axioms

Article

Applications of the q-Derivative Operator to New Families of Bi-Univalent Functions Related to the Legendre Polynomials

Ying Cheng [1], Rekha Srivastava [2,*] and Jin-Lin Liu [1]

[1] Department of Mathematics, Yangzhou University, Yangzhou 225002, China
[2] Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
* Correspondence: rekhas@math.uvic.ca

Abstract: By using the q-derivative operator and the Legendre polynomials, some new subclasses of q-starlike functions and bi-univalent functions are introduced. Several coefficient estimates and Fekete–Szegö-type inequalities for functions in each of these subclasses are obtained. The results derived in this article are shown to extend and generalize those in some earlier works.

Keywords: analytic functions; bi-univalent functions; q-derivative operator; subordination between analytic functions; Legendre polynomials; Fekete–Szegö inequality

MSC: 30C45; 30C80; 33C45

check for updates

Citation: Cheng, Y.; Srivastava, R.; Liu, J. Applications of the q-Derivative Operator to New Families of Bi-Univalent Functions Related to the Legendre Polynomials. *Axioms* **2022**, *11*, 595. https://doi.org/10.3390/axioms11110595

Academic Editor: Clemente Cesarano

Received: 05 October 2022
Accepted: 23 October 2022
Published: 27 October 2022

1. Introduction

In the development of the Geometric Function Theory of Complex Analysis, the q-derivative is an important research tool. The application of q-calculus was first considered by Jackson (see [1–4]). Recently, many scholars have defined new subclasses of analytic functions by combining the q-derivative operator with the principle of differential subordination and studied their geometric properties (see [5–15]). In this article, we investigate two new subclasses $I^q_{\mathcal{A}}[A,B,\lambda,\beta]$ and $I^q_{\Sigma}[\phi,\lambda,\beta]$ of the class of q-starlike functions and bi-univalent functions associated with the q-derivative operator and the Legendre polynomials. For each of these subclasses, we obtain certain coefficient estimates and Fekete–Szegö-type inequalities. The results obtained in this article are also shown to extend and generalize those in some earlier works. For motivation and incentive for further researches, the reader's attention is drawn toward some of the related recent developments in [12,16–19] dealing with the coefficient inequalities and coefficient estimates of various subclasses of analytic, univalent, and bi-univalent functions involving the second, third, and fourth Hankel determinants and the Fekete–Szegö functional.

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

which have the following normalized form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Also, let $\mathcal{S} \subset \mathcal{A}$ be the class of functions that are univalent in $\mathbb{U}$. Obviously, each function $f \in \mathcal{S}$ has an inverse f^{-1}, so that

$$f^{-1}\big(f(z)\big) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote this class using Σ. We remark that the study of the normalized class Σ of analytic and bi-univalent functions in $\mathbb{U}$ was revived in recent years by a pioneering article on the subject by Srivastava et al. [20], which has flooded the literature on the Geometric Function Theory of Complex Analysis with a large number of sequels to [20].

For a function $f \in \mathcal{A}$, given by (1), and a function $g \in \mathcal{A}$, given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathbb{U}),$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z) \qquad (z \in \mathbb{U}).$$

Let $\mathcal{P}$ be the class of Carathéodory functions h that are analytic in $\mathbb{U}$ and that satisfy

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$\Re\big(h(z)\big) > 0 \quad (z \in \mathbb{U}).$$

For two analytic functions f and g, we say that f is subordinate to g and it is written as $f \prec g$ or $f(z) \prec g(z)$, if there is a Schwarz function w such that $f(z) = g\big(w(z)\big)$. Further, if g is univalent in $\mathbb{U}$, then

$$f \prec g \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ as follows:

$$[\lambda]_q := \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\[2ex] 1 + \sum\limits_{j=1}^{n-1} q^j & (\lambda = n \in \mathbb{N}). \end{cases}$$

Especially, we note that $[0]_q = 0$.

Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! := \begin{cases} 1 & (n = 0) \\[2ex] \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Let $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Define $[r]_{q,n}$ by

$$[r]_{q,n} := \begin{cases} 1 & (n = 0) \\ \prod_{k=r}^{r+n-1} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Now, we recall here the q-derivative (or the q-difference) operator D_q $(0 < q < 1)$ of a function $f \in \mathcal{A}$ as follows:

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases}$$

where $f'(0)$ exists. Also, we write

$$(D_q^{(2)} f)(z) = \big(D_q(D_q f)\big)(z).$$

The Legendre polynomials $P_n(x)$ are the particular solutions to the Legendre differential equation:

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (n \in \mathbb{N}_0).$$

The Legendre polynomials $P_n(x)$ are defined by

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2 - 1)^n \qquad (n \in \mathbb{N}_0) \tag{3}$$

for arbitrary real or complex values of the variable x. The Legendre polynomials $P_n(x)$ are generated by (see, for details, [21])

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

where $(1 - 2xt + t^2)^{-\frac{1}{2}}$ is taken to be 1 when $t \to 0$. The first few Legendre polynomials are given by

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1) \ \text{ and } \ P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

The function $\phi(z)$ given by

$$\phi(z) = \frac{1 - z}{\sqrt{1 - 2z\cos\alpha + z^2}} \tag{4}$$

belongs to the class $\mathcal{P}$ for every real number α (see [22]). By using (3), it is easy to see that

$$\phi(z) = 1 + \sum_{n=1}^{\infty} [P_n(\cos\alpha) - P_{n-1}(\cos\alpha)] z^n = 1 + \sum_{n=1}^{\infty} l_n\, z^n,$$

where

$$l_n = P_n(\cos\alpha) - P_{n-1}(\cos\alpha).$$

We also note that

$$l_1 = \cos\alpha - 1 \quad \text{and} \quad l_2 = \frac{1}{2}(\cos\alpha - 1)(1 + 3\cos\alpha).$$

For more details, one can see the earlier work [23].

