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Boundedness of Riesz Potential Operator on Grand Herz-Morrey Spaces

Babar Sultan [1], Fatima Azmi [2], Mehvish Sultan [3], Mazhar Mehmood [4] and Nabil Mlaiki [2,*]

1 Department of Mathematics, Quaid-I-Azam University, Islamabad 45320, Pakistan
2 Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
3 Department of Mathematics, Capital University of Science and Technology, Islamabad 44000, Pakistan
4 Department of Mathematics, Government Post Graduate College, Haripur 22620, KPK, Pakistan
* Correspondence: nmlaiki@psu.edu.sa or nmlaiki2012@gmail.com

Abstract: In this paper, we introduce grand Herz–Morrey spaces with variable exponent and prove the boundedness of Riesz potential operators in these spaces.

Keywords: Riesz potential operator; grand Herz–Morrey spaces; grand Lebesgue spaces; grand Herz spaces

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1. Introduction

In the last two decades, under the influence of some applications revealed in [1], there was a vast boom of research of the so-called variable exponent spaces, and the operator in them. For the time being, the theory of such variable exponent Lebesgue, Orlicz, Lorentz and Sobolev function spaces is widely developed, and we refer to the books [2,3] and the surveying papers [4–7]. Herz spaces with variable exponents have been recently introduced in [8–10]. In [11], variable parameters were used to define continual Herz spaces, and proved the boundedness of sublinear operators in these spaces. The boundedness of other operators such as Riesz potential operator and the Marcinkiewicz integrals was proved in [12,13].

The concept of Morrey spaces $L^{p,\lambda}$ was introduced by C. Morrey in 1938 (see [14]) in order to study regularity questions that appear in the calculus of variations. They describe local regularity more precisely than Lebesgue spaces and are widely used not just in harmonic analysis, but also in PDEs. Meskhi introduced the idea of grand Morrey spaces $L^{r),\theta,\lambda}$ and derived the boundedness of a class of integral operators (Hardy–Littlewood maximal functions, Calderón–Zygmund singular integrals and potentials) in these spaces, see ([15]). Moreover, Izuki [16] defined the Herz–Morrey spaces with a variable exponent and investigated the boundedness of fractional integrals on these spaces.

In [17], the idea of grand variable Herz spaces $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$ was introduced and proved the boundedness of sublinear operators $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(\mathbb{R}^n)$. Motivated by the concept, in this article, we introduce the concept of grand Herz–Morrey spaces, and prove the boundedness of the Riesz potential operator on grand Herz–Morrey spaces with variable exponents. There are four sections in this article; the first section is dedicated to the introduction, the second section contains some basic definitions and lemmas, we introduce the concept of grand Herz–Morrey spaces in part three, and the boundedness of the Riesz potential operator on grand variable Herz–Morrey spaces is proved in the last section.

2. Preliminaries

For this section, we refer to [2,3,9,10,18].

2.1. Lebesgue Space with Variable Exponent

Assume that $G \subseteq \mathbb{R}^n$ is an open set and $p(\cdot) : G \to [1, \infty)$ is a real-valued measurable function. Let the following condition hold:

$$1 \leq p_-(G) \leq p_+(G) < \infty, \tag{1}$$

where

(i) $p_- := \operatorname{ess}\inf_{g \in G} p(g)$

(ii) $p_+ := \operatorname{ess}\sup_{g \in G} p(g)$.

Lebesgue space $L^{p(\cdot)}(G)$ is the space of measurable functions f_1 on G such that,

$$I_{L^{p(\cdot)}}(f_1) = \int_G |f_1(g)|^{p(g)} dg < \infty,$$

norm is defined as,

$$\|f_1\|_{L^{p(\cdot)}(G)} = \operatorname{ess}\inf\left\{\gamma > 0 : I_{L^{p(\cdot)}}\left(\frac{f_1}{\gamma}\right) \leq 1\right\},$$

this is the Banach function space, $p'(g) = \frac{p(g)}{p(g)-1}$ denotes the conjugate exponent of $p(g)$.

Next, we will define the space $L^{p(\cdot)}_{\text{loc}}(G)$ as,

$$L^{p(\cdot)}_{\text{loc}}(G) := \left\{\kappa : \kappa \in L^{p(\cdot)}(K) \textit{ for all compact subsets } K \subset G\right\}.$$

Now to define the log-condition,

$$|\eta(z_1) - \eta(z_2)| \leq \frac{C}{-\ln|z_1 - z_2|}, \quad |z_1 - z_2| \leq \frac{1}{2}, \quad z_1, z_2 \in G, \tag{2}$$

where $C = C(\eta) > 0$ is not dependent on z_1, z_2.

