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Reich-Type and (α, F)-Contractions in Partially Ordered Double-Controlled Metric-Type Spaces with Applications to Non-Linear Fractional Differential Equations and Monotonic Iterative Method

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Abstract: In this manuscript, we defined (α, F)-contractions in the context of double-controlled metric spaces and partially ordered double-controlled metric spaces. We established new fixed-point results and defined the notion of double-controlled metric space on a Reich-type contraction. Our findings are generalizations of a few well-known findings in the literature. Some non-trivial examples and certain consequences are also provided to illustrate the significance of the presented results. The existence and uniqueness of the solution of non-linear fractional differential equations and the monotone iterative method are also determined using the fixed-point method.

Keywords: double-controlled metric space; partial order metric space; F-contraction; α-admissible; fractional differential equation; fixed point

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1. Introduction and Preliminaries

In the second quarter of the 18th century, a paper establishing the existence of solutions to differential equations introduced fixed-point theory (Joseph Liouville, 1837). Later, this method was enhanced as a sequential approximation method (Charles Emile Picard, 1890), and in the context of complete normed space, it was extracted and abstracted as a fixed-point theorem (Stefan Banach, 1922). It provides an approximate method to actually locate the fixed point as well as the a priori and a posteriori estimates for the rate of convergence. It also guarantees the presence and uniqueness of a fixed point. This tool is important to the understanding of metric spaces. After that, it is said that Stefan Banach established fixed-point theory. The presence of a fixed point for a given function is guaranteed by fixed-point theorems, which also allow us to guarantee that the original problem has been solved. The existence of a solution is equivalent to the existence of a fixed point for an appropriate mapping in a wide range of scientific problems that start from many fields of mathematics.

In 1993, Czerwik [1] presented the more dominant and widespread idea of metric-type space, called b-metric space. In the definition of metric space, he introduced a constant in the right-hand side of the triangular inequality and also proved the more generalized form of the Banach Contraction theorem. Alharbi et al. [2] extended the previous work and proved many fixed-point results in rectangular b-metric space. They also used α-admissible function on a rectangular b-metric space and proved many results in more generalized form than the existing literature. In addition, they presented an application and some examples to illustrate the results. In 2012, Aydi et al. [3] extended this work and used set-valued quasi-contraction maps in b-metric spaces. They also generalized several well-known comparable results in the existing literature. Furthermore, Aydi et al. [4] proved common fixed-point results of single and multi-valued mappings, which satisfy a weak φ-contraction in b-metric space. In 2018, Karapinar et al. [5] proved several fixed-point results for Meir–Keeler contraction mappings in generalized-metric spaces. In 2016, Shatanawi [6] used the notion of c-comparison function with base s and established some common fixed-point theorems for nonlinear contractions in a complete b-metric space (see [7–9] for more details). Alqahtani et al. [10] established nonlinear F-contractions in a more general framework of b-metric spaces and studied the existence and uniqueness of such contractions. By utilizing nonlinear F-contractions, they also examined the solutions of differential equations in the setting of fractional derivatives involving Mittag–Leffler kernels (Atangana–Baleanu fractional derivative).

In 2017, Kamran et al. [11] introduced the more generalized metric space called extended b-metric space and proved some results from the literature. Mukheimer et al. [12] defined the notion of an α-ψ-contractive mapping and generalized the results defined on extended b-metric spaces. Many other researchers (see [13,14]) also proved fixed-point results on such spaces. Mlaiki et al. [15] introduced the notion of controlled-type metric spaces by replacing b ≥ 1 with a controlled function $\beta : \Xi \times \Xi \rightarrow [1, +\infty)$ in the triangular inequality of b-metric space. Lateef [16] defined a Fisher-type contractive condition by using the idea of controlled metric-type spaces and obtained some generalized fixed-point results. In addition, he established some interesting examples to show the authenticity of the established results. Ahmad et al. [17] introduced Reich-type contractions and (α, F)-contractions on a controlled metric-type space and generalized some known results from the literature. In 2012, the structure of the F-contraction was presented by Wardowski [18] and established new remarkable results in the context of complete metric spaces and established a more generalized form of the Banach contraction principle. Wardowski provided new guidance for researchers, so they can add additional work in the field of fixed-point theory. Secelean [19] extended the idea of the F-contraction given by Wardowski [18] and provided new properties of F-contractions. They also examined the iterated function systems (IFS) composed of F-contractions, and then, from the classical Hutchinson–Barnsley theory of IFS consisting of Banach contractions, extended several fixed-point results as an application. Some other results related to F-contractions can be seen in [20–22]. In 1971, Reich [23] established some interesting results in non-linear analysis. In 2018, Abdeljawad et al. [24] established the notion of double-controlled metric-type spaces and some fixed-point results. Samet et al. [25] derived several fixed-point theorems for α-ψ-contractive-type mappings. Jankowski [26] solved fractional equations of the Volterra type involving a Riemann–Liouville derivative.

Ali et al. [27] solved nonlinear fractional differential equations for contractive and weakly compatible mappings in the context of neutrosophic metric spaces. Huang et al. [28] proved some fixed-point results for generalized F-contractions in b-metric-like spaces. Saleem et al. [29,30] established numerous fixed-point theorems and worked on some interesting applications. Asjad et al. [31,32] generalized the Hermite–Hadamard-type inequality with exp-convexity involving non-singular fractional operator and the fractional comparative study of the non-linear directional couplers in non-linear optics. Ishtiaq et al. [33] introduced the notion of orthogonal neutrosophic metric spaces and proved several fixed-point theorems.

In this manuscript, we define (α, F)-contractions in the context of double-controlled metric spaces and partially ordered double-controlled metric spaces. We establish new fixed-point results and define the notion of double-controlled metric space on a Reich-type contraction. Some non-trivial examples and certain consequences are also provided to illustrate the significance of the presented results. The existence and uniqueness of the solution of non-linear fractional differential equations and the monotone iterative method are also examined using the fixed-point method.

Some of the following notions are used throughout this article: CMS for controlled metric space, DCMS for double-controlled metric space, and CDCMS for complete double-controlled metric space.

Definition 1. [1] *Consider a non-empty set* Ξ *and* $s \geq 1$*. A function* $\Delta_b : \Xi \times \Xi \to [0, \infty)$ *is said to be a b-metric if for all* $\varkappa, \omega, z \in \Xi$

(B1) $\Delta_b(\varkappa, \omega) = 0$ *iff* $\varkappa = \omega$;
(B2) $\Delta_b(\varkappa, \omega) = \Delta_b(\omega, \varkappa)$;
(B3) $\Delta_b(\varkappa, z) \leq s[\Delta_b(\varkappa, \omega) + \Delta_b(\omega, z)]$.

The pair (Ξ, Δ_b) *is called the b-metric space.*

Definition 2. [11] *Consider a non-empty set* Ξ *and* $\alpha : \Xi \times \Xi \to [1, \infty)$ *be a function. A function* $\Delta_e : \Xi \times \Xi \to [0, \infty)$ *is said to be extended b-metric if for all* $\varkappa, \omega, z \in \Xi$

(E1) $\Delta_b(\varkappa, \omega) = 0$ *iff* $\varkappa = \omega$;
(E2) $\Delta_b(\varkappa, \omega) = \Delta_b(\omega, \varkappa)$;
(E3) $\Delta_b(\varkappa, z) \leq \alpha(\varkappa, z)[\Delta_b(\varkappa, \omega) + \Delta_b(\omega, z)]$.

The pair (Ξ, Δ_e) *is called the extended b-metric space.*

Definition 3. [24] *Let* Ξ *be a non-empty set and* α, $\beta : \Xi \times \Xi \to [1, \infty)$ *be a function. A function* $\Delta : \Xi \times \Xi \to [0, \infty)$ *is said to be DCMS if for all* $\varkappa, \omega, z \in \Xi$

(D1) $\Delta(\varkappa, \omega) = 0$ *iff* $\varkappa = \omega$;
(D2) $\Delta(\varkappa, \omega) = \Delta(\omega, \varkappa)$;
(D3) $\Delta(\varkappa, z) \leq \alpha(\varkappa, \omega)\Delta(\varkappa, \omega) + \beta(\omega, z)\Delta(\omega, z)$.