For $f \in \mathcal{A}$, the q-Ruscheweyh operator $R_{q,\lambda}$ is defined as follows (see [24]):

$$R_{q,\lambda}f(z) = f(z) * F_{q,\lambda+1}(z) \quad (z \in \mathbb{U};\ \lambda > -1),$$

where

$$F_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{q,n-1}}{[n-1]_q!}\, z^n.$$

Let $f \in \mathcal{A}$. The q-integral operator $R_{q,\lambda}^{-1}$ is defined by (see [5] and [25])

$$R_{q,\lambda}^{-1}(z) * R_{q,\lambda}(z) = z\big(D_q f(z)\big).$$

Further, we have

$$R_{q,\lambda}^{-1}(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} z^n,$$

where

$$\psi_{n-1} = \frac{[n]_q!}{[\lambda+1]_{q,n-1}} \quad (n \geqq 2). \tag{5}$$

When $q \to 1-1$, the q-integral operator $R_{q,\lambda}^{-1}$ reduces to an integral operator studied by Noor [26].

For $f \in \mathcal{A}$, the q-integral operator I_q^{λ} is defined by (see [5])

$$I_q^{\lambda} f(z) = f(z) * R_{q,\lambda}^{-1}(z) = z + \sum_{n=2}^{\infty} \psi_{n-1} a_n z^n, \tag{6}$$

where ψ_{n-1} is given by (5). Clearly, one can see that

$$I_q^{0} f(z) = z\big(D_q f(z)\big) \quad \text{and} \quad I_q^{1} f(z) = f(z).$$

Next, we will define the analytic function class $I_{\mathcal{A}}^{q}[A,B,\lambda,\beta]$ and the bi-univalent function class $I_{\Sigma}^{q}[\phi,\lambda,\beta]$.

Definition 1. *Let $\lambda > -1$, $-1 \leqq B < A \leqq 1$, and $0 \leqq \beta \leqq 1$. A function $f \in \mathcal{A}$ is said to be in the class $I_{\mathcal{A}}^{q}[A,B,\lambda,\beta]$ if*

$$\frac{z(D_q I_q^{\lambda} f)(z) + \beta z^2 (D_q^{(2)} I_q^{\lambda} f)(z)}{(1-\beta) I_q^{\lambda} f(z) + \beta z (D_q I_q^{\lambda} f)(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{7}$$

or equivalently,

$$\left| \frac{\frac{z(D_q I_q^{\lambda} f)(z) + \beta z^2 (D_q^{(2)} I_q^{\lambda} f)(z)}{(1-\beta) I_q^{\lambda} f(z) + \beta z (D_q I_q^{\lambda} f)(z)} - 1}{A - B\frac{z(D_q I_q^{\lambda} f)(z) + \beta z^2 (D_q^{(2)} I_q^{\lambda} f)(z)}{(1-\beta) I_q^{\lambda} f(z) + \beta z (D_q I_q^{\lambda} f)(z)}} \right| < 1. \tag{8}$$

Remark 1. *(i) For $\lambda = 1$ and $\beta = 0$, we have*

$$I_{\mathcal{A}}^{q}[A,B,1,0] = \mathcal{S}_q^{*}[A,B],$$

where the class $\mathcal{S}_q^{*}[A,B]$ was introduced by Srivastava et al. [27].

(ii) For $\lambda = 1$, $\beta = 0$, and $q \to 1^{-}$, we get

$$\lim_{q \to 1^{-}} I_{\mathcal{A}}^{q}[A,B,1,0] = \mathcal{S}^{*}[A,B],$$

where the class $\mathcal{S}^{*}[A,B]$ was considered by Janowski [28].

(iii) *For $A = 1 - 2\alpha$ $(0 \leqq \alpha < 1)$ and $B = -1$, the class $\mathcal{S}^*[A, B]$ reduces to the class $\mathcal{S}^*(\alpha)$, which was studied by Silverman [29].*

Definition 2. *Let $\lambda > -1$ and $0 \leqq \beta \leqq 1$. A function $f \in \Sigma$ is said to be in the class $I_{\Sigma}^{q}[\phi, \lambda, \beta]$ if*

$$\begin{cases} \frac{z(D_q I_q^{\lambda} f)(z)+\beta z^2 (D_q^{(2)} I_q^{\lambda} f)(z)}{(1-\beta) I_q^{\lambda} f(z)+\beta z (D_q I_q^{\lambda} f)(z)} \prec \phi(z) \\ \\ \frac{w(D_q I_q^{\lambda} g)(w)+\beta w^2 (D_q^{(2)} I_q^{\lambda} g)(w)}{(1-\beta) I_q^{\lambda} g(w)+\beta w (D_q I_q^{\lambda} g)(w)} \prec \phi(w), \end{cases} \tag{9}$$

where the functions g and ϕ are given by (2) and (4), respectively.