For the decay condition: let $\eta_\infty \in (1, \infty)$, such that

$$|\eta(z_1) - \eta_\infty| \leq \frac{C}{\ln(e + |z_1|)}, \tag{3}$$

$$|\eta(z_1) - \eta_0| \leq \frac{C}{\ln|z_1|}, |z_1| \leq \frac{1}{2}, \tag{4}$$

inequality (4) holds for $\eta_0 \in (1, \infty)$ in case of homogenous Herz spaces. We adopted the following notations in this paper:

(i) The Hardy–Littlewood maximal operator M for $f \in L^1_{\text{loc}}(G)$ is defined as

$$Mf(g) := \sup_{t>0} t^{-n} \int_{D(g,r)} |f(g)| dg \quad (g \in G),$$

where $D(g, t) := \{y \in G : |g - y| < t\}$.

(ii) The set $\mathcal{P}(G)$ is the collection of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

(iii) $\mathcal{P}^{\log} = \mathcal{P}^{\log}(G)$ is the class of functions $p \in \mathcal{P}(G)$ satisfying (1) and (2).

(iv) When G is unbounded, $\mathcal{P}_\infty(G)$ and $\mathcal{P}_{0,\infty}(G)$ are the subsets of $\mathcal{P}(G)$ and its values lies in $[1, \infty)$ satisfying (3) and (4), respectively.

(v) In the case G is bounded, $\mathcal{P}_{\infty}(G)$ and $\mathcal{P}_{0,\infty}(G)$ are the subsets of $\mathcal{P}(G)$.
(vi) In the case S is unbounded, $\mathcal{P}_{\infty}(S)$ are the subsets of exponents in $L^{\infty}(S)$ and its values lies in $[1,\infty]$, which satisfy both conditions (2) and (3), respectively, and $\mathcal{P}_{\infty}^{\log}(S)$ is the set of exponents $p \in \mathcal{P}_{\infty}(S)$, satisfying condition (1).

C is a constant that is independent of the main parameters involved, and its value varies from line to line.

Lemma 1 ([11]). *Let $D > 1$ and $\eta \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then,*

$$\frac{1}{k_0} t^{\frac{n}{\eta(0)}} \leq \|\chi_{R_{t,D_t}}\|_{\eta(\cdot)} \leq k_0 t^{\frac{n}{\eta(0)}}, \text{ for } 0 < t \leq 1 \tag{5}$$

and

$$\frac{1}{k_\infty} t^{\frac{n}{\eta_\infty}} \leq \|\chi_{R_{t,D_t}}\|_{\eta(\cdot)} \leq k_\infty t^{\frac{n}{\eta_\infty}}, \text{ for } t \geq 1, \tag{6}$$

respectively, where $k_0 \geq 1$ and $k_\infty \geq 1$ depend on D and do not depend on t.

Lemma 2 (Generalized Hölder's inequality [2]). *Assume that G is a measurable subset of $\mathbb{R}^n$, and $1 \leq p_-(G) \leq p_+(G) \leq \infty$. Then,*

$$\|fg\|_{L^{r(\cdot)}(G)} \leq C\|f\|_{L^{p(\cdot)}(G)}\|g\|_{L^{q(\cdot)}(G)}$$

holds, where $f \in L^{p(\cdot)}(G)$, $g \in L^{q(\cdot)}(G)$ and $\frac{1}{r(z)} = \frac{1}{p(z)} + \frac{1}{q(z)}$ for every $z \in G$.

2.2. Herz Spaces with Variable Exponent

We adopted the following notations in this subsection:

(a) $\chi_k = \chi_{R_k}$;
(b) $R_k = D_k \setminus D_{k-1}$;
(c) $D_k = D(0,2^k) = \{x \in \mathbb{R}^n : |x| < 2^k\}$ for all $k \in \mathbb{Z}$.

Definition 1. *Let $r, s \in [1,\infty)$, $\alpha \in \mathbb{R}$, the classical versions of Herz spaces, commonly known as non-homogenous and homogenous Herz spaces, can be defined by the norms,*

$$\|g\|_{K^{\alpha}_{r,s}(\mathbb{R}^n)} := \|g\|_{L^r(D(0,1))} + \left\{ \sum_{k\in\mathbb{N}} 2^{k\alpha s} \left(\int\limits_{R_{2^{k-1},2^k}} |g(z)|^r dz \right)^{\frac{s}{r}} \right\}^{\frac{1}{s}}, \tag{7}$$

$$\|g\|_{\dot{K}^{\alpha}_{r,s}(\mathbb{R}^n)} := \left\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha s} \left(\int\limits_{R_{2^{k-1},2^k}} |g(z)|^r dz \right)^{\frac{s}{r}} \right\}^{\frac{1}{s}}, \tag{8}$$

respectively, where $R_{t,\tau}$ stands for the annulus $R_{t,\tau} := D(0,\tau) \backslash D(0,t)$.