The pair (Ξ, Δ) *is called DCMS.*

In DCMS, the Cauchy and convergent sequences are defined as follows.

Definition 4. *Let* (Ξ, Δ) *be a DCMS and* $\{\varkappa_n\}_{n \in N}$ *be a sequence in* Ξ*, then*

(a) *A sequence* $(\varkappa_n)$ *is called convergent to a point* $\varkappa \in \Xi$ *if, for every* $\varepsilon > 0$*, there exist a* $N = N(\varepsilon)$ *such that* $\Delta(\varkappa_n, \varkappa) < \varepsilon$ *for all* $n \geq N$*. Then, we write*

$$\lim_{n \to \infty} \varkappa_n = \varkappa.$$

(b) *A sequence* $(\varkappa_n)$ *is said to be Cauchy if, for every* $\varepsilon > 0$*, there exists* $N = N(\varepsilon)$ *such that* $\Delta(\varkappa_n, \varkappa_m) < \varepsilon$ *for all* $m, n \geq N$.
(c) *The DCMS* (Ξ, Δ) *is called complete if every Cauchy sequence is convergent.*

Definition 5. *Assume* (Δ, Ξ) *be DCMS,* $\varkappa \in \Xi$ *and* $\varepsilon > 0$*. Then*

(a) *The open ball is denoted and defined by*

$$B(\varkappa, \varepsilon) = \{\varkappa_o \in \Xi, \ \Delta(\varkappa, \varkappa_o) < r\}.$$

(b) *The mapping* $G : \Xi \to \Xi$ *is continuous at point* $\varkappa \in \Xi$ *if, for every* $\varepsilon > 0$ *and* $\delta > 0$*, such that*

$$G(B(\varkappa, \delta)) \subseteq B(G(\varkappa, \gamma)).$$

Definition 6. [18] *Let* $F : \mathbb{R}^+ \to \mathbb{R}$ *be a function that satisfies the following conditions:*

(F1) *F is strictly increasing, i.e., for all* $\varkappa_1, \varkappa_2 \in \mathbb{R}^+$ *with* $\varkappa_1 < \varkappa_2$ *implies* $F(\varkappa_1) < F(\varkappa_2)$.

(F2) *For every sequence* $(\varkappa_n)$ *of positive real numbers* $\lim_{n\to\infty} \varkappa_n = 0$ *and* $\lim_{n\to\infty} F(\varkappa_n) = -\infty$ *are equivalent.*

(F3) *There is* $\theta \in (0,1)$ *so that* $\lim_{t\to 0^+} t^\theta F(\varkappa) = 0$.

Let $\mathcal{F}$ be the class of all functions that satisfy (F1)–(F3). A self-mapping $T : \Xi \to \Xi$ is said to be the F-contraction on a metric space (Ξ, Δ) if there is a function F that satisfies (F1)–(F3) and a constant $\lambda > 0$, $\Delta(T\varkappa, T\omega) > 0$,

$$\lambda + F(\Delta(T\varkappa, T\omega)) \leq F(\Delta(\varkappa, \omega)) \text{ for all } \varkappa, \omega \in \Xi.$$

2. Result on Reich-Type Contraction

In this section, we establish the Reich-type contraction [23] on a double-controlled metric space and provide some new fixed-point results. To further demonstrate the significance of the established results, we also offer several examples.

Theorem 1. *Let* (Ξ, Δ) *be a CDCMS. Let* $T : \Xi \to \Xi$ *be self-mapping so that there are* $p,\ q,\ r \in (0,1)$ *with* $k = \frac{p+q}{1-r} < 1$

$$\Delta(T\varkappa, T\omega) \leq p\Delta(\varkappa, \omega) + q\Delta(\varkappa, T\varkappa) + r\Delta(\omega, T\omega). \tag{1}$$

for all $\varkappa, \omega \in \Xi$. *For* $\varkappa_o \in \Xi$, *take* $T^n\varkappa_o = \varkappa_n$. *Assume that*

$$\sup_{m\geq i} \lim_{i\to\infty} \frac{\alpha(\varkappa_{i+1}, \varkappa_{i+2})\beta(\varkappa_{i+1}, \varkappa_m)}{\alpha(\varkappa_i, \varkappa_{i+1})} < \frac{1}{k}. \tag{2}$$

Suppose that $\lim_{n\to\infty} \alpha(\varkappa_n, \varkappa)$ *and* $\lim_{n\to\infty} \beta(\varkappa, \varkappa_n)$ *exist and are finite, and* $r \lim_{n\to\infty} \alpha(\varkappa, \varkappa_n) < 1$ *for every* $\varkappa \in \Xi$, *then* T *possesses a unique fixed point.*

Proof: Let $\{\varkappa_n\}$ be a sequence in Ξ such that $\varkappa_n = \varkappa_{n+1}$ $\forall\ n \in N$. If there exist $n_o \in N$ for which $\varkappa_{n_o+1} = \varkappa_{n_o}$, then $T\varkappa_{n_o} = \varkappa_{n_o}$ and the proof is complete. We suppose that $\varkappa_n = \varkappa_{n+1}$ for every $n \in N$, then we have

$$\begin{aligned} \Delta(\varkappa_n, \varkappa_{n+1}) &= \Delta(T\varkappa_{n-1}, T\varkappa_n) \leq p\Delta(\varkappa_{n-1},\ \varkappa_n) + q\Delta(\varkappa_{n-1},\ T\varkappa_{n-1}) + r\Delta(\varkappa_n,\ T\varkappa_n) \\ &= p\Delta(\varkappa_{n-1},\ \varkappa_n) + q\Delta(\varkappa_{n-1},\ \varkappa_n) + r\Delta(\varkappa_n,\ \varkappa_{n+1}) \end{aligned} \tag{3}$$

$$\Delta(\varkappa_n, \varkappa_{n+1}) \leq \frac{p+q}{1-r}\Delta(\varkappa_{n-1},\ \varkappa_n) = k\Delta(\varkappa_{n-1},\ \varkappa_n), \tag{4}$$

where $k = \frac{p+q}{1-r}$. Thus, we have

$$\Delta(\varkappa_n, \varkappa_{n+1}) \leq k\Delta(\varkappa_{n-1},\ \varkappa_n) \leq k^2\Delta(\varkappa_{n-2},\ \varkappa_{n-1}) \leq \ldots \leq k^n\Delta(\varkappa_0,\ \varkappa_1). \tag{5}$$

For all $n,\ m \in N$ $(m > n)$, we have

$$\begin{gathered} \Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \beta(\varkappa_{n+1},\ \varkappa_m)\Delta(\varkappa_{n+1},\ \varkappa_m) \\ \Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \beta(\varkappa_{n+1},\ \varkappa_m)\alpha(\varkappa_{n+1},\ \varkappa_{n+2})\Delta(\varkappa_{n+1},\ \varkappa_{n+2}) \\ +\beta(\varkappa_{n+1},\ \varkappa_m)\beta(\varkappa_{n+2},\ \varkappa_m)\Delta(\varkappa_{n+2},\ \varkappa_m) \\ \Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \beta(\varkappa_{n+1},\ \varkappa_m)\alpha(\varkappa_{n+1},\ \varkappa_{n+2})\Delta(\varkappa_{n+1},\ \varkappa_{n+2}) \\ +\beta(\varkappa_{n+1},\ \varkappa_m)\beta(\varkappa_{n+2},\ \varkappa_m)\alpha(\varkappa_{n+2},\ \varkappa_{n+3})\Delta(\varkappa_{n+2},\ \varkappa_{n+3}) + \beta(\varkappa_{n+1},\ \varkappa_m)\beta(\varkappa_{n+2},\ \varkappa_m) \\ \beta(\varkappa_{n+3},\ \varkappa_m)\Delta(\varkappa_{n+3},\ \varkappa_m) \end{gathered}$$

$$\Delta(\varkappa_n,\varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \sum_{i=n+1}^{m-2}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})\Delta(\varkappa_i,\ \varkappa_{i+1}) + \sum_{i=n+1}^{m-1}\beta(\varkappa_j,\ \varkappa_m)\Delta(\varkappa_{m-1},\ \varkappa_m). \tag{6}$$