To derive our main results, we need the following lemmas.

Lemma 1 (see [30])**.** *Let $\varphi(z) = 1 + \omega_1 z + \omega_2 z^2 + \cdots \in \mathcal{P}$. Then,*

$$|\omega_2 - \nu\omega_1^2| \leqq \begin{cases} -4\nu + 2 & (\nu < 0) \\ \\ 2 & (0 \leqq \nu \leqq 1) \\ \\ 4\nu - 2 & (\nu > 1). \end{cases} \tag{10}$$

Lemma 2 (see [13])**.** *Let*

$$M(z) = 1 + \sum_{n=1}^{\infty} C_n z^n \prec H(z) = 1 + \sum_{n=1}^{\infty} d_n z^n.$$

If $H(z)$ is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is convex, then

$$|C_n| \leqq |d_1| \quad (n \in \mathbb{N}).$$

Lemma 3 (see [31])**.** *If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$, then*

$$|c_n| \leqq 2 \qquad (n \in \mathbb{N}).$$

2. Main Results

In this section, we derive certain coefficient estimates and the Fekete–Szegö-type inequalities for functions in the classes $I_{\mathcal{A}}^{q}[A, B, \lambda, \beta]$ and $I_{\Sigma}^{q}[\phi, \lambda, \beta]$, which are defined above (see Definition 1 and Definition 2). Many special cases and consequences of our main findings are pointed out.

Theorem 1. *Let a function $f \in I_{\mathcal{A}}^{q}[A, B, \lambda, \beta]$ be of the form given by (1). Then,*

$$|a_n| \leqq \frac{A - B}{\big(q(1-\beta) + \beta[n]_q\big)\psi_{n-1}} \cdot \prod_{j=1}^{n-1} \frac{\{q(1-\beta) + \beta[j]_q\}[j-1]_q + \{(1-\beta) + \beta[j]_q\}(A - B)}{\{q(1-\beta) + \beta[j]_q\}[j]_q} \quad (n \geqq 2), \tag{11}$$

where ψ_{n-1} is given by (5).

Proof. For $f \in I_{\mathcal{A}}^{q}[A, B, \lambda, \beta]$, we have

$$\nu(z) := \frac{z(D_q I_q^{\lambda} f)(z) + \beta z^2 (D_q^{(2)} I_q^{\lambda} f)(z)}{(1-\beta) I_q^{\lambda} f(z) + \beta z (D_q I_q^{\lambda} f)(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{12}$$

where

$$\frac{1+Az}{1+Bz} = 1 + \sum_{n=0}^{\infty}(A-B)(-B)^n z^{n+1} = 1 + (A-B)z - B(A-B)z^2 + \cdots .$$

Since $\nu(z) = 1 + \sum_{n=1}^{\infty} \nu_n z^n$, we get from Lemma 2 that

$$|\nu_n| \leqq A - B \quad (n \in \mathbb{N}). \tag{13}$$

From (12), we have

$$z(D_q I_q^\lambda f)(z) + \beta z^2 (D_q^{(2)} I_q^\lambda f)(z) = \nu(z)[(1-\beta) I_q^\lambda f(z) + \beta z\big(D_q I_q^\lambda f(z)\big)]$$

which shows that

$$\begin{aligned} z &+ \sum_{n=2}^{\infty} [n]_q [1 + \beta[n-1]_q] a_n \psi_{n-1} z^n \\ &= \left(1 + \sum_{n=1}^{\infty} \nu_n z^n\right)\left(z + \sum_{n=2}^{\infty} [1 + \beta([n]_q - 1)] a_n \psi_{n-1} z^n\right). \end{aligned} \tag{14}$$

Comparing the coefficients of z^n on both sides of the equation (14), we get

$$[n-1]_q [q(1-\beta) + \beta[n]_q] a_n \psi_{n-1} = \sum_{l=1}^{n-1} [1 + \beta([l]_q - 1)] a_l \psi_{l-1} \nu_{n-l},$$

where $a_1 = 1$, $\nu_1 = 1$ and $\psi_0 = 1$. The above equation gives

$$|a_n| \leqq \frac{A-B}{[n-1]_q [q(1-\beta) + \beta[n]_q] \psi_{n-1}} \sum_{l=1}^{n-1} [1 + \beta([l]_q - 1)] |a_l| \psi_{l-1}.$$