Definition 2. *Let $r \in [1,\infty)$, $\alpha \in \mathbb{R}$ and $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogenous Herz space $\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)$ is defined by*

$$\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{s(\cdot)}{}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{9}$$

where

$$\|g\|_{\dot{K}^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\alpha} g\chi_k\|^r_{L^{s(\cdot)}} \right)^{\frac{1}{r}}.$$

Definition 3. *Let $r \in [1,\infty)$, $\alpha \in \mathbb{R}$ and $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The non-homogenous Herz space $K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)$ is defined by*

$$K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{s(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} < \infty \right\}, \tag{10}$$

where

$$\|g\|_{K^{\alpha,r}_{s(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{k=\infty} \|2^{k\alpha} g\chi_k\|^r_{L^{s(\cdot)}} \right)^{\frac{1}{r}} + \|g\|_{L^{s(\cdot)}(D(0,1))}.$$

2.3. Herz–Morrey Spaces

Next, we define Herz–Morrey spaces with variable exponent.

Definition 4. *Let $\alpha \in \mathbb{R}$, $0 \le \lambda < \infty$, $0 < r < \infty$ and $s(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A Herz–Morrey spaces with variable exponent $M\dot{K}^{\alpha\lambda}_{r,s(\cdot)}(\mathbb{R}^n)$ is defined by,*

$$M\dot{K}^{\alpha\lambda}_{r,s(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{s(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{\alpha\lambda}_{r,s(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}^{\alpha\lambda}_{r,s(\cdot)}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha r} \|g\chi_k\|^r_{L^{s}(\cdot)}(\mathbb{R}^n) \right)^{\frac{1}{r}}.$$

2.4. Grand Lebesgue Sequence Space

Now, we will define the grand Lebesgue sequence space. $\mathbb{G}$ is representing one of the sets $\mathbb{N}_0, \mathbb{Z}^n, \mathbb{N}, \mathbb{Z}$ in the following definitions (see [19]).

Definition 5. *Let $r \in [1,\infty)$ and $\theta > 0$. The grand Lebesgue sequence space $l^{r)\theta}$ can be defined by the norm*

$$\begin{aligned} \|\{x_k\}_{k\in\mathbb{G}}\|_{l^{r)\theta}(\mathbb{G})} &= \|x\|_{l^{r)\theta}(\mathbb{G})} \\ &:= \sup_{\delta>0} \left(\delta^\theta \sum_{k\in\mathbb{X}} |x_k|^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} = \sup_{\delta>0} \delta^{\frac{\theta}{r(1+\delta)}} \|x\|_{l^{r(1+\delta)\theta}(\mathbb{G})}, \end{aligned}$$

where $x = \{x_k\}_{k\in\mathbb{G}}$. The following nesting properties hold:

$$l^{r(1-\delta)} \hookrightarrow l^r \hookrightarrow l^{r),\theta_1} \hookrightarrow l^{r),\theta_2} \hookrightarrow l^{r(1+\delta)} \tag{11}$$

for $0 < \delta < \frac{1}{r}$, $\delta > 0$ and $0 < \theta_1 \le \theta_2$.

3. Grand Variable Herz–Morrey Spaces

Grand variable Herz–Morrey spaces are introduced in this section.

Definition 6. *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $r \in [1,\infty)$, $s : \mathbb{R}^n \to [1,\infty)$, $\theta > 0$, $0 \le \lambda < \infty$. We define the homogeneous grand variable Herz–Morrey spaces can be defined by the norm:*

$$M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n) = \left\{ g \in L^{s(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)} = \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} 2^{k\alpha(\cdot)r(1+\delta)} \|g\chi_k\|^{r(1+\delta)}_{L^{s(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{r(1+\delta)}}.$$

For $\lambda = 0$, the grand Herz–Morrey spaces become grand Herz spaces.

Non-homogeneous grand variable Herz–Morrey spaces can be defined in a similar way.