This implies that

$$\Delta(\varkappa_n,\varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \sum_{i=n+1}^{m-2}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})\Delta(\varkappa_i,\ \varkappa_{i+1}) + \sum_{i=n+1}^{m-1}\beta(\varkappa_j,\ \varkappa_m)\Delta(\varkappa_{m-1},\ \varkappa_m)$$

$$\Delta(\varkappa_n,\varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})k^n\Delta(\varkappa_o,\ \varkappa_1) + \sum_{i=n+1}^{m-2}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})k^i\Delta(\varkappa_o,\ \varkappa_1) + \sum_{i=n+1}^{m-1}\beta(\varkappa_{m-1},\ \varkappa_m)k^{m-1}\Delta(\varkappa_o,\ \varkappa_1)$$

$$\Delta(\varkappa_n,\varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})k^n\Delta(\varkappa_o,\ \varkappa_1) + \sum_{i=n+1}^{m-1}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})k^i\Delta(\varkappa_o,\ \varkappa_1) \tag{7}$$

Now, let

$$S_l = \sum_{i=0}^{l}\left(\sum_{j=0}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})k^i\Delta(\varkappa_o,\ \varkappa_1). \tag{8}$$

Consider

$$v_i = \left(\sum_{j=0}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})k^i\Delta(\varkappa_o,\ \varkappa_1). \tag{9}$$

We have

$$\frac{v_{i+1}}{v_i} = \beta(\varkappa_{i+1},\varkappa_m)\frac{\alpha(\varkappa_{i+1},\varkappa_{i+2})}{\alpha(\varkappa_i,\varkappa_{i+1})}\cdot k. \tag{10}$$

In view of (2) and the ratio test, we assure that the series $\sum_i v_i$ converges. Thus, $\lim_{i\to\infty} s_n$ exists. Hence $\{s_n\}$ is the real sequence, which is Cauchy.

Now, using (7), we get

$$\Delta(\varkappa_n,\varkappa_m) \leq \Delta(\varkappa_o,\varkappa_1)[k^n\alpha(\varkappa_n,\varkappa_{n+1}) + (S_{m-1},S_n)]. \tag{11}$$

In the above, we used $\alpha(\varkappa,\omega) \geq 1$, and letting $n,\ m \to \infty$ in (13), we obtain

$$\lim_{n,m\to\infty}\Delta(\varkappa_n,\varkappa_m) = 0.$$

So, $\{\varkappa_n\}$ is the sequence, which is a Cauchy sequence in CDCMS $(\Xi,\ \Delta)$. So, a point $\varkappa^* \in \Xi$ so that

$$\lim_{n\to\infty}\Delta(\varkappa_n,\varkappa^*) = 0,$$

i.e., $\varkappa^n \to \varkappa^*$ as $n \to \infty$. Now we need to prove that $\varkappa^*$ is a fixed point of Ξ. By (1) and condition (c), we get

$$\Delta(\varkappa^*,T\varkappa^*) \leq \alpha(\varkappa^*,\varkappa_{n+1})\Delta(\varkappa^*,\varkappa_{n+1}) + \beta(\varkappa_{n+1},T\varkappa^*)\Delta(\varkappa_{n+1},T\varkappa^*)$$

$$\Delta(\varkappa^*,T\varkappa^*) = \alpha(\varkappa^*,\varkappa_{n+1})\Delta(\varkappa^*,\varkappa_{n+1}) + \beta(\varkappa_{n+1},T\varkappa^*)\Delta(T\varkappa_n,T\varkappa^*)$$

$$\Delta(\varkappa^*,T\varkappa^*) \leq \alpha(\varkappa^*,\varkappa_{n+1})\Delta(\varkappa^*,\varkappa_{n+1}) + \beta(\varkappa_{n+1},T\varkappa^*)[p\Delta(\varkappa_n,\varkappa^*) + q\Delta(\varkappa_n,T\varkappa_n) + r\Delta(\varkappa^*,T\varkappa^*)]$$

$$\Delta(\varkappa^*, T\varkappa^*) = \alpha(\varkappa^*, \varkappa_{n+1})\Delta(\varkappa^*, \varkappa_{n+1}) + \beta(\varkappa_{n+1}, T\varkappa^*)[p\Delta(\varkappa_n, \varkappa^*) + q\Delta(\varkappa_n, \varkappa_{n+1}) + r\Delta(\varkappa^*, T\varkappa^*)].$$

Taking the limit as $n \to \infty$ and using 3 and 4, the fact that $\lim_{n\to\infty} \alpha(\varkappa_n, \varkappa)$ and $\lim_{n\to\infty} \beta(\varkappa, \varkappa_n)$ exist and are finite.

We have

$$\Delta(\varkappa^*, T\varkappa^*) \leq \left[r \lim_{n\to\infty} \Delta(\varkappa_{n+1}, T\varkappa^*)\right] \Delta(\varkappa^*, T\varkappa^*).$$

Suppose that $\varkappa^* \neq T\varkappa^*$, bearing in mind that $\left[r \lim_{n\to\infty} \Delta(\varkappa_{n+1}, T\varkappa^*)\right] < 1$, so

$$0 < \Delta(\varkappa^*, T\varkappa^*) \leq \left[r \lim_{n\to\infty} \Delta(\varkappa_{n+1}, T\varkappa^*)\right] \Delta(\varkappa^*, T\varkappa^*) < \Delta(\varkappa^*, T\varkappa^*),$$

which is a contradiction. Thus, it provides that $\varkappa^* = T\varkappa^*$. The uniqueness of the proof is obvious. This completes the proof. □

Corollary 1. *Let (Ξ, Δ) be a DCMS. Let $T : \Xi \to \Xi$ be such that there is $\mu \in (0, 1)$ and*

$$\Delta(T\varkappa, T\omega) \leq \mu\Delta(\varkappa, \omega)$$

For all $\in \Xi$. For $\varkappa_0 \in \Xi$, take $T^n\varkappa_0 = \varkappa_n$. Assume that

$$\sup_{m\geq i} \lim_{i\to\infty} \frac{\alpha(\varkappa_{i+1}, \varkappa_{i+2})\beta(\varkappa_{i+1}, \varkappa_m)}{\alpha(\varkappa_i, \varkappa_{i+1})} < \frac{1}{\mu}.$$

Assume that $\lim_{n\to\infty} \alpha(\varkappa_n, \varkappa)$ and $\lim_{n\to\infty} \beta(\varkappa, \varkappa_n)$ exist and are finite, and $\left[r \lim_{n\to\infty} \Delta(\varkappa_{n+1}, T\varkappa^*)\right] < 1$ for every $\varkappa \in \Xi$; then, T has a distinct fixed point.

Proof: Taking $p = q = 0$ in Theorem 1. □

Example 1. *Assume that $\Xi = \{0, 1, 2\}$. We define the double-controlled metric as follows:*

$\Delta(\varkappa, \omega)$	0	1	2
0	0	$\frac{5}{3}$	$\frac{15}{18}$
1	$\frac{5}{3}$	0	$\frac{10}{14}$
2	$\frac{15}{18}$	$\frac{10}{14}$	0

where $\alpha, \beta : \Xi \times \Xi \to [1, \infty)$ is defined as

$\alpha(\varkappa, \omega)$	0	1	2
0	0	2	2
1	2	1	2
2	2	2	1

and

$\beta(\varkappa, \omega)$	0	1	2
0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$

Given $T : \Xi \to \Xi$ as $T(0) = 2$, $T(1) = T(1) = 1$, and considering $p = \frac{4}{7}$, $q = \frac{1}{8}$, $r = \frac{2}{5}$, then it is evident that each condition of Theorem 1 is true, so T has a unique fixed point, which is 1.

Example 2. *Assume that Y = [0, 1]. Consider the DCMS, which is defined as*

$$\Delta(\varkappa,\omega) = |\varkappa-\omega|^2.$$

Choose $\alpha(\varkappa,\omega) = 1+\varkappa+\omega$ and $\beta(\varkappa,\omega) = 2\{1+\max(\varkappa,\ \omega)\}$ for all $\varkappa,\omega \in \Xi$. Take $T\varkappa = \frac{\varkappa^2}{6}$. Consider $p = \frac{1}{7}$, $q = \frac{1}{8}$, $r = \frac{2}{5}$, and also choose $\varkappa_o = 0$. Then, clearly all conditions of Theorem 1 are satisfied and "0" is the unique fixed point of T.