Thus, we get

$$\begin{aligned} |a_2| &\leqq \frac{A-B}{(q+\beta)\psi_1}; \\ |a_3| &\leqq \frac{A-B}{(q(1-\beta) + \beta[3]_q)\psi_2} \cdot \frac{(q+\beta) + (1+\beta q)(A-B)}{[2]_q (q+\beta)}; \\ |a_4| &\leqq \frac{A-B}{(q(1-\beta) + \beta[4]_q)\psi_3} \cdot \left(\frac{(q+\beta) + (1+\beta q)(A-B)}{[2]_q (q+\beta)}\right) \\ &\qquad \cdot \left(\frac{[2]_q \{q(1-\beta) + \beta[3]_q\} + \big((1-\beta) + \beta[3]_q\big)(A-B)}{[3]_q}\right); \\ &\cdots\cdots \\ |a_n| &\leqq \frac{A-B}{(q(1-\beta) + \beta[n]_q)\psi_{n-1}} \\ &\qquad \cdot \prod_{j=1}^{n-1} \frac{\{q(1-\beta) + \beta[j]_q\}[j-1]_q + \{(1-\beta) + \beta[j]_q\}(A-B)}{\{q(1-\beta) + \beta[j]_q\}[j]_q}. \end{aligned}$$

This proves Theorem 1. $\square$

For $\lambda = 1$ and $\beta = 0$ in Theorem 1, we obtain a result of the class $\mathcal{S}_q^*[A,B]$, which was considered by Srivastava et al. [27].

Corollary 1. *Let a function $f \in \mathcal{S}_q^*[A,B]$ be of the form given by* (1)*. Then,*

$$|a_n| \leqq \frac{1}{q}\prod_{j=1}^{n-1}\frac{q[j-1]_q+(A-B)}{q[j]_q} \qquad (n \geqq 2).$$

Theorem 2. *Let a function $f \in I_\Sigma^q[\phi,\lambda,\beta]$ be given by* (1)*. Then,*

$$|a_2| \leqq \min\left\{\begin{array}{l}\dfrac{|\cos\alpha-1|}{(q+\beta)\psi_1}, \\ \dfrac{\sqrt{2}|\cos\alpha-1|}{\sqrt{|2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2-(q+\beta)[2(1+\beta q)(\cos\alpha-1)+(q+\beta)(3\cos\alpha-1)]\psi_1^2|}}\end{array}\right. \tag{15}$$

and

$$|a_3| \leqq \min\left\{\begin{array}{l}\dfrac{|\cos\alpha-1|^2}{(q+\beta)^2\psi_1^2}+\dfrac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2}, \\ \frac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2}+\frac{2(\cos\alpha-1)^2}{|\{2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2-(q+\beta)[2(1+\beta q)(\cos\alpha-1)+(q+\beta)(3\cos\alpha-1)]\psi_1^2\}|}.\end{array}\right. \tag{16}$$

Proof. From (9), we know that there are two Schwarz functions $u(z)$ and $\nu(w)$, such that

$$\frac{z(D_qI_q^\lambda f)(z)+\beta z^2(D_q^{(2)}I_q^\lambda f)(z)}{(1-\beta)I_q^\lambda f(z)+\beta z\big(D_qI_q^\lambda f(z)\big)} = \phi\big(u(z)\big) \tag{17}$$

and

$$\frac{w(D_qI_q^\lambda g)(w)+\beta w^2(D_q^{(2)}I_q^\lambda g)(w)}{(1-\beta)I_q^\lambda g(w)+\beta w\big(D_qI_q^\lambda g(w)\big)} = \phi\big(\nu(w)\big). \tag{18}$$

Now we define the functions $s(z)$ and $t(w)$ by

$$s(z) = \frac{1+u(z)}{1-u(z)} = 1+s_1z+s_2z^2+\cdots \in \mathcal{P}$$

and

$$t(w) = \frac{1+\nu(w)}{1-\nu(w)} = 1+t_1z+t_2z^2+\cdots \in \mathcal{P}.$$

Since

$$\phi(z) = 1+\sum_{n=1}^{\infty}[P_n(\cos\alpha)-P_{n-1}(\cos\alpha)]z^n = 1+\sum_{n=1}^{\infty}l_nz^n,$$

we get

$$\begin{aligned}\phi\big(u(z)\big) &= 1+\tfrac{1}{2}l_1s_1z+\left[\tfrac{1}{2}l_1\left(s_2-\tfrac{s_1^2}{2}\right)+\tfrac{1}{4}l_2s_1^2\right]z^2+\cdots, \\ \phi\big(\nu(w)\big) &= 1+\tfrac{1}{2}l_1t_1w+\left[\tfrac{1}{2}l_1\left(t_2-\tfrac{t_1^2}{2}\right)+\tfrac{1}{4}l_2t_1^2\right]w^2+\cdots.\end{aligned} \tag{19}$$

Using the Taylor series formula, we have

$$\begin{aligned}\frac{z(D_qI_q^\lambda f)(z)+\beta z^2(D_q^{(2)}I_q^\lambda f)(z)}{(1-\beta)I_q^\lambda f(z)+\beta z\big(D_qI_q^\lambda f(z)\big)} &= 1+(q+\beta)\psi_1a_2z \\ &\quad +\{[q(q+1)+(q^2+1)[2]_q\beta]\psi_2a_3-(1+\beta q)(q+\beta)\psi_1^2a_2^2\}z^2+\cdots\end{aligned}$$

and

$$\frac{w(D_q I_q^\lambda g)(w) + \beta w^2 (D_q^{(2)} I_q^\lambda g)(w)}{(1-\beta) I_q^\lambda g(w) + \beta w \big(D_q I_q^\lambda g(w)\big)} = 1 - (q+\beta)\psi_1 a_2 w$$
$$+ \{[q(q+1) + (q^2+1)[2]_q \beta]\psi_2(2a_2^2 - a_3) - (1+\beta q)(q+\beta)\psi_1^2 a_2^2\} w^2 + \cdots .$$