Theorem 1. *If $0 < r_i < \infty, 1 \le q_- \le q_+ < \infty$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $i = 1,2$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, $1 = \frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)}$, $\lambda = \lambda_1 + \lambda_2$ and $\alpha(\cdot) = \alpha(\cdot)_1(\cdot) + \alpha_2(\cdot)$. Then*

$$\|fg\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,1}} \le \|f\|_{M\dot{K}^{\alpha_1(\cdot),r_1),\theta}_{\lambda,q(\cdot)}} \|g\|_{M\dot{K}^{\alpha_2(\cdot),r_2),\theta}_{\lambda,q'(\cdot)}}.$$

Proof. We have

$$\begin{aligned}\|fg\|_{M\dot{K}^{\alpha(\cdot),),\theta}_{\lambda,1}(\mathbb{R}^n)} &= \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} 2^{k\alpha(\cdot)r(1+\delta)} \|f\chi_k\|_{L^1}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\ &= \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} 2^{k\alpha(\cdot)r(1+\delta)} \left(\int_{2^k}^{2^{k+1}} |fg| \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}},\end{aligned}$$

by using Hölder's inequality

$$\begin{aligned}&\le C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} 2^{k(\alpha_1(\cdot)+\alpha_2(\cdot))r(1+\delta)} \|f\chi_k\|_{L^{q(\cdot)}}^{r(1+\delta)} \|g\chi_k\|_{L^{q'(\cdot)}}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\ &= C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} 2^{k(\alpha_1(\cdot)+\alpha_2(\cdot))r(1+\delta)} \|f\chi_k\|_{L^{q(\cdot)}}^{r(1+\delta)} \|g\chi_k\|_{L^{q'(\cdot)}}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\ &= C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} \left(2^{k\alpha_1(\cdot)} \|f\chi_k\|_{L^{q(\cdot)}}\right)^{r(1+\delta)} \left(2^{k\alpha_2(\cdot)} \|g\chi_k\|_{L^{q'(\cdot)}}\right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\ &\le C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} \left(2^{k\alpha_1(\cdot)} \|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)} \left(2^{k\alpha_2(\cdot)} \|g\chi_k\|_{L^{q'(\cdot)}}\right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}},\end{aligned}$$

by using generalized Hölder's inequality

$$\begin{aligned}&\le C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} \left(2^{k\alpha_1(\cdot)} \|f\chi_k\|_{L^{q(\cdot)}}\right)^{r_1(1+\delta)} \right)^{\frac{1}{r_1(1+\delta)}} \\ &\times \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} \left(2^{k\alpha_1(\cdot)} \|f\chi_k\|_{L^{q'(\cdot)}}\right)^{r_2(1+\delta)} \right)^{\frac{1}{r_2(1+\delta)}} \\ &= C\|f\|_{M\dot{K}^{\alpha_1(\cdot),r_1),\theta}_{\lambda,q(\cdot)}} \|f\|_{M\dot{K}^{\alpha_2(\cdot),r_2),\theta}_{\lambda,q'(\cdot)}}.\end{aligned}$$

□

4. Boundedness of the Riesz Potential Operator

Now Riesz potential operator can be defined as

$$I^{\gamma} f(z_1) = \frac{1}{\eta_n(\gamma)} \int\limits_{\mathbb{R}^n} \frac{f(z_2)}{|z_1 - z_2|^{n-\gamma}} dz_2 \tag{12}$$

with the normalizing constant $\eta_n(\gamma) = 2^{\gamma} \pi^{\frac{n}{2}} \frac{\Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)}$.

Whenever $\gamma q_1(z_1) < n$, we can define the Sobolev conjugate of q_1 by the usual relation

$$\frac{1}{q_2(z_1)} := \frac{1}{q_1(z_1)} - \frac{\gamma}{n}, z_1 \in \mathbb{R}^n \tag{13}$$

The well-known Sobolev theorem was extended to variable exponents in [20] for bounded sets in $\mathbb{R}^n$ under the assumption that the maximal operator is bounded in $L^{p(\cdot)}(\Omega)$; for unbounded sets, proved in [21], the Sobolev theorem runs as follows.

Theorem 2. *Let $s_2 \in \mathcal{P}_{\infty}^{\log}(\mathbb{R}^n)$ and $\gamma s_1^{+} < n$,*

$$\|I^{\gamma} g\|_{s_2(\cdot)} \leq C \|g\|_{s_1(\cdot)}.$$

Theorem 3. *Let $1 \leq r < \infty$, $\alpha(\cdot), q(\cdot) \in \mathcal{P}_{\infty}^{\log}(\mathbb{R}^n)$, q_1 is sobolev conjugate defined by the relation (13) such that*

(i) $\gamma - \frac{n}{q(0)} < \alpha(0) < \frac{n}{q'(0)}$;

(ii) $\gamma - \frac{n}{q_{\infty}} < \alpha_{\infty} < \frac{n}{q'_{\infty}}$.