3. Results on (π, *F*)-Contraction

In this section, we establish the (π, F)-contraction on a double-controlled metric space and provide some new fixed-point results.

Definition 7. [26] *Assume a non-empty set Ξ and $\pi : \Xi\times\Xi \to [0,\infty)$ be given a function. A self-mapping T* on *Ξ is called π-admissible if*

$$\pi(\varkappa,\omega) \geq 1 \Rightarrow \pi(T\varkappa,T\omega) \geq 1\ \forall\ \varkappa,\omega \in \Xi.$$

Definition 8. *Let (Ξ,Δ) be a DCMS. Let $T : \Xi \to \Xi$ is said to be a (π, F)-contraction if there is some*
$\pi : \Xi\times\Xi \to [0,\infty)$, $F \in \mathcal{F}$, $\lambda > 0$ so that

$$\lambda + \pi(\varkappa,\omega)F(\Delta(T\varkappa,T\omega)) \leq F(\Delta(\varkappa,\omega)) \tag{12}$$

$\varkappa,\omega \in \Xi$ with $\Delta(T\varkappa,T\omega) > 0$.

Theorem 2. *Let (Ξ,Δ) be a DCMS and $T : \Xi \to \Xi$ be a (π, F)-contraction; then, the following conditions hold:*

(a) *T is π-admissible.*
(b) *There is point $\varkappa_o \in \Xi$ so that $\pi(\varkappa_o,T\varkappa_o) \geq 1$.*
(c) *T is continuous.*
(d) *For $\varkappa_o \in \Xi$, define a Picard sequence $\{\varkappa_n = T^n\varkappa_o\}$ such that*

$$\sup_{m\geq i}\lim_{i\to\infty} \frac{\alpha(\varkappa_{i+1},\varkappa_{i+2})\beta(\varkappa_{i+1},\varkappa_m)}{\alpha(\varkappa_i,\varkappa_{i+1})} < 1. \tag{13}$$

Suppose that $\lim_{n\to\infty}\alpha(\varkappa_n,\varkappa)$ and $\lim_{n\to\infty}\beta(\varkappa,\varkappa_n)$ exist and are finite for every $\varkappa \in \Xi$; then, T has a fixed point, which is unique.

Proof: Assume that $\varkappa_o \in \Xi$ be a point such that $\pi(\varkappa_o,T\varkappa_o) \geq 1$. We define $\{\varkappa_n\}$ as a sequence in Ξ such that $\varkappa_n = \varkappa_{n+1}\ \forall\ n \in N$. If there exists $n_o \in N$ for which $\varkappa_{n_o+1} = \varkappa_{n_o}$, then $T\varkappa_{n_o} = \varkappa_{n_o}$ and the proof is finished. We suppose that $\varkappa_n = \varkappa_{n+1}$ for every $n \in N$ and then by (I) and (II), it is obvious that $\pi(\varkappa_n,T\varkappa_n) \geq 1$. □

Now, for all $n \in N$, we have

$$\lambda + F(\Delta(\varkappa_n,\varkappa_{n+1})) = \lambda + F(\Delta(\varkappa_{n-1},T\varkappa_n))$$
$$\lambda + F(\Delta(\varkappa_n,\varkappa_{n+1})) \leq \lambda + \pi(\varkappa_n,\varkappa_{n+1})F(\Delta(\varkappa_{n-1},T\varkappa_n)).$$

Since T is a (π, F)-contraction, we can write

$$\lambda + F(\Delta(\varkappa_n,\varkappa_{n+1})) \leq \lambda + \pi(\varkappa_n,\varkappa_{n+1})F(\Delta(T\varkappa_{n-1},T\varkappa_n))$$
$$\lambda + F(\Delta(\varkappa_n,\varkappa_{n+1})) \leq\ F(\Delta(\varkappa_{n-1},\varkappa_n)).$$

Thus, we get

$$\begin{aligned} F(\Delta(\varkappa_n, \varkappa_{n+1})) &\leq F(\Delta(\varkappa_{n-1}, \varkappa_n)) - \lambda \\ F(\Delta(\varkappa_n, \varkappa_{n+1})) &\leq F(\Delta(\varkappa_{n-2}, \varkappa_{n-1})) - 2\lambda \\ F(\Delta(\varkappa_n, \varkappa_{n+1})) &\leq F(\Delta(\varkappa_{n-3}, \varkappa_{n-2})) - 3\lambda \\ &\vdots \\ F(\Delta(\varkappa_n, \varkappa_{n+1})) &\leq F(\Delta(\varkappa_o, \varkappa_1)) - n\lambda. \end{aligned}$$

Letting $n \to \infty$ in above, we get $\lim_{n\to\infty} F(\Delta(\varkappa_n, \varkappa_{n+1})) = -\infty$. By (F2), we get

$$\lim_{n\to\infty} F(\Delta(\varkappa_n, \varkappa_{n+1})) = 0$$

By condition (F3), there is $h \in (0,1)$, such that

$$\lim_{n\to\infty} [\Delta(\varkappa_n, \varkappa_{n+1})]^h F(\Delta(\varkappa_n, \varkappa_{n+1})) = 0$$

Now, from (12), we have

$$(\Delta(\varkappa_n, \varkappa_{n+1}))^h F(\Delta(\varkappa_n, \varkappa_{n+1})) - (\Delta(\varkappa_n, \varkappa_{n+1}))^h F(\Delta(\varkappa_o, \varkappa_{n+1})) \leq -n\lambda(\Delta(\varkappa_n, \varkappa_{n+1}))^h \leq 0.$$

Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} [\Delta(\varkappa_n, \varkappa_{n+1})]^h = 0$$

Hence,

$$\lim_{n\to\infty} (n)^{\frac{1}{h}} \Delta(\varkappa_n, \varkappa_{n+1}) = 0$$

and there exists $n_1 \in N$ such that $(n)^{\frac{1}{h}} \Delta(\varkappa_n, \varkappa_{n+1}) \leq 1$ for all $n \geq n_1$. So, we have

$$\Delta(\varkappa_n, \varkappa_{n+1}) \leq \frac{1}{(n)^{\frac{1}{h}}} \quad n \geq n_1.$$

Now considering the inequality for $q \geq 1$, we have

$$\Delta(\varkappa_n, \varkappa_{n+q}) \leq \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + \beta(\varkappa_{n+1}, \varkappa_{n+q})\Delta(\varkappa_{n+1}, \varkappa_{n+q})$$

$$\begin{aligned} \Delta(\varkappa_n, \varkappa_{n+q}) \leq{}& \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + \beta(\varkappa_{n+1}, \varkappa_{n+q})\alpha(\varkappa_{n+1}, \varkappa_{n+2})\Delta(\varkappa_{n+1}, \varkappa_{n+2}) \\ &+\beta(\varkappa_{n+1}, \varkappa_{n+q})\beta(\varkappa_{n+2}, \varkappa_{n+q})\Delta(\varkappa_{n+2}, \varkappa_{n+q}) \end{aligned}$$

$$\begin{aligned} \Delta(\varkappa_n, \varkappa_{n+q}) \leq{}& \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + \beta(\varkappa_{n+1}, \varkappa_{n+q})\alpha(\varkappa_{n+1}, \varkappa_{n+2})\Delta(\varkappa_{n+1}, \varkappa_{n+2}) \\ &+\beta(\varkappa_{n+1}, \varkappa_{n+q})\beta(\varkappa_{n+2}, \varkappa_{n+q})\alpha(\varkappa_{n+2}, \varkappa_{n+3})\Delta(\varkappa_{n+2}, \varkappa_{n+3}) + \beta(\varkappa_{n+1}, \varkappa_{n+q}) \\ &\beta(\varkappa_{n+2}, \varkappa_{n+q})\, \beta(\varkappa_{n+3}, \varkappa_{n+q})\Delta(\varkappa_{n+3}, \varkappa_{n+q}) \\ &\vdots \end{aligned}$$