Comparing the left-side and right-side coefficients of (17) and (18), we obtain

$$(q+\beta)\psi_1 a_2 = \frac{1}{2} l_1 s_1, \tag{20}$$

$$[2]_q[q + \beta(q^2+1)]\psi_2 a_3 - (1+\beta q)(q+\beta)\psi_1^2 a_2^2 = \frac{1}{2} l_1 \left(s_2 - \frac{s_1^2}{2}\right) + \frac{1}{4} l_2 s_1^2, \tag{21}$$

$$-(q+\beta)\psi_1 a_2 = \frac{1}{2} l_1 t_1 \tag{22}$$

and

$$[2]_q[q + \beta(q^2+1)]\psi_2(2a_2^2 - a_3) - (1+\beta q)(q+\beta)\psi_1^2 a_2^2 = \frac{1}{2} l_1 \left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4} l_2 t_1^2. \tag{23}$$

From (20) and (22), we have

$$a_2 = \frac{l_1 s_1}{2(q+\beta)\psi_1} = \frac{-l_1 t_1}{2(q+\beta)\psi_1}. \tag{24}$$

Thus, we find that

$$s_1 = -t_1 \tag{25}$$

and

$$8(q+\beta)^2 \psi_1^2 a_2^2 = l_1^2 (s_1^2 + t_1^2). \tag{26}$$

Using Lemma 3, we find from (24) that

$$|a_2| \leqq \frac{|\cos\alpha - 1|}{(q+\beta)\psi_1}. \tag{27}$$

Now from (21), (23), (24), and (25), we have

$$4\{[2]_q[q + \beta(q^2+1)] l_1^2 \psi_2 - (q+\beta)[(1+\beta q) l_1^2 + (q+\beta)(l_2 - l_1)]\psi_1^2\} a_2^2 = l_1^3 (s_2 + t_2).$$

Since

$$l_1 = \cos\alpha - 1 \quad \textit{and} \quad l_2 = \frac{1}{2}(\cos\alpha - 1)(1 + 3\cos\alpha),$$

we obtain

$$\begin{aligned} a_2^2 &= \frac{l_1^3 (s_2 + t_2)}{4\{[2]_q[q + \beta(q^2+1)] l_1^2 \psi_2 - (q+\beta)[(1+\beta q) l_1^2 + (q+\beta)(l_2 - l_1)]\psi_1^2\}} \\ &= \frac{(\cos\alpha - 1)^2 (s_2 + t_2)}{2\{2[2]_q[q + \beta(q^2+1)](\cos\alpha - 1)\psi_2 - (q+\beta)[2(1+\beta q)(\cos\alpha - 1) + (q+\beta)(3\cos\alpha - 1)]\psi_1^2\}}. \end{aligned} \tag{28}$$

Applying Lemma 3 to the coefficients s_2 and t_2, we have

$$|a_2| \leqq \frac{\sqrt{2}|\cos\alpha - 1|}{\sqrt{|2[2]_q[q + \beta(q^2+1)](\cos\alpha - 1)\psi_2 - (q+\beta)[2(1+\beta q)(\cos\alpha - 1) + (q+\beta)(3\cos\alpha - 1)]\psi_1^2|}}.$$

By subtracting (23) from (21), we have

$$2[2]_q[q+\beta(q^2+1)](a_3-a_2^2)\psi_2 = \frac{1}{2}l_1(s_2-t_2)+\frac{1}{4}(l_2-l_1)(s_1^2-t_1^2). \tag{29}$$

From (24), (25), and (29), we obtain

$$a_3 = a_2^2+\frac{l_1(s_2-t_2)}{4[2]_q[q+\beta(q^2+1)]\psi_2}. \tag{30}$$

Now taking the modulus of (30) and using Lemma 3, we get

$$|a_3| \leqq |a_2|^2+\frac{l_1}{[2]_q[q+\beta(q^2+1)]\psi_2}. \tag{31}$$

Further, by using (27) and (31), we find

$$\begin{aligned}|a_3| &\leqq \frac{l_1^2}{(q+\beta)^2\psi_1^2}+\frac{|l_1|}{[2]_q[q+\beta(q^2+1)]\psi_2}\\ &= \frac{|\cos\alpha-1|}{(q+\beta)^2\psi_1^2}+\frac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2}.\end{aligned}$$

Also, using (26) and (31), we derive

$$\begin{aligned}|a_3| &\leqq |a_2^2|+\frac{l_1}{[2]_q[q+\beta(q^2+1)]\psi_2}\\ &= \frac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2}\\ &\quad+\frac{2(\cos\alpha-1)^2}{|\{2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2-(q+\beta)[2(1+\beta q)(\cos\alpha-1)+(q+\beta)(3\cos\alpha-1)]\psi_1^2\}|}.\end{aligned}$$