Suppose that Riesz potential operator I^{γ} is bounded on Lebesgue spaces and satisfies the size condition (12). Then, I^{γ} from $M\dot{K}_{q_1(\cdot)}^{\alpha(\cdot),r),\theta}(\mathbb{R}^n)$ to $M\dot{K}_{q_2(\cdot)}^{\alpha(\cdot),r),\theta}(\mathbb{R}^n)$.

Proof. Let $f \in M\dot{K}_{q_2(\cdot)}^{\alpha(\cdot),p),\theta}(\mathbb{R}^n)$, and $f(z_1) = \Sigma_{l=-\infty}^{\infty} f(z_1)\chi_l(z_1) = \Sigma_{l=-\infty}^{\infty} f_l(z_1)$, we have

$$\begin{aligned}
\|I^{\gamma} f\|_{M\dot{K}_{q_2(\cdot)}^{\alpha(\cdot),p,\theta}(\mathbb{R}^n)} &= \sup_{\delta>0} \sup_{k_o \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\delta^{\theta} \sum_{k \in \mathbb{Z}} 2^{k\alpha(\cdot) r(1+\delta)} \|\chi_k I^{\gamma} f\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0} \sup_{k_o \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\delta^{\theta} \sum_{k \in \mathbb{Z}} 2^{k\alpha(\cdot) r(1+\delta)} \left(\sum_{l=-\infty}^{\infty} \|\chi_k I^{\gamma} f(\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{r(1+\delta)} \right) \right)^{\frac{1}{r(1+\delta)}} \\
&\leq \sup_{\delta>0} \sup_{k_o \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\delta^{\theta} \sum_{k \in \mathbb{Z}} 2^{k\alpha(\cdot) r(1+\delta)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k I^{\gamma} (f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&+ \sup_{\delta>0} \sup_{k_o \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\delta^{\theta} \sum_{k \in \mathbb{Z}} 2^{k\alpha(\cdot) r(1+\delta)} \left(\sum_{l=k-1}^{k+1} \|\chi_k I^{\gamma} (f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&+ \sup_{\delta>0} \sup_{k_o \in \mathbb{Z}} 2^{-k_0 \lambda} \left(\delta^{\theta} \sum_{k \in \mathbb{Z}} 2^{k\alpha(\cdot) r(1+\delta)} \left(\sum_{l=k+2}^{\infty} \|\chi_k I^{\gamma} (f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
&= E_1 + E_2 + E_3.
\end{aligned}$$

As operator I^{γ} is bounded on the Lebesgue space $L^{q_2(\cdot)}(\mathbb{R}^n)$ so for E_2,

$$\begin{aligned}
E_2 \leq & \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k-1}^{k+1}\|I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k-1}^{k+1}\|I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
& + \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{\infty}2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k-1}^{k+1}\|I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
& = E_{21}+E_{22}.
\end{aligned}$$

By using the fact $2^{k\alpha(z_1)}=2^{k\alpha(0)}$, $k<0$, $z_1\in R_k$ implies that

$$\|2^{k\alpha(\cdot)}f\chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}=2^{k\alpha(0)}\|f\chi_k\|_{L^{q_1(\cdot)}(\mathbb{R}^n)},$$

$$\begin{aligned}
E_{21} \leq & \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k-1}^{k+1}\|I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)r(1+\delta)}\left(\sum_{l=k-1}^{k+1}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1}2^{k\alpha(0)r(1+\delta)}\|f\chi_k\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha(\cdot)r(1+\delta)}\|f\chi_k\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{r(1+\delta)}}\\
= & C\|f\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For E_{22}, we use the fact $2^{k\alpha(z_1)}=2^{k\alpha_\infty}$, $k\geq 0$, $z_1\in R_k$, we obtain

$$\begin{aligned}
E_{22} \leq & \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{\infty}2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k-1}^{k+1}\|I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{\infty}2^{k\alpha_\infty r(1+\delta)}\left(\sum_{l=k-1}^{k+1}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{\infty}2^{k\alpha_\infty r(1+\delta)}\|f\chi_k\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}}2^{k\alpha(\cdot)r(1+\delta)}\|f\chi_k\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{r(1+\delta)}}\\
= & C\|f\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