$$\begin{aligned} \Delta(\varkappa_n, \varkappa_{n+q}) \leq{}& \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + \sum_{i=n+1}^{n+q-2}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\Delta(\varkappa_i, \varkappa_{i+1}) \\ &+\sum_{i=n+1}^{n+q-1}\beta(\varkappa_j, \varkappa_{n+q})\Delta(\varkappa_{m-1}, \varkappa_{n+q}) \end{aligned}$$

$$\begin{aligned} \Delta(\varkappa_n, \varkappa_{n+q}) \leq{}& \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + \sum_{i=n+1}^{n+q-2}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\Delta(\varkappa_i, \varkappa_{i+1}) \\ &+\left(\sum_{i=n+1}^{n+q-1}\beta(\varkappa_j, \varkappa_{n+q})\right)\beta(\varkappa_{n+q-1}, \varkappa_{n+q})\Delta(\varkappa_{n+q-1}, \varkappa_{n+q}) \end{aligned}$$

$$\Delta(\varkappa_n, \varkappa_{n+q}) = \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + \sum_{i=n+1}^{n+q-1}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\Delta(\varkappa_i, \varkappa_{i+1})$$

$$\Delta(\varkappa_n, \varkappa_{n+q}) \leq \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + \sum_{i=n+1}^{n+q-1}\left(\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\Delta(\varkappa_i, \varkappa_{i+1})$$

$$\Delta(\varkappa_n, \varkappa_{n+q}) \leq \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + \sum_{i=n+1}^{n+q-1}\left(\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\frac{1}{(i)^{\frac{1}{k}}}. \tag{14}$$

Now consider

$$\sum_{i=n+1}^{n+q-1}\left(\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\frac{1}{(i)^{\frac{1}{k}}} = \sum_{i=n+1}^{n+q-1}\frac{1}{(i)^{\frac{1}{k}}}\left(\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})$$
$$\sum_{i=n+1}^{n+q-1}\left(\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\frac{1}{(i)^{\frac{1}{k}}} \leq \sum_{i=n+1}^{\infty}\frac{1}{(i)^{\frac{1}{k}}}\left(\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})$$
$$\sum_{i=n+1}^{n+q-1}\left(\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\frac{1}{(i)^{\frac{1}{k}}} = \sum_{i=n+1}^{\infty}U_iV_i,$$

where $U_i = \frac{1}{(i)^{\frac{1}{k}}}$ and $V_i = \alpha(\varkappa_i, \varkappa_{i+1})\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})$.

Since $\frac{1}{k} > 0$, $\sum_{i=n+1}^{\infty}\frac{1}{(i)^{\frac{1}{k}}}$ converges and also $(V_i)_i$ is increasing and bounded above, thus $\lim_{i\to\infty}\{V_i\}$, exists, which is non-zero. Hence, $\left\{\sum_{i=n+1}^{\infty}U_iV_i\right\}_n$ converges.

Now, we assume the partial sum

$$S_q = \sum_{j=0}^{q}\left(\sum_{j=0}^{i}\beta(\varkappa_j, \varkappa_{n+q})\right)\alpha(\varkappa_i, \varkappa_{i+1})\frac{1}{(i)^{\frac{1}{k}}}.$$

Now, from (14), we have

$$\Delta(\varkappa_n, \varkappa_{n+q}) \leq \alpha(\varkappa_n, \varkappa_{n+1})\Delta(\varkappa_n, \varkappa_{n+1}) + (S_{n+q-1} - S_n) \tag{15}$$

using the condition (13) and by the ratio test, we assure that the existence of $\lim_{n\to\infty}S_n$. Hence, by the real sequence, $\{S_n\}$is a Cauchy.

Now, by taking $n \to +\infty$ in (15), we get $\lim_{n\to\infty}\Delta(\varkappa_n, \varkappa_{n+q}) = 0$. Hence, $\{\varkappa_n\}$ is a Cauchy sequence in (Ξ, Δ), which is complete, so $\{\varkappa_n\}$ converges to some $u \in \Xi$. We claim that $Tu = u$. Since

$\varkappa_n \to u$ as $n \to \infty$ and T is continuous, we have $T\varkappa_n \to Tu$ as $n \to \infty$. Hence, we have

$$\Delta(u, Tu) = \lim_{n\to\infty}\Delta(\varkappa_{n+1}, Tu) = \lim_{n\to\infty}\Delta(T\varkappa_n, Tu) = 0$$

and hence $Tu = u$. Thus, u is a fixed point of T. It is obvious that it is unique.

Corollary 2. *Let* (Ξ, Δ) *be a CDCMS, and let* $T : \Xi \to \Xi$ *be continuous, so that*

$$\lambda + \pi(\varkappa, \omega)F(\Delta(T\varkappa, T\omega)) \leq F(\Delta(\varkappa, \omega))$$

$\varkappa, \omega \in \Xi$. *For* $\varkappa_o \in \Xi$, *take* $\{\varkappa_n = T^n\varkappa_o\}$. *Suppose that*

$$\sup_{m\geq i}\lim_{i\to\infty}\frac{\alpha(\varkappa_{i+1}, \varkappa_{i+2})\beta(\varkappa_{i+1}, \varkappa_m)}{\alpha(\varkappa_i, \varkappa_{i+1})} < 1.$$

Suppose that $\lim_{n\to\infty}\alpha(\varkappa_n, \varkappa)$ and $\lim_{n\to\infty}\beta(\varkappa, \varkappa_n)$ exist and are finite for every $\in \Xi$; then, T has a unique fixed point.

Proof: Taking $\alpha,\ \beta : \Xi \times \Xi \to [1, \infty)$ for all $\varkappa, \omega \in \Xi$ by $\alpha(\varkappa, \omega) = 1$ in Theorem 2. □

Corollary 3. *Let (Ξ, Δ) be a complete extended b-metric space and let $T : \Xi \to \Xi$ be continuous α-admissible and (α, F)-contraction so that there is $\varkappa_o \in \Xi$ in order that $\alpha(\varkappa_o, T\varkappa_o) \geq 1$. Suppose that*

$$\sup_{m \geq i} \lim_{i \to \infty} \frac{\alpha(\varkappa_{i+1}, \varkappa_{i+2})\beta(\varkappa_{i+1}, \varkappa_m)}{\alpha(\varkappa_i, \varkappa_{i+1})} < 1.$$

In addition, $\lim_{n\to\infty} \alpha(\varkappa_n, \varkappa)$ and $\lim_{n\to\infty} \beta(\varkappa, \varkappa_n)$ exist and are finite for every $\in \Xi$, so T has a unique fixed point.

Corollary 4. *Let (Ξ, Δ) be a complete b-metric space. Let $T : \Xi \to \Xi$ be continuous α-admissible and (α, F)-contraction so that there is $\varkappa_o \in \Xi$ in order that $\alpha(\varkappa_o, T\varkappa_o) \geq 1$. Then, T possesses a unique fixed point.*

Proof: Taking $\alpha,\ \beta : \Xi \times \Xi \to [1, \infty)$ for all $\varkappa, \omega \in \Xi$ by $\alpha(\varkappa, \omega) = \alpha(\omega, z)$ in Theorem 2. □

Corollary 5. *Let (Ξ, Δ) be a complete metric space, and let $T : \Xi \to \Xi$ be continuous α-admissible and (α, F)-contraction so there is a point $\varkappa_o \in \Xi$ in order that $\alpha(\varkappa_o, T\varkappa_o) \geq 1$. Then, T possesses a unique fixed point.*

Proof: Taking $\alpha,\ \beta : \Xi \times \Xi \to [1, \infty)$ for all $\varkappa, \omega \in \Xi$ by $\alpha(\varkappa, \omega) = \beta(\varkappa, \omega) = 1$ in Theorem 2. □

4. Fixed-Point Results in Partially Ordered Double-Controlled Metric Spaces

In this section, we provide some new fixed-point results in the context of partially ordered double-controlled metric spaces. To further demonstrate the significance of the established results, we also offer several examples.

