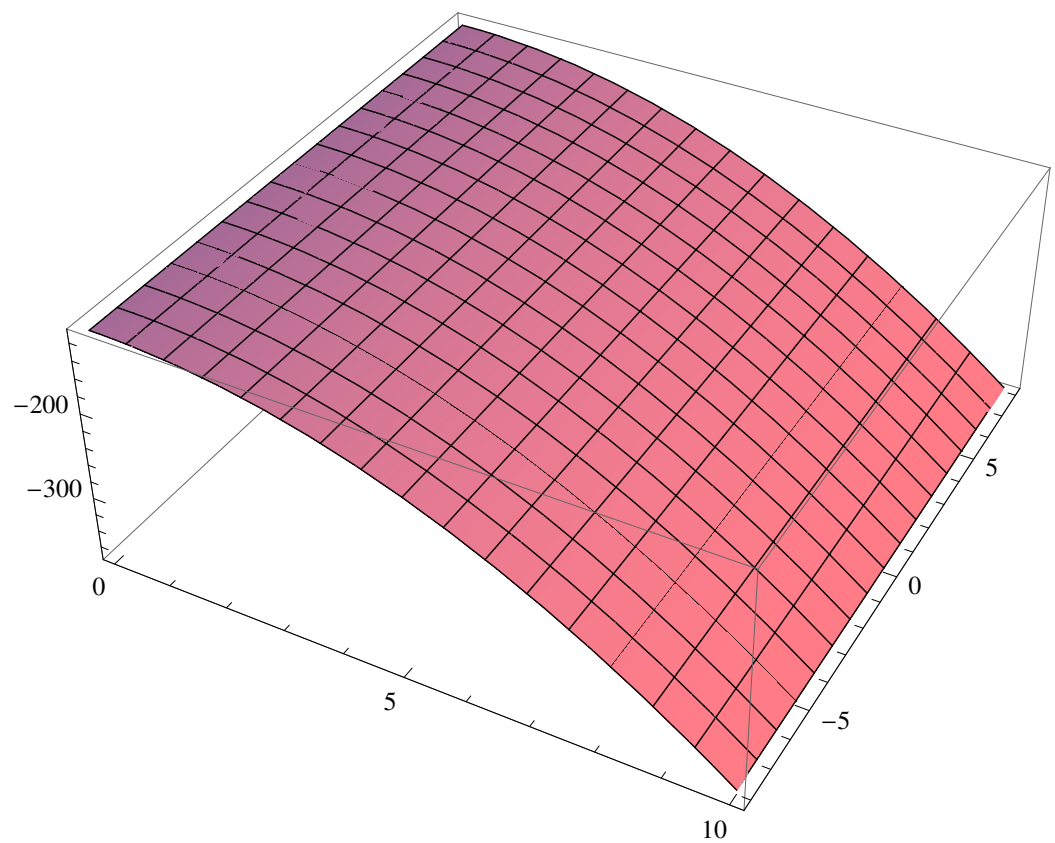


Figure 24. Variation of cosmological constant in presence of quintessence field ($\omega = -\frac{1}{3}$) is shown against r with the values of the arbitrary constants $\lambda = -0.1, C_3 = 2, C_4 = 0.001, C_5 = 10, C_6 = 1$.

Case(iii)(a): $\Lambda = \Lambda(R)$: In perfect fluid matter case, we observe from Equations (13) and (25) that

$$\Lambda = \frac{1}{2}\rho\lambda R, \quad (35)$$

which is plotted below in Figures 25 and 26. We observe here that the cosmological constant varies with respect to the scalar factor R , of our expanding universe, as well