For each $k \in \mathbb{Z}$ and $l \le k-2$ and a.e. $z_1 \in R_k$, $z_2 \in R_l$, we have

$$E_1 \le \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k\in\mathbb{Z}} 2^{k\alpha(\cdot)r(1+\delta)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k I^\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}}$$

$$\begin{aligned} |I^\gamma(f\chi_l)(z_1)| \le & \int_{R_l} |z_1-z_2|^{\gamma-n} |f(z_2)| dz_2 \\ \le & C2^{k(\gamma-n)} \int_{R_l} |f(z_2)| dz_2 \\ \le & C2^{k(\gamma-n)} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}, \end{aligned}$$

splitting E_1 by using Minkowski's inequality we have

$$\begin{aligned} E_1 \le & \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(\cdot)r(1+\delta)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k I^\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\ & + \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k=0}^{\infty} 2^{k\alpha(\cdot)r(1+\delta)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k I^\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\ & = E_{11} + E_{12}. \end{aligned}$$

By using Lemma 1, we have

$$2^{k(\gamma-n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \le C2^{k(\gamma-n)} 2^{\frac{kn}{q_2(0)}} 2^{\frac{ln}{q_1'(0)}} \le C2^{\frac{(l-k)n}{q_1'(0)}}, \tag{14}$$

applying above estimates to E_{11}, we can obtain

$$\begin{aligned} E_{11} \le & \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(\cdot)r(1+\delta)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k I^\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\ & \le C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left[\delta^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(\cdot)r(1+\delta)} \left(\sum_{l=-\infty}^{k-2} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} 2^{k(\gamma-n)} \right.\right. \\ & \left.\left. \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right]^{\frac{1}{r(1+\delta)}}, \end{aligned}$$

let $b = \frac{n}{q_1'(0)} - \alpha(0)$,

$$E_{11} \le C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left[\delta^\theta \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-2} 2^{\alpha(0)l} \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} 2^{b(l-k)} \right)^{r(1+\delta)} \right]^{\frac{1}{r(1+\delta)}}, \tag{15}$$

by using Hölder's inequality, Fubini's theorem and the inequality $2^{-r(1+\delta)} < 2^{-r}$, we obtain

$$\begin{aligned}
E_{11} \leq & C \sup_{\delta>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \Bigg[\delta^{\theta} \sum_{k=-\infty}^{-1} \Bigg(\sum_{l=-\infty}^{k-2} 2^{\alpha(0)r(1+\delta)l} \|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} 2^{br(1+\delta)(l-k)/2} \\
& \times \sum_{l=-\infty}^{k-2} 2^{br(1+\delta)'(l-k)/2} \Bigg)^{\frac{r(1+\delta)}{r(1+\delta)'}} \Bigg]^{\frac{1}{r(1+\delta)}} \\
= & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k=-\infty}^{-1} \sum_{l=-\infty}^{k-2} 2^{\alpha(0)r(1+\delta)l} \|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} 2^{br(1+\delta)(l-k)/2} \right)^{\frac{1}{r(1+\delta)}} \\
= & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{l=-\infty}^{-1} 2^{\alpha(0)r(1+\delta)l} \|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \sum_{k=l+2}^{-1} 2^{br(1+\delta)(l-k)/2} \right)^{\frac{1}{r(1+\delta)}} \\
< & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{l=-\infty}^{-1} 2^{\alpha(0)r(1+\delta)l} \|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \sum_{k=l+2}^{-1} 2^{bp(l-k)/2} \right)^{\frac{1}{r(1+\delta)}} \\
\leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{l=-\infty}^{-1} 2^{\alpha(0)r(1+\delta)l} \|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{r(1+\delta)}} \\
= & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{l\in\mathbb{Z}} 2^{\alpha(\cdot)r(1+\delta)l} \|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{r(1+\delta)}} \\
\leq & C\|f\|_{M\dot{K}^{\alpha,r),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Now, for E_{12} using Minkowski's inequality, we have

$$\begin{aligned}
E_{12} \leq & \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k=0}^{\infty} 2^{k\alpha(\cdot)r(1+\delta)} \left(\sum_{l=-\infty}^{-1} \|\chi_k I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}} \\
& + \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda} \left(\delta^{\theta} \sum_{k=0}^{\infty} 2^{k\alpha(\cdot)r(1+\delta)} \left(\sum_{l=0}^{k-2} \|\chi_k I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{r(1+\delta)} \right)^{\frac{1}{r(1+\delta)}}
\end{aligned}$$