This completes the proof of Theorem 2. □

Theorem 3. *Let a function $f \in I_{\mathcal{A}}^q[A,B,\lambda,\beta]$ be of the form given by (1). Then,*

$$|a_3-\mu a_2^2| \leqq \begin{cases} \dfrac{A-B}{(1+q)[q+(q^2+1)\beta]\psi_2}\Lambda(q) & (\mu<\sigma_1)\\[2ex] \dfrac{A-B}{(1+q)[q+(q^2+1)\beta]\psi_2} & (\sigma_1 \leqq \mu \leqq \sigma_2)\\[2ex] \frac{B-A}{(1+q)[q+(q^2+1)\beta]\psi_2}\Lambda(q) & (\mu>\sigma_2),\end{cases}$$

where μ is real and

$$\Lambda(q) = \frac{(q+\beta)[(1+\beta q)(A-B)-(q+\beta)B]\psi_1^2-\mu(1+q)[q+(q^2+1)\beta](A-B)\psi_2}{(q+\beta)^2\psi_1^2},$$

$$\sigma_1 = \frac{(q+\beta)[(1+\beta q)(A-B)-(q+\beta)(B+1)]\psi_1^2}{(1+q)[q+(q^2+1)\beta](A-B)\psi_2}$$

and

$$\sigma_2 = \frac{(q+\beta)[(1+\beta q)(A-B)-(q+\beta)(B-1)]\psi_1^2}{(1+q)[q+(q^2+1)\beta](A-B)\psi_2}.$$

Proof. Let $f \in I_{\mathcal{A}}^{q}[A, B, \lambda, \beta]$. Using the Taylor series formula, we have

$$\frac{z(D_q I_q^{\lambda} f)(z) + \beta z^2 (D_q^{(2)} I_q^{\lambda} f)(z)}{(1-\beta) I_q^{\lambda} f(z) + \beta z \big(D_q I_q^{\lambda} f(z)\big)} = 1 + (q+\beta)\psi_1 a_2 z \\ + \{[q(q+1) + (q^2+1)[2]_q \beta]\psi_2 a_3 - (1+\beta q)(q+\beta)\psi_1^2 a_2^2\} z^2 + \cdots . \tag{32}$$

From (7), we know that there exists a Schwarz function h such that

$$\frac{z(D_q I_q^{\lambda} f)(z) + \beta z^2 (D_q^{(2)} I_q^{\lambda} f)(z)}{(1-\beta) I_q^{\lambda} f(z) + \beta z \big(D_q I_q^{\lambda} f(z)\big)} = \frac{1 + Ah(z)}{1 + Bh(z)}.$$

We now define a function $w \in \mathcal{P}$ by

$$w(z) = \frac{1 + h(z)}{1 - h(z)} = 1 + w_1 z + w_2 z^2 + \cdots .$$

This implies that

$$h(z) = \frac{w(z) - 1}{w(z) + 1} = 1 + \frac{1}{2} w_1 z + \left(\frac{1}{2} w_2 - \frac{1}{4} w_1^2\right) z^2 + \cdots .$$

Also, we have

$$\frac{1 + Ah(z)}{1 + Bh(z)} = 1 + \frac{1}{2}(A-B) w_1 z + \left[\frac{1}{2}(A-B) w_2 - \frac{1}{4}(B+1)(A-B) w_1^2\right] z^2 + \cdots . \tag{33}$$

Therefore, we obtain

$$\begin{aligned} a_2 &= \left(\tfrac{A-B}{2(q+\beta)\psi_1}\right) w_1, \\ a_3 &= \left(\tfrac{A-B}{2(1+q)[q+(q^2+1)\beta]\psi_2}\right) \left\{ w_2 - \tfrac{1}{2}\left[(B+1) - \left(\tfrac{1+\beta q}{q+\beta}\right)(A-B)\right] w_1^2 \right\}. \end{aligned} \tag{34}$$

Now, we can find that

$$\begin{aligned} |a_3 - \mu a_2^2| &= \frac{A-B}{2(1+q)[q+(q^2+1)\beta]\psi_2} \Bigg| \left\{ w_2 - \frac{1}{2}\left[(B+1) - \left(\frac{1+\beta q}{q+\beta}\right)(A-B)\right] w_1^2 \right\} \\ &\quad - \mu \frac{(A-B)^2}{4(q+\beta)^2 \psi_1^2} w_1^2 \Bigg| \\ &= \frac{A-B}{2(1+q)[q+(q^2+1)\beta]\psi_2} \Bigg| \Bigg\{ w_2 - \frac{1}{2}[(B+1) \\ &\quad - \left(\frac{(1+\beta q)(q+\beta)\psi_1^2 + \mu(1+q)[q+(q^2+1)\beta]\psi_2}{(q+\beta)^2 \psi_1^2}\right)(A-B)\Bigg] w_1^2 \Bigg\} \Bigg| \\ &= \frac{A-B}{2(1+q)[q+(q^2+1)\beta]\psi_2} |\{ w_2 - k_1(q) w_1^2 \}|, \end{aligned} \tag{35}$$

where

$$k_1(q) = \frac{(q+\beta)[(q+\beta)(B+1) - (1+\beta q)(A-B)]\psi_1^2 + \mu(1+q)[q+(q^2+1)\beta](A-B)\psi_2}{2(q+\beta)^2 \psi_1^2}.$$

Applying Lemma 1 in (35), we get the desired results. The proof of Theorem 3 is completed. □

For $\lambda = 1, \beta = 0$, and $q \to 1^-$, we get a result of the class $\mathcal{S}^*[A, B]$ that was considered by Janowski [28].