Definition 9. *Consider X to be a non-empty set. If (Ξ, Δ) is a DCMS and $(\Xi, \prec)$ is a partially ordered set, then $(\Xi, \Delta, \prec)$ is called a partially ordered double-controlled metric space. Then, $\varkappa_1, \varkappa_2 \in \Xi$ are said to be comparable if $\varkappa_1 \prec \varkappa_2$ and $\varkappa_2 \prec \varkappa_1$ holds.*

Theorem 3. *Assume $(\Xi, \Delta, \prec)$ is called a partially ordered double-controlled metric space. Let $T : \Xi \to \Xi$ be an increasing mapping. Assume that there exists $\varkappa_o \prec T(\varkappa_o)$ and define the sequence $\{\varkappa_n\}$ by $\varkappa_1 = T(\varkappa_o)$, $\varkappa_2 = T(\varkappa_1)$, $\varkappa_3 = T(\varkappa_2), \ldots\ \varkappa_n = T(\varkappa_{n-1})$. Suppose there exists a function $\mu : [1, \infty) \to [0, k)$ where $0 < k < 1$, satisfying $\mu(\varkappa_n) \to 1$ implies $\varkappa_n \to 0$ such that*

$$\Delta(T(\varkappa), T(\omega)) \leq \mu(\Delta(\varkappa, \omega))\Delta(\varkappa, \omega) \text{ for each } \in \Xi \text{ with } \varkappa \prec \omega. \tag{16}$$

Assume that T is continuous or Ξ is such that:
If a sequence $(\varkappa_n) \to \varkappa$ is an increasing sequence, then $\varkappa_n \prec \varkappa\ \forall\ n$. Moreover, if for each $\varkappa,\ \omega \in \Xi$ there exists $z \in \Xi$, which is comparable to $\varkappa$ and ω.

In addition, for every $\varkappa \in \Xi$, we have

$$\lim_{n\to\infty} \alpha(\varkappa_{n+1}, \varkappa) \text{ and } \lim_{n\to\infty} \alpha(\varkappa, \varkappa_{n+1}),\ \lim_{n\to\infty} \beta(\varkappa_{n+1}, \varkappa) \text{ and } \lim_{n\to\infty} \beta(\varkappa, \varkappa_{n+1}), \tag{17}$$

which exist and are finite. Suppose that

$$\sup_{m \geq i} \lim_{i \to \infty} \frac{\alpha(\varkappa_{i+1}, \varkappa_{i+2})}{\alpha(\varkappa_i, \varkappa_{i+1})} \beta(\varkappa_{i+1}, \varkappa_m) < \frac{1}{k} \tag{18}$$

then T has a unique fixed point.

Proof: Since $\varkappa_o \prec T(\varkappa_o)$ and T is an increasing function, then by induction, we obtain $\varkappa_o \prec T(\varkappa_o) \prec T^2(\varkappa_o) \prec T^3(\varkappa_o) \ldots \prec T^n(\varkappa_o) \prec T^{n+1}(\varkappa_o)$. We denote $T^n(\varkappa_o) = \varkappa_n$, $n = 1,2,\ldots$ □

Since $\varkappa_n \prec \varkappa_{n+1}$ for each $n \in N$, then by (1), we get

$$\begin{gathered}
\Delta(\varkappa_{n+1}, \varkappa_{n+2}) = \Delta\big(T^{n+1}(\varkappa_o), T^{n+2}(\varkappa_o)\big) \\
\Delta(\varkappa_{n+1}, \varkappa_{n+2}) \leq \mu(\Delta(\varkappa_n, \varkappa_{n+1}))\Delta(\varkappa_n, \varkappa_{n+1}) \\
\Delta(\varkappa_{n+1}, \varkappa_{n+2}) \leq k\Delta(\varkappa_n, \varkappa_{n+1}) \\
\Delta(\varkappa_{n+1}, \varkappa_{n+2}) \leq k^2\Delta(\varkappa_{n-1}, \varkappa_n) \\
\vdots \\
\Delta(\varkappa_{n+1}, \varkappa_{n+2}) \leq k^n\Delta(\varkappa_0, \varkappa_1).
\end{gathered} \tag{19}$$

Therefore, we can conclude from (19) that

$$\lim_{n\to\infty} \Delta(\varkappa_n, \varkappa_{n+1}) = 0$$

Now we show that $\{\varkappa_n\}$ is a Cauchy sequence. Now, using triangular inequality,

$$\Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \beta(\varkappa_{n+1},\ \varkappa_m)\Delta(\varkappa_{n+1},\ \varkappa_m)$$

$$\begin{gathered}
\Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \beta(\varkappa_{n+1},\ \varkappa_m)\alpha(\varkappa_{n+1},\ \varkappa_{n+2})\Delta(\varkappa_{n+1},\ \varkappa_{n+2}) \\
+\beta(\varkappa_{n+1},\ \varkappa_m)\beta(\varkappa_{n+2},\ \varkappa_m)\Delta(\varkappa_{n+2},\ \varkappa_m)
\end{gathered}$$

$$\begin{gathered}
\Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \beta(\varkappa_{n+1},\ \varkappa_m)\alpha(\varkappa_{n+1},\ \varkappa_{n+2})\Delta(\varkappa_{n+1},\ \varkappa_{n+2}) \\
+\beta(\varkappa_{n+1},\ \varkappa_m)\beta(\varkappa_{n+2},\ \varkappa_m)\alpha(\varkappa_{n+2},\ \varkappa_{n+3})\Delta(\varkappa_{n+2},\ \varkappa_{n+3})+ \\
\beta(\varkappa_{n+1},\ \varkappa_m)\beta(\varkappa_{n+2},\ \varkappa_m) \\
\beta(\varkappa_{n+3},\ \varkappa_m)\Delta(\varkappa_{n+3},\ \varkappa_m) \\
\vdots
\end{gathered}$$

$$\begin{gathered}
\Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})\Delta(\varkappa_n,\ \varkappa_{n+1}) + \sum_{i=n+1}^{m-2}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})\Delta(\varkappa_i,\ \varkappa_{i+1}) \\
+\sum_{i=n+1}^{m-1}\beta(\varkappa_i,\ \varkappa_m)\Delta(\varkappa_{m-1},\ \varkappa_m)
\end{gathered}$$

$$\begin{gathered}
\Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})k^n\Delta(\varkappa_0,\ \varkappa_1) + \sum_{i=n+1}^{m-2}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})k^i\Delta(\varkappa_0,\ \varkappa_1)+ \\
\sum_{i=n+1}^{m-1}\beta(\varkappa_i,\ \varkappa_m)k^m\Delta(\varkappa_0,\ \varkappa_1)
\end{gathered}$$

$$\Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})k^n\Delta(\varkappa_o,\ \varkappa_1) + \sum_{i=n+1}^{m-1}\left(\sum_{j=n+1}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})k^i\Delta(\varkappa_o,\ \varkappa_1)$$

$$\Delta(\varkappa_n, \varkappa_m) \leq \alpha(\varkappa_n,\ \varkappa_{n+1})k^n\Delta(\varkappa_o,\ \varkappa_1) + \sum_{i=n+1}^{m-1}\left(\sum_{j=0}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})k^i\Delta(\varkappa_o,\ \varkappa_1). \tag{20}$$

We denote

$$\psi_s = \sum_{i=0}^{s}\left(\sum_{j=0}^{i}\beta(\varkappa_j,\ \varkappa_m)\right)\alpha(\varkappa_i,\ \varkappa_{i+1})k^i\Delta(\varkappa_o,\ \varkappa_1).$$

Then from (20), we have

$$\Delta(\varkappa_n, \varkappa_m) \le \Delta(\varkappa_o,\ \varkappa_1)[k^n \Delta(\varkappa_n,\ \varkappa_{n+1}) + (\psi_{m-1} - \psi_n)]. \tag{21}$$

Using (17) and by taking into account (17) and (18), we deduce that $\lim_{n\to\infty} \psi_s$ exists and is finite. The sequence $\{\psi_s\}$is a Cauchy sequence. Hence, if we take the limit in the inequality (21) as $n, m \to \infty$, we conclude that

$$\lim_{n,\ m\to\infty} \Delta(\varkappa_n, \varkappa_m) = 0.$$

which affirms that $\{\varkappa_n\}$ is a Cauchy sequence in the complete partially ordered double-controlled metric space (X, d, $\prec$), and then $\{\varkappa_n\}$ converges to some point $\varkappa \in \Xi$.