$= A_1 + A_2.$

The estimate for A_2 can be obtained by similar way to E_{11} by replacing $q_1'(0)$ with $q_{1\infty}'$ and using the fact $\frac{n}{q_{1\infty}'} - \alpha_\infty > 0$. For A_1 using Lemma 1, we obtain

$$\begin{aligned}
2^{k(\gamma-n)} \|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} &\leq C 2^{k(\gamma-n)} 2^{\frac{kn}{q_{2\infty}}} 2^{\frac{ln}{q_1'(0)}} \\
&\leq C 2^{\frac{-kn}{q_{1\infty}'}} 2^{\frac{ln}{q_1'(0)}},
\end{aligned}$$

as $\alpha_\infty - \frac{n}{q'_{1\infty}} < 0$, we have

$$
\begin{aligned}
A_1 \le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty r(1+\delta)}\left(\sum_{l=-\infty}^{-1}\|\chi_k I^\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\le C \sup_{\epsilon>0}\left[\delta^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty r(1+\delta)} \times \left(\sum_{l=-\infty}^{-1} 2^{\frac{-kn}{q_{1\infty}}} 2^{\frac{ln}{q_1'(0)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right]^{\frac{1}{r(1+\delta)}}\\
&\le C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left[\delta^\theta \sum_{k=0}^{\infty} 2^{(k\alpha-kn/q'_{1\infty})r(1+\delta)} \times \left(\sum_{l=-\infty}^{-1} 2^{\frac{ln}{q_1'(0)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right]^{\frac{1}{r(1+\delta)}}\\
&\le C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \left(\sum_{l=-\infty}^{-1} 2^{\frac{ln}{q'(0)}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
&\le C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \left(\sum_{l=-\infty}^{-1} 2^{\frac{ln}{q_1'(0)}-\alpha(0)l}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\alpha(0)l}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}.
\end{aligned}
$$

Now, by using Hölder's inequality and the fact $\frac{n}{q_1'(0)} - \alpha(0) > 0$, we have

$$
\begin{aligned}
A_1 \le & \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \left(\sum_{l=-\infty}^{-1} 2^{\frac{ln}{q_1'(0)}-\alpha(0)l}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{\alpha(0)l}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
\le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left[\delta^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)lr(1+\delta)}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right.\\
&\left.\times\left(\sum_{l=-\infty}^{-1} 2^{(\frac{ln}{q_1'(0)}-\alpha(0)l)r(1+\delta)'}\right)^{\frac{r(1+\delta)}{r(1+\delta)'}}\right]^{\frac{1}{r(1+\delta)}}\\
\le & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \left(\sum_{l\in\mathbb{Z}} 2^{\alpha(\cdot)lr(1+\delta)}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)\right)^{\frac{1}{r(1+\delta)}}\\
\le & C\|f\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Now, we estimate E_3, for every $k \in \mathbb{Z}$ and $l \ge k+2$ and a.e. $z_1 \in R_k$; the size condition and Hölder's inequality imply

$$
\begin{aligned}
|I^\gamma(f\chi_l)(z-1)| \le & \int_{R_l} |z_1 - z_2|^{-n}|f(z_2)|dz_2\\
\le & C2^{l(\gamma-n)}\int_{R_l}|f(z_2)|dz_2\\
\le & C2^{l(\gamma-n)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

splitting E_3 by applying the Minkowski's inequality we have

$$\begin{aligned}
E_3 \leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k\in\mathbb{Z}} 2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=-\infty}^{-1} 2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
& + C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{\infty} 2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
= & E_{31}+E_{32}.
\end{aligned}$$

For E_{32} Lemma 1 yields

$$2^{l(\gamma-n)}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \leq C2^{l(\gamma-n)}2^{\frac{kn}{q_{2\infty}}}2^{\frac{ln}{q_{1\infty}'}} \leq C2^{\frac{(k-l)n}{q_{1\infty}}}, \tag{16}$$

we get

$$\begin{aligned}
E_{32} \leq & \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{\infty} 2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k+2}^{\infty}\|\chi_k I^{\gamma}(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg[\delta^{\theta}\sum_{k=0}^{\infty} 2^{k\alpha(\cdot)r(1+\delta)}\Bigg(\sum_{l=k+2}^{\infty}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}2^{l(\gamma-n)}\cdot\\
& \qquad \|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\Bigg)^{r(1+\delta)}\Bigg]^{\frac{1}{r(1+\delta)}}\\
\leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{(\alpha_\infty)l}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{d(k-l)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}},
\end{aligned}$$

where $d=\frac{n}{q_{1\infty}}+\alpha_\infty>0$. Then, we use Hölder's theorem for series and $2^{-r(1+\delta)}<2^{-r}$ to obtain