Corollary 2. *Let a function $f \in \mathcal{S}^*[A,B]$ be of the form given by (1). Then,*

$$|a_3-\mu a_2^2| \leqq \begin{cases} \left(\frac{A-B}{2}\right)[(A-2B)-\mu(A-B)] & \left(\mu < \frac{A-2B-1}{2(A-B)}\right) \\ \frac{A-B}{2} & \left(\frac{A-2B-1}{2(A-B)} \leqq \mu \leqq \frac{A-2B+1}{2(A-B)}\right) \\ \left(\frac{B-A}{2}\right)[(A-2B)-\mu(A-B)] & \left(\mu > \frac{A-2B+1}{2(A-B)}\right). \end{cases}$$

Theorem 4. *Let a function $f \in I^q_\Sigma[\phi,\lambda,\beta]$ be of the form given by (1). Then,*

$$|a_3-\mu a_2^2| \leqq \begin{cases} \frac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2} & \left(0 \leqq |h(\mu)| \leqq \frac{|\cos\alpha-1|}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right) \\ 4|h(\mu)| & \left(|h(\mu)| \geqq \frac{|\cos\alpha-1|}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right), \end{cases}$$

where μ is real and

$$h(\mu) = \frac{(1-\mu)(\cos\alpha-1)^2}{2\{2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2-(q+\beta)[2(1+\beta q)(\cos\alpha-1)+(q+\beta)(3\cos\alpha-1)]\psi_1^2}. \tag{36}$$

Proof. From (30), we have

$$a_3-\mu a_2^2 = (1-\mu)a_2^2 + \frac{l_1(s_2-t_2)}{4[2]_q[q+\beta(q^2+1)]\psi_2}. \tag{37}$$

Using (28) and (37), we get

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{(\cos\alpha-1)(s_2-t_2)}{4[2]_q[q+\beta(q^2+1)]\psi_2} \\ &+ \frac{(1-\mu)(\cos\alpha-1)^2(s_2+t_2)}{2\{2[2]_q[q+\beta(q^2+1)](\cos\alpha-1)\psi_2-(q+\beta)[2(1+\beta q)(\cos\alpha-1)+(q+\beta)(3\cos\alpha-1)]\psi_1^2\}} \\ &= \left(h(\mu)+\frac{\cos\alpha-1}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right)s_2 + \left(h(\mu)-\frac{\cos\alpha-1}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right)t_2, \end{aligned} \tag{38}$$

where $h(\mu)$ is given by (36).

Taking the modulus of each side in (38), we get

$$|a_3-\mu a_2^2| \leqq \begin{cases} \frac{|\cos\alpha-1|}{[2]_q[q+\beta(q^2+1)]\psi_2} & \left(0 \leqq |h(\mu)| \leqq \frac{|\cos\alpha-1|}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right) \\ 4|h(\mu)| & \left(|h(\mu)| \geqq \frac{|\cos\alpha-1|}{4[2]_q[q+\beta(q^2+1)]\psi_2}\right). \end{cases}$$

This proves Theorem 4. □

3. Conclusions

In our present investigation, we have used the q-derivative (or the q-difference) operator D_q, as well as the Legendre polynomials $P_n(x)$ to introduce and study two new subclasses of the class of q-starlike functions and the class of analytic and bi-univalent functions. For each of these subclasses, we have derived a number of coefficient estimates and Fekete–Szegö-type inequalities. The results derived in this article are also shown to extend and generalize those in some earlier works. For motivation and incentive for further research, the reader's attention is drawn toward some of the related recent developments in [16–19] dealing with the coefficient inequalities and coefficient estimates of various subclasses of analytic, univalent, and bi-univalent functions involving the second, third, and fourth Hankel determinants, and the Fekete–Szegö functional.

In concluding this article, we choose to discourage the current trend of some amateurish-type publications in which there are falsely-claimed "generalizations" of known q-theory and known q-results by forcing-in an obviously superfluous (or redundant) parameter p. In this connection, the reader should refer to [32] (p. 340) and [33] (pp. 1511–1512) for a detailed exposition and demonstration, where it is stated clearly that the current trend of trivially and inconsequentially translating known q-results into the corresponding (p,q)-results leads to no more than a straightforward and shallow publication involving an additional forced-in parameter p that is obviously redundant (or superfluous).

Author Contributions: All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (Grant No. 11571299).