Now we need to prove that $\varkappa$ is a fixed point of T. Since T is continuous, we have

$$\varkappa = \lim_{n\to\infty} \varkappa_n = \lim_{n\to\infty} T^n(\varkappa_o) = \lim_{n\to\infty} T^{n+1}(\varkappa_o) = T\Big(\lim_{n\to\infty} T^n(\varkappa_o)\Big) = T(\varkappa).$$

Then, $\varkappa$ is a fixed point of T.

Uniqueness:

Let u be another fixed point of T. Then,

$$\Delta(\varkappa, u) = \Delta(T(\varkappa), T(u)) \le \mu(\Delta(\varkappa, u))\Delta(\varkappa, u).$$

which holds unless $\Delta(\varkappa, u) = 0$, and then $\varkappa = u$; hence, T has a fixed point, which is unique.

Example 3. *Assume that* $\Xi = \{\ 1,\ 2,\ 3\}$. *We define the double control metric* $d\ : X \times X \to \mathbb{R}$ *as follows:*

$$d(1,\ 1) = d(2,\ 2) = d(3,\ 3) = 0,\ d(1,\ 2) = d(2,\ 1) = 5e^t,$$

$$d(2,\ 3) = d(3,\ 2) = 2e^t,\ d(1,\ 3) = d(3,\ 1) = e^t,$$

where $\alpha,\ \beta\ : \Xi \times \Xi \to [1, \infty)$ *is defined as*

$\alpha(\varkappa, \omega)$	1	2	3
1	0	4	4
2	4	1	4
3	4	4	1

and

$\beta(\varkappa, \omega)$	1	2	3
1	$\frac{1}{2}$	$\frac{6}{4}$	$\frac{6}{4}$
2	$\frac{6}{4}$	$\frac{1}{2}$	$\frac{6}{4}$
3	$\frac{6}{4}$	$\frac{6}{4}$	$\frac{1}{2}$

Given $T\ : \Xi \to \Xi$ as $T(1) = 1$, $T(2) = 3$, $T(3) = 1$, with partial order

$$\prec = \{(1,\ 1), (2,\ 2), (3,\ 3), (1,\ 2), (2,\ 3), (1,\ 3)\}$$

and considering $p = q = \frac{1}{4}$, $r = \frac{1}{5}$, then it is evident that each condition of Theorem 3 is true, so T has a unique fixed point, which is 1.

5. Fractional Differential Equation

Let $C_{1-\vartheta}(J, \mathbb{R}) = \{f \in C(0, T], \mathbb{R})\ :\ t^{1-\vartheta} \in C(J, \mathbb{R})\}$. We define the following weight norm:

$$\|f\| = \max_{[0,T]} t^{1-\vartheta} |\varkappa(t)|.$$

Theorem 4. *Let $\vartheta \in (0,1)$, $f \in C(J \times \mathbb{R}, \mathbb{R})$ increasing and $\Lambda : [0,\infty) \to [0,k)$ where $0 < k < 1$. In addition, we assume the following hypothesis:*

(a) $|f(\varkappa_1(t), \omega_1(t)) - f(\varkappa_2(t), \omega_2(t))| \leq \frac{\Gamma(2\vartheta)}{T^{\vartheta}} \Lambda\big(t^{1-\vartheta}(v_1 - v_2)\big)|v_1 - v_2|$

(b) $\frac{\Gamma(2\vartheta)}{T^{\vartheta}} \leq \frac{1}{k}$.

Then the problem $\mathcal{P}$ has a unique solution.

Proof: Problem $\mathcal{P}$ is equivalent to the problem $M\varkappa = \varkappa$, where

$$M\varkappa(t) = rt^{1-\vartheta} + \frac{1}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} F\varkappa(s)\Delta s.$$

□

In fact, proving that the operator T has a fixed point is sufficient to say that problem M has a unique solution. We use Banach fixed-point theorem. Therefore, we need to check that hypothesis in Theorem 3 is satisfied.

Indeed $A = C_{1-\vartheta}(J, \mathbb{R})$ is a partially ordered set. Now, if we define the following order relation in A,

$$U,\ V \in C_{1-\vartheta}(J,\mathbb{R}),\ U \leq V \text{ iff } U(t) \leq V(t)\ \forall\ t \in J.$$

In addition, (A, Δ) is a complete metric space. If we choose

$$\Delta(\varkappa, \omega) = \max_{[0,T]} t^{1-\vartheta}|\varkappa(t) - \omega(t)|^2,\ \varkappa,\ \omega\ \in C_{1-\vartheta}(J,\mathbb{R}).$$

the mapping M is increasing, since f is increasing.

Now we can prove M is a contraction map. Let $\varkappa,\ \omega \in C_{1-\vartheta}(J,\mathbb{R}),\ 0 < \vartheta < 1.$

$$\|M\varkappa - M\omega\| = \frac{1}{\Gamma(\vartheta)} \max_{t\in[0,T]} t^{1-\vartheta} \int_0^t (t-s)^{\vartheta-1}|f(t,\varkappa(s)) - f(t,\omega(s))|^2 \Delta s.$$

Since

$$\|M\varkappa - M\omega\| = \max t^{1-\vartheta}|\varkappa(s) - \omega(s)|^2$$

then

$$|\varkappa(s) - \omega(s)|^2 = \|\varkappa(s) - \omega(s)\| \max s^{\vartheta-1}.$$

Subsequently, using the first hypothesis of the theorem, we get

$$\|M\varkappa - M\omega\| \leq \tfrac{1}{\Gamma(\vartheta)} \max_{t\in[0,T]} t^{1-\vartheta} \int_0^t (t-s)^{\vartheta-1} \tfrac{\Gamma(2\vartheta)}{\Gamma(\vartheta)} \Lambda\Big(s^{\vartheta-1}|\varkappa(s) - \omega(s)|^2\Big)|\varkappa(s) - \varkappa(s)|^2$$

$$\|M\varkappa - M\omega\| = \tfrac{1}{\Gamma(\vartheta)} \max_{t\in[0,T]} t^{1-\vartheta} \int_0^t [(t-s)^{\vartheta-1} \tfrac{\Gamma(2\vartheta)}{\Gamma(\vartheta)} \Lambda\big(s^{1-\vartheta}\|\varkappa(s) - \omega(s)\| \max s^{\vartheta-1}\big)$$

$$\|\varkappa(s) - \omega(s)\|^2 \max s^{\vartheta-1}]\Delta s$$

$$\|M\varkappa - M\omega\| \leq \frac{1}{\Gamma(\vartheta)} \max_{t\in[0,T]} t^{1-\vartheta}\|\varkappa(s) - \omega(s)\|\ \Lambda(\|\varkappa(s) - \omega(s)\|) \frac{\Gamma(2\vartheta)}{\Gamma(\vartheta)} \int_0^t (t-s)^{\vartheta-1} s^{\vartheta-1} \Delta s.$$

From the Riemann–Liouville fractional integral, we have

$$\int_0^t [(t-s)^{\vartheta-1} s^{\vartheta-1} \Delta s = \frac{\Gamma(2\vartheta)}{\Gamma(\vartheta)} (t)^{2\vartheta-1}.$$

Therefore, we have

$$\|M\varkappa - M\omega\| \leq \Lambda(\|\varkappa(s) - \omega(s)\|)\|\varkappa(s) - \omega(s)\|.$$

6. Monotone Iterative Method

First, we present the following hypothesis:

Hypothesis 1.

(1) $L(t) = L,\ t \in J$*or*
(2) *The function L is non-constant on J and*

$\frac{(\vartheta)^{(\vartheta)}}{\Gamma(2\vartheta)}$*max*$|L(t)| < 1$ *Only if* $\vartheta \in \left(0, \frac{1}{2}\right)$.

Next, we present the following consequence of Theorem 4.

Lemma 1. *If* $\vartheta \in \left(0, \frac{1}{2}\right),\ L \in C(J,\mathbb{R}),\ z \in C_{1-\vartheta}(J,\mathbb{R})$, *and hypothesis 1 holds, the problem* $\mathcal{P}$ *has a unique solution.*

Hypothesis 2.