$$\begin{aligned}
E_{32} \leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg[\delta^{\theta}\sum_{k=0}^{\infty}\left(\sum_{l=k+2}^{\infty}2^{l(\alpha_\infty)r(1+\delta)}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{dr(1+\delta)(k-l)/2}\right)\\
& \qquad \times\left(\sum_{l=k+2}^{\infty}2^{dr(1+\delta)'(k-l)/2}\right)^{\frac{r(1+\delta)}{r(1+\delta)'}}\Bigg]^{\frac{1}{r(1+\delta)}}\\
\leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{k=0}^{\infty}\sum_{l=k+2}^{\infty}2^{l(\alpha_\infty)r(1+\delta)}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{dr(1+\delta)(k-l)/2}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l=0}^{\infty}2^{l(\alpha_\infty)r(1+\delta)}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=0}^{l-2}2^{dr(1+\delta)(k-l)/2}\right)^{\frac{1}{r(1+\delta)}}\\
< & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l\in\mathbb{Z}}2^{l(\alpha_\infty)r(1+\delta)}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=-\infty}^{l-2}2^{dp(k-l)/2}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C \sup_{\delta>0} \sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^{\theta}\sum_{l\in\mathbb{Z}}2^{\alpha(\cdot)r(1+\delta)l}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{r(1+\delta)}}\\
\leq & C\|f\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Now, for E_{31} using Monkowski's inequality, we have

$$E_{31} \leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=k+2}^{-1}\|\chi_k I^\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}$$
$$+\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(\cdot)r(1+\delta)}\left(\sum_{l=0}^{\infty}\|\chi_k I^\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}$$
$$=B_1+B_2.$$

The estimate for B_1 can be obtained similar to E_{32} by replacing $q_{1\infty}$ with $q_1(0)$ and applying the fact that $\frac{n}{q_1(0)}+\alpha(0)>0$. For B_2 using Lemma 1, we obtain

$$2^{l(\gamma-n)}\|\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq C2^{l(\gamma-n)}2^{\frac{kn}{q_2(0)}}2^{\frac{ln}{q'_{1\infty}}} \leq C2^{\frac{kn}{q_1(0)}}2^{\frac{l(-n)}{q_{1\infty}}} \tag{17}$$

$$B_2 \leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=\infty}^{-1} 2^{k\alpha(0)r(1+\delta)}\left(\sum_{l=0}^{\infty}\|\chi_k I^\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=\infty}^{-1} 2^{k\alpha(0)r(1+\delta)}\times\left(\sum_{l=0}^{\infty}2^{l(\gamma-n)}2^{\frac{kn}{q_1(0)}}2^{\frac{ln}{q_{1\infty}}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=\infty}^{-1} 2^{k\alpha(0)r(1+\delta)}\times\left(\sum_{l=0}^{\infty}2^{\frac{kn}{q_1(0)}}2^{\gamma l}2^{\frac{-ln}{q_{1\infty}}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \sum_{k=\infty}^{-1} 2^{k(\alpha(0)+n)/q_1(0)r(1+\delta)}\times\left(\sum_{l=0}^{\infty}2^{\gamma l}2^{\frac{-ln}{q_{1\infty}}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \left(\sum_{l=0}^{\infty}2^{\gamma l}2^{\frac{-ln}{q_{1\infty}}}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \left(\sum_{l=0}^{\infty}2^{l(\alpha_\infty)}\|f\chi_l\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}2^{l(nq_{1\infty}+\alpha_\infty)}\right)^{r(1+\delta)}\right)^{\frac{1}{r(1+\delta)}}.$$

Now, by using Hölder's inequality and the fact that $\frac{n}{q_\infty}+\alpha_\infty>0$, we have

$$B_2 \leq \sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta \left(\sum_{l=0}^{\infty}2^{2^{l(\alpha_\infty)}r(1+\delta)}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{r(1+\delta)}\right.$$
$$\left.\times\left(\sum_{l=0}^{\infty}2^{l(nq_{1\infty}+\alpha_\infty)r(1+\delta)}\right)^{\frac{r(1+\delta)}{r(1+\delta)'}}\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq C\sup_{\delta>0}\sup_{k_o\in\mathbb{Z}} 2^{-k_0\lambda}\left(\delta^\theta\left(\sum_{l\in\mathbb{Z}}2^{l(\alpha_\infty)r(1+\delta)}\|f\chi_l\|^{r(1+\delta)}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)\right)^{\frac{1}{r(1+\delta)}}$$
$$\leq C\|f\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)}$$

Combining the estimates for E_1, E_2 and E_3 yields

$$\|I^{\gamma} f\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,q_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,q_1(\cdot)}(\mathbb{R}^n)}$$

□

5. Conclusions

We have defined a new type of space called variable exponents grand Herz–Morrey spaces, where we used discrete grand spaces, and we have proved the boundedness of the Riesz potential operator on these spaces.

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