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their sincere thanks to the referees for their careful reading and suggestions, which has helped us to improve the presentation of this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Jackson, F.H. On q-functions and a certain difference operator. *Trans. Roy. Soc. Edinb.* **1908**, *46*, 64–72
2. Jackson, F.H. On q-definite integrals, Quart. *J. Pure Appl. Math.* **1910**, *41*, 193–203
3. Jackson, F.H. q-difference equations. *Am. J. Math.* **1910**, *32*, 305–314
4. Jackson, F.H. The application of basic numbers to Bessel's and Legendre's functions. *Proc. Lond. Math. Soc.* **1905**, *3*, 1–23
5. Srivastava, H.M.; Ahmad, Q.Z.; Khan, N.; Khan, N.; Khan, B. Hankel and Toeplitz determinants for a subclass of q-starlike funtions associated with a general conic domain. *Mathematics* **2019**, *7*, 181–196
6. Ahuja, O.P.; Cetinkaya, A.; Polatoğlu, Y. Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of q-convex and q-close-to-convex functions. *J. Comput. Anal. Appl.* **2019**, *26*, 639–649
7. Aouf, M.K.; Mostafa, A.O.; Al-Quhali, F.Y. Properties for class of bi-uniformly univalent functions defined by Sălăgean type q-difference operator. Internat. *J. Open Probl. Complex Anal.* **2019**, *11*, 1–16
8. Khan, N.; Ahmad, Q.Z.; Khalid, T. Results on spirallike p-valent functions. AIMS Math. **2017**, *3*, 12–20
9. Khan, S.; Hussain, S.; Darus, M. Inclusion relations of q-Bessel functions associated with generalized conic domain. *AIMS Math.* **2021**, *6*, 3624–3640
10. Pommerenke, C. *Univalent Functions*; Vandenhoeck & Ruprecht: Göttingen, Germany, 1975.
11. Rehman, M.S.U.; Ahmad, Q.Z.; Srivastava, H.M.; Khan, N.; Darus, M.; Khan, B. Applications of higher-order q-derivatives to the subclass of q-starlike functions associated with the Janowski functions. *AIMS Math.* **2021**, *6*, 1110–1125
12. Srivastava, H.M.; Tahir, M.; Khan, B.; Ahmad, Q.Z.; Khan, N. Some general families of q-starlike functions associated with the Janowski functions. *Filomat* **2019**, *33*, 2613–2626.
13. Rogosinski, W. On the coefficients of subordinate functions. *Proc. Lond. Math. Soc.* **1943**, *48*, 48–82
14. Srivastava, H.M.; Tahir, M.; Khan, B.; Ahmad, Q.Z.; Khan, N. Some general classes of q-starlike functions associated with the Janowski functions. *Symmetry* **2019**, *11*, 292.
15. Güney, H. Ö.; Oros, G.I.; Owa S. An application of Sălăgean operator concerning starlike functions. *Axioms* **2022**, *11*, 50.
16. Srivastava, H.M.; Kaur, G.; Singh, G. Estimates of the fourth Hankel determinant for a class of analytic functions with bounded turnings involving cardioid domains. *J. Nonlinear Convex Anal.* **2021**, *22*, 511–526
17. Srivastava, H.M.; Shaba, T.G.; Murugusundaramoorthy, G.; Wanas, A.K.; Oros, G.I. The Fekete-Szegö functional and the Hankel determinant for a certain class of analytic functions involving the Hohlov operator. *AIMS Math.* **2023**, *8*, 340–360
18. Srivastava, H.M.; Murugusundaramoorthy, G.; Bulboacă, T. The second Hankel determinant for subclasses of bi-univalent functions associated with a nephroid domain. *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM)* **2022**, *116*, 145.
19. Shi, L.; Srivastava, H.M.; Rafiq, A.; Arif, M.; Ihsan, M. Results on Hankel determinants for the inverse of certain analytic functions subordinated to the exponential function. *Mathematics* **2022**, *10*, 3429.
20. Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. Appl. Math. Lett. **2010**, *23*, 1188–1192

21. Srivastava, H.M. Some families of generating functions associated with orthogonal polynomials and other higher transcendental functions, *Mathematics* **2022**, *10*, 3730.
22. Goodman, A.W. *Univalent Functions*, Mariner Publishing Company Incorporated: Tampa, FL, USA, 1983.
23. Murugusundaramoorthy, G.; Cotîrlă, L.-I. Bi-univalent functions of complex order defined by Hohlov operator associated with Legendre polynomial. *AIMS Math.* **2022**, *7*, 8733–8750
24. Kanas, S.; Răducanu, D. Some class of analytic functions related to conic domains. *Math. Slov.* **2014**, *64*, 1183–1196
25. Srivastava, H.M.; Wanas, A.K.; Srivastava, R. Applications of the q-Srivastava-Attiya operator involving a certain family of bi-univalent functions associated with the Horadam polynomials. *Symmetry* **2021**, *13*, 1230.
26. Noor, K.I. On new classes of integral operators. *J. Natur. Geom.* **1999**, *16*, 71–80
27. Srivastava, H.M.; Khan, B.; Khan, N.; Ahmad, Q.Z. Coefficient inequalities for q-starlike functions associated with the Janowski functions. *Hokkaido Math. J.* **2019**, *48*, 407–425
28. Janowski, W. Some extremal problems for certain families of analytic functions. *Ann. Polon. Math.* **1973**, *28*, 297–326
29. Silverman, H. Univalent functions with negative coefficients. *Proc. Amer. Math. Soc.* **1975**, *51*, 109–116
30. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceeding of the International Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; pp. 157–169.
31. Duren, P.L. *Univalent Functions (Grundlehren der Mathematischen Wissenschaften 259)*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
32. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344
33. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.