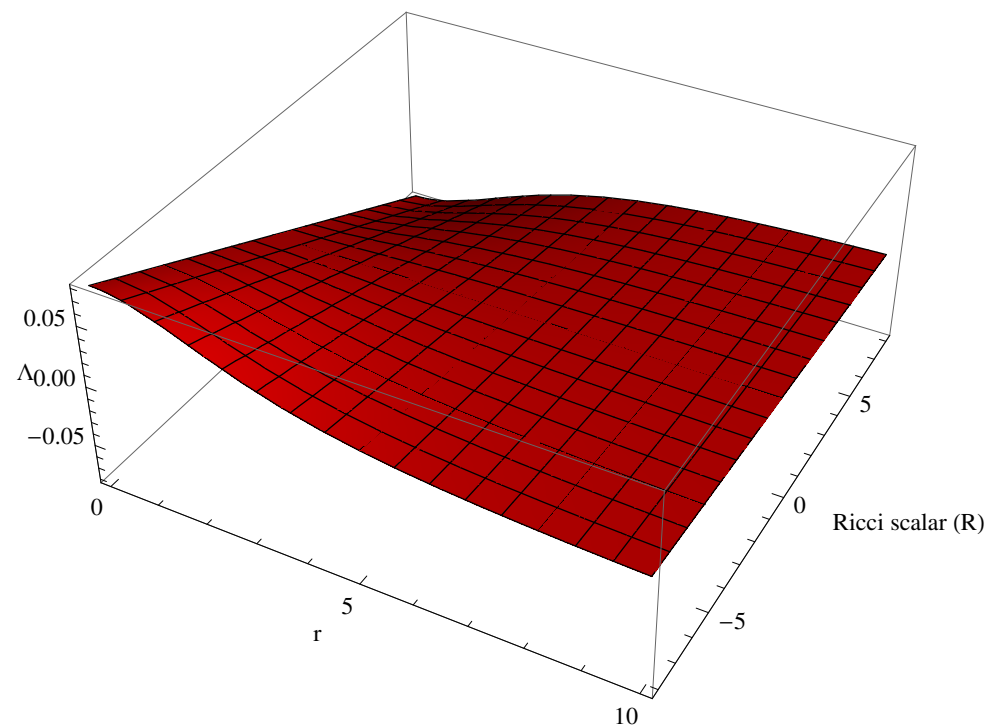


Figure 25. The behaviour of the cosmological constant in presence of perfect fluid matter source is plotted against the Ricci scalar R with the values of the arbitrary constants $\lambda = -0.1, C_3 = 2, C_4 = 0.001, C_5 = 10$.

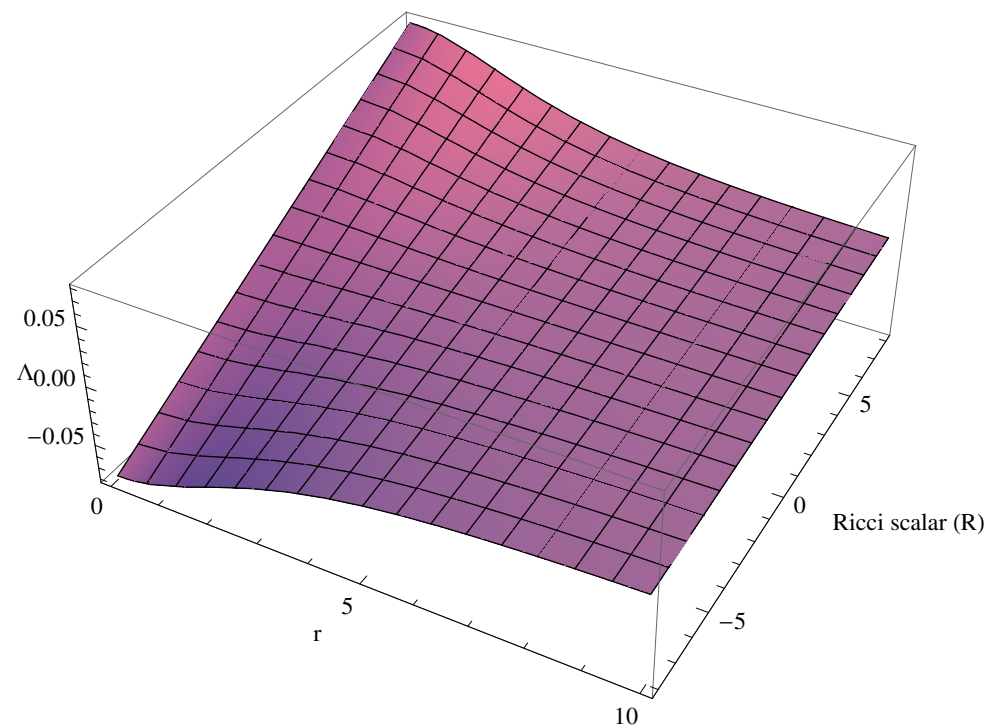


Figure 26. The behaviour of the cosmological constant in presence of perfect fluid matter source is shown against the Ricci scalar R with the values of the arbitrary constants $\lambda = 0.1, C_3 = 2, C_4 = 0.001, C_5 = 10$.

4. Effective Mass and Redshift Function

We obtain the effective mass of an object with radius R for dark energy matter case, ($\omega = -1, \Lambda = 0$) using Equation (22), as in [45]

$$M = \int_0^R 2\pi r \rho(r) dr = \int_0^R \frac{2\pi}{C_3} r dr = \frac{\pi R^2}{C_3}. \quad (36)$$

We observe the same result when ($\omega = -1, \Lambda = \Lambda(r)$). The corresponding gravitational redshift using Equation (36) is given by

$$Z = \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} - 1 = \left(1 - \frac{2\pi R}{C_3}\right)^{-\frac{1}{2}} - 1. \quad (37)$$

Hence, the mass function $M(r)$ and redshift function $Z(r)$ are shown graphically in Figures 27 and 28, respectively.

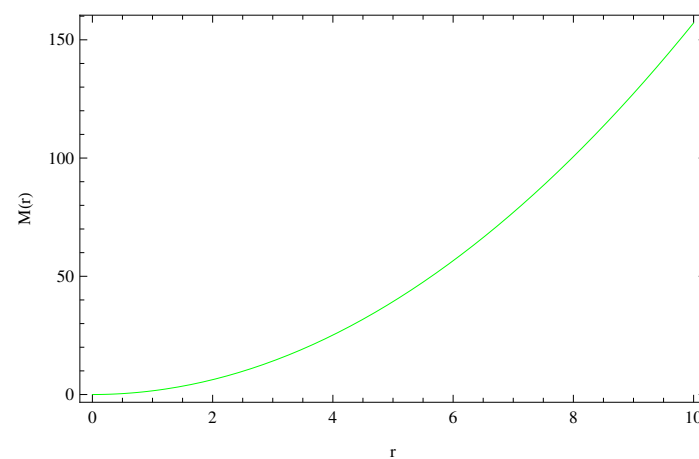


Figure 27. The mass function is shown against r for the value of constant $C_3 = 2$.