(1) $L(t) = L,\ t \in J$ *or*
(2) *The function L is non-constant and if L(t) is negative, then there exists*$\overline{L}$*non-decreasing where* $-L(t) \leq \overline{L}(t)$*on J and for every*$\in J$*, we have*

$$\frac{1}{\Gamma(\vartheta)}\int_0^a (a-\tau)^{\vartheta-1}\overline{L}(t)\Delta\tau < 1.$$

Now, we prove the following lemma to fulfill our requirements.

Lemma 2. *Assume that* $\vartheta \in \left(0, \frac{1}{2}\right)$*, and* $L \in C(J,[0,\infty))$ *or*$L \in C(J,(-\infty,0])$. *Assume that* $q \in C_{1-\vartheta}((J,\ \mathbb{R}))$ *is the solution of following problem:*

$$D^{\vartheta}q(t) \leq -L(t)q(t),\ t \in J_0\overline{q}(0) < 0. \tag{22}$$

If hypothesis 2 holds, then $q(t) \leq 0$ for all $t \in J$.

Proof: Contrarily assume that there exists $\varkappa,\omega \in (-\infty,a]$ such that $q(\varkappa) = 0, q(\omega) > 0$ and $q(t) \leq 0$ for $t \in (0,\varkappa]$; $q(t) > 0$ and for $t \in (\varkappa,\omega]$. Let $\varkappa_o$ be the first maximal point of q on $[\varkappa,\ \omega]$. □

Case 1. *Consider* $L(t) \geq 0$ *for all* $t \in J$. *Therefore,* $D^{\vartheta,\rho}q(t) \leq 0$ *for* $t \in [\varkappa,\ \omega]$. *Hence*

$$\int_{\varkappa}^{\varkappa_o} D^{\vartheta}q(t) \leq 0.$$

Therefore, $B \equiv I^{1-\vartheta}q(\varkappa_o) - I^{1-\vartheta}q(\varkappa_o) \leq 0$. However,

$$B = \frac{1}{\Gamma(1-\vartheta)}\left[\int_0^{\varkappa_o}(\varkappa_o-\tau)^{-\vartheta}q(\tau)\Delta\tau - \int_0^{\varkappa}(\varkappa_o-\tau)^{-\vartheta}q(\tau)\Delta\tau\right]$$
$$B = \frac{1}{\Gamma(1-\vartheta)}\left\{\int_0^{\varkappa_o}\left[(\varkappa_o-\tau)^{-\vartheta} - (\varkappa-\tau)^{-\vartheta}\right]q(\tau)\Delta\tau + \int_0^{\varkappa_o}(\varkappa_o-\tau)^{-\vartheta}q(\tau)\Delta\tau\right\}$$
$$B \geq \frac{1}{\Gamma(1-\vartheta)}\int_0^{\varkappa_o}(\varkappa_o-\tau)^{-\vartheta}q(\tau)\Delta\tau > 0.$$

which contradicts the fact that $B \leq 0$.

Case 2. *Assume that $L(t) \geq 0$ for all $t \in J$. Consider $\overline{L}$ to be non-decreasing on J. Now, if we apply I^{ϑ} on problem (22), we obtain*

$$q(t) - \overline{q}(0)\frac{t^{\vartheta-1}}{\Gamma(\vartheta)} \leq -I^{\vartheta}[L(t)q(t)] \text{ for } t \in [\varkappa,\ \varkappa_0].$$

Notice that $\overline{q}(0)\frac{t^{\vartheta-1}}{\Gamma(\vartheta)} \leq 0$, and that is due to the fact that $\overline{q}(0) \leq 0$. Thus,

$$q(\varkappa_o) \leq -\frac{1}{\Gamma(\vartheta)}\int\limits_0^{\varkappa_o}(\varkappa_o - \tau)^{\vartheta-1}L(\tau)q(\tau)\Delta\tau$$
$$q(\varkappa_o) = -\frac{1}{\Gamma(\vartheta)}\left[\int\limits_0^{\varkappa}(\varkappa_o - \tau)^{\vartheta-1}L(\tau)q(\tau)\Delta\tau + \int\limits_0^{\varkappa_o}(\varkappa_o - \tau)^{\vartheta-1}L(\tau)q(\tau)\Delta\tau\right]$$
$$q(\varkappa_o) < -\frac{q(\varkappa_o)}{\Gamma(\vartheta)}\int\limits_0^{\varkappa_o}(\varkappa_o - \tau)^{\vartheta-1}L(\tau)q(\tau)\Delta\tau.$$

Let $\sigma = \frac{\tau}{\varkappa_o}$

$$q(\varkappa_o) = -\frac{q(\varkappa_o)\varkappa_o^{\vartheta}}{\Gamma(\vartheta)}\int\limits_0^1(1-\tau)^{\vartheta-1}L(\sigma\varkappa_o)\Delta\sigma$$
$$q(\varkappa_o) \leq \frac{q(\varkappa_o)\varkappa_o^{\vartheta}}{\Gamma(\vartheta)}\int\limits_0^1(1-\tau)^{\vartheta-1}\widetilde{L}(\sigma a)\Delta\sigma$$
$$q(\varkappa_o) = \frac{q(\varkappa_o)\varkappa_o^{\vartheta}}{\Gamma(\vartheta)a^{\vartheta}}\int\limits_0^1(a-\tau)^{\vartheta-1}\widetilde{L}(\tau)\Delta\tau$$
$$q(\varkappa_o) \leq \frac{q(\varkappa_o)}{\Gamma(\vartheta)}\int\limits_0^a(a-\tau)^{\vartheta-1}\widetilde{L}(\tau)\Delta\tau.$$

Hence,

$$q(\varkappa_o)\left[1 - \frac{1}{\Gamma(\vartheta)}\int_0^a(a-\tau)^{\vartheta-1}\widetilde{L}(\tau)\Delta\tau\right] \leq 0.$$

Using hypothesis 2, this implies that $q(\varkappa_o) \leq 0$, and it completes our proof by leading us to a contradiction.

Now we say that ω is a lower solution of problem $(\mathcal{P})$, if

$$D^{\vartheta}\omega(t) \leq \mathcal{F}\omega(t),\ t \in J_o;\ \widetilde{\omega}(0) \leq 0,$$

We say that ω is an upper solution of problem $(\mathcal{P})$, if

$$D^{\vartheta}\omega(t) \geq \mathcal{F}\omega(t),\ t \in J_o;\ \widetilde{\omega}(0) \leq 0,$$

Now we define the following hypothesis:

Hypothesis 3. *There exists a function $L \in C(J, \mathbb{R})$, where*

$$|g(t,\varkappa_1,\varkappa_2) - g(t,\omega_1,\omega_2)| \leq L(t)|\omega_1 - \varkappa_1|,$$

whenever $\varkappa_o \leq \varkappa_1 \leq \omega_1 \leq \omega_o$, $\varkappa_2 \leq \omega_2$.

Theorem 5. *Assume that $\varkappa_0$ is a lower solution of problem $(\mathcal{P})$, and ω_0 is an upper solution of problem $(\mathcal{P})$, where $\varkappa_0, \omega_0 \in C_{1-\alpha}(J,\ \mathbb{R})$. Moreover, assuming that hypotheses 6.1, 6.2, and 6.3 hold, the problem $(\mathcal{P})$ has solutions in*

$$[\varkappa_0,\ \omega_0] = \{\text{y} \in C_{1-\alpha}(\text{J},\ \mathbb{R}), \varkappa_0(t) \leq \omega(t) \leq \omega_0(t),\ t \in J_0, \widetilde{\varkappa}_0(0) \leq \widetilde{\omega}(0) \leq \widetilde{\omega}_0(0)\}$$

Proof: By using Lemmas 1 and 2, we can prove in a similar way to Theorem 1 in [27]. □

7. Conclusions

In this manuscript, we proved the uniqueness and existence of fixed-point theorems for a contractive mapping in DCMSs and partially ordered DCMSs, using Reich-type and (α, F)-contractions. Several non-trivial examples are also provided to show the validity of our main results. We were also able to use our results to show that the fractional differential equation has a solution. Moreover, we also used the monotone iterative method to find the existence of a solution. This work can extend the context of triple-controlled metric-type spaces, complex valued triple-controlled type metric spaces, double-controlled fuzzy metric spaces, and pentagonal fuzzy-controlled metric spaces.

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