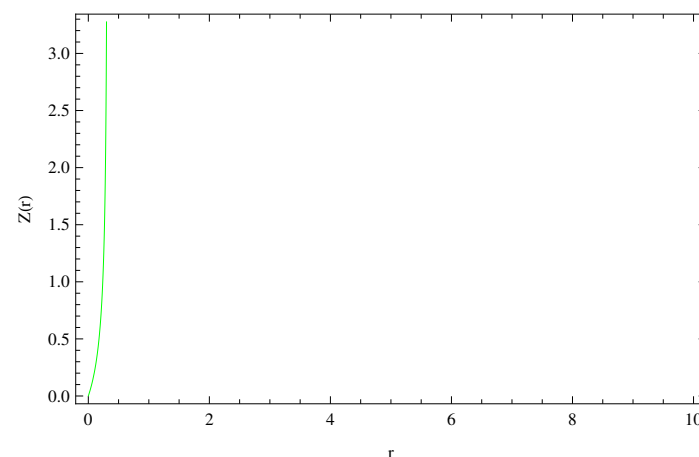


Figure 28. The redshift function is shown against r for the value of constant $C_3 = 2$.

We find that both mass function and redshift function are positive. As per latest observations, galaxies and the objects producing gamma ray bursts have the highest known redshifts. The most reliable redshifts are confirmed from spectroscopic data, and the highest spectroscopic redshift of a galaxy is that of GN-z11 [46], with a redshift of $Z = 11.1$, which corresponds to 400 million years after the Big Bang.

5. Concluding Remarks

We have investigated a feasible combined cosmological model under the $f(R, T)$ gravity that satisfies the equation of the state for dark energy matter, quintessence matter or perfect matter fluid. Various physical properties of the model are elaborated, which satisfies all the energy conditions.

It is believed that the energy conditions do not represent the physical constraints but are rather mathematically imposed boundary conditions. We know that the energy should be positive [47].

Most energy conditions do not correspond with physical reality. The dark energy and its observable effects are very well-known, and they violate the strong energy condition (SEC) [48,49].

Figure 9 (Case i) and Figure 14 (Case ii) refer to the Dark Energy matter source ($\Lambda = 0, \omega = -1$ and $\Lambda \neq 0, \omega = -1$, respectively), where we observe that the SEC is not satisfied or is violated.

However, we observe that all the energy conditions (including SEC) are satisfied for quintessence matter in Figure 10 ($\Lambda = 0, \omega = -\frac{1}{3}$) and Figure 20 ($\Lambda \neq 0, \omega = -\frac{1}{3}$), as well as for perfect matter in Figure 21 ($\Lambda \neq 0, \omega = 1$).

We have analysed the properties in the framework of $f(R, T)$ theory. The density is positive throughout in all the three cases, whereas the pressure remains negative. The equation of state $p = \omega\rho$, yields an exact model where the proportionality constant ω is expressed accordingly. For stability, we find that the energy conditions pertaining to matter content are satisfied. The cosmological constant does not exist if, however, we take $\omega = 0$. Such vanishing of ω also gives rise to an infinite positive energy density and vanishing of pressure, as is evident from Equation (18). This is comparable to the dust model of the universe [$\omega = 0$]. Dust models find its place in cosmology as models of a toy universe, where the dust particles are considered as highly idealized models of galaxies, clusters, or superclusters [50]. Hence, the $f(R, T)$ model displays the necessary features corresponding to expectations.

This study indicates that the cosmological constant can vary with respect to the radial parameter r , in the presence of the perfect fluid matter or quintessence field. On the other hand, the actual cosmological constant Λ is almost static, with a fixed energy density when $\omega = -1$, that is in the presence of dark energy matter source. The cosmological constant also varies with respect to the scalar factor R of our expanding universe, in the presence of the perfect fluid matter source. Hence, we are able to portray a clear picture of the cosmological constant Λ .

The effective mass and gravitational redshift are discussed for the dark energy matter case ($\omega = -1$), both when ($\Lambda = 0$) and $\Lambda = \Lambda(r)$, as it is most relevant according to present day observations for accelerating the universe.

We intend to further study the $f(R, T)$ modified gravity combined with the cosmological model, in $(3 + 1)$ and higher dimensions. Such study is in progress.

Author Contributions: Conceptualization, S.I., M.A. and I.R.; Data curation, D.B.; Formal analysis, D.B.; Investigation, S.I.; Methodology, S.I., M.A., I.R. and D.B.; Software, S.I.; Supervision, S.I.; Visualization, I.R.; Writing—review and editing, S.I. All authors have read and agreed to the published version of the manuscript.

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