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Article

Weighted Integral Inequalities for Harmonic Convex Functions in Connection with Fejér's Result

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Abstract: In this study, on the subject of harmonic convex functions, we introduce some new functionals linked with weighted integral inequalities for harmonic convex functions. In addition, certain new inequalities of the Fejér type are discovered.

Keywords: Hermite–Hadamard inequality; convex function; harmonic convex function; Fejér inequality

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1. Introduction

For convex functions the following double inequality has great significance in the literature on mathematical inequalities and is known as Hermite–Hadamard's inequality [1]:

Let $\nu : I \longrightarrow \mathbb{R}, \varnothing \neq I \subseteq \mathbb{R}, \alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$, be a convex function. Then

$$\nu\left(\frac{\alpha_1+\alpha_2}{2}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \nu(\tau)d\tau \leq \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}. \quad (1)$$

The inequalities (1) hold in reversed direction if ν is a concave function.

Interested readers are referred to [2–22] for some generalizations, improvements, and variants of the well-known Hermite–Hadamard integral inequalities.

Fejér [23] established the following double inequality as a weighted generalization of (1):

$$\nu\left(\frac{\alpha_1+\alpha_2}{2}\right)\int_{\alpha_1}^{\alpha_2} \varrho(\tau)d\tau \leq \int_{\alpha_1}^{\alpha_2} \nu(\tau)\varrho(\tau)d\tau \leq \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2} \varrho(\tau)d\tau, \quad (2)$$

where $\nu : I \longrightarrow \mathbb{R}, \varnothing \neq I \subseteq \mathbb{R}, \alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$ is any convex function and $\varrho : [\alpha_1, \alpha_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\tau = \frac{\alpha_1+\alpha_2}{2}$.

The inequalities (2) have many extensions and generalizations as well, see [3–12,14–34].

Consider the following mappings on $[0,1]$:

$$G(\sigma) = \frac{1}{2}\left[\nu\left(\sigma\alpha_1 + (1-\sigma)\frac{\alpha_1+\alpha_2}{2}\right) + \nu\left(\sigma\alpha_2 + (1-\sigma)\frac{\alpha_1+\alpha_2}{2}\right)\right],$$

$$F(\sigma) = \frac{1}{(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\alpha_2} \nu(\sigma\tau + (1-\sigma)\lambda)d\tau d\lambda,$$

$$H(\sigma) = \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \nu\left(\sigma\tau + (1-\sigma)\frac{\alpha_1+\alpha_2}{2}\right)d\tau,$$

$$H_{\varrho}(\sigma) = \int_{\alpha_1}^{\alpha_2} \nu\left(\sigma\tau + (1-\sigma)\frac{\alpha_1+\alpha_2}{2}\right)\varrho(\tau)d\tau,$$

$$I(\sigma) = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\sigma\frac{\alpha_1+\tau}{2} + (1-\sigma)\frac{\alpha_1+\alpha_2}{2}\right)\right.$$
$$\left. +\nu\left(\sigma\frac{\alpha_2+\tau}{2} + (1-\sigma)\frac{\alpha_1+\alpha_2}{2}\right)\right]\varrho(\tau)d\tau$$

$$K(\sigma) = \int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\sigma\frac{\tau+\alpha_1}{2} + (1-\sigma)\frac{\lambda+\alpha_1}{2}\right)\right.$$
$$+\nu\left(\sigma\frac{\tau+\alpha_1}{2} + (1-\sigma)\frac{\lambda+\alpha_2}{2}\right) + \nu\left(\sigma\frac{\tau+\alpha_2}{2} + (1-\sigma)\frac{\lambda+\alpha_1}{2}\right)$$
$$\left. +\nu\left(\sigma\frac{\tau+\alpha_2}{2} + (1-\sigma)\frac{\lambda+\alpha_2}{2}\right)\right]\varrho(\tau)\varrho(\lambda)d\tau d\lambda,$$

$$N(\sigma) = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\sigma\alpha_1 + (1-\sigma)\frac{\alpha_1+\tau}{2}\right) + \nu\left(\sigma\alpha_2 + (1-\sigma)\frac{\tau+\alpha_2}{2}\right)\right]\varrho(\tau)d\tau,$$

$$L(\sigma) = \frac{1}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}[\nu(\sigma\alpha_1 + (1-\sigma)\tau) + \nu(\sigma\alpha_2 + (1-\sigma)\tau)]d\tau,$$

$$L_{\varrho}(\sigma) = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}[\nu(\sigma\alpha_1 + (1-\sigma)\tau) + \nu(\sigma\alpha_2 + (1-\sigma)\tau)]\varrho(\tau)d\tau$$

and

$$S_{\varrho}(\sigma) = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\sigma\alpha_1 + (1-\sigma)\frac{\alpha_1+\tau}{2}\right) + \nu\left(\sigma\alpha_1 + (1-\sigma)\frac{\tau+\alpha_2}{2}\right)\right.$$
$$\left. +\nu\left(\sigma\alpha_2 + (1-\sigma)\frac{\alpha_1+\tau}{2}\right) + \nu\left(\sigma\alpha_2 + (1-\sigma)\frac{\tau+\alpha_2}{2}\right)\right]\varrho(\tau)d\tau$$

where $\nu : [\alpha_1, \alpha_2] \to \mathbb{R}$ is a convex function and $\varrho : [\alpha_1, \alpha_2] \to \mathbb{R}$ is non-negative integrable and symmetric about $\tau = \frac{\alpha_1+\alpha_2}{2}$.

Remark 1. *It should be noted that* $H = H_{\varrho} = I$, $F = K$ *and* $L = L_{\varrho} = S_{\varrho}$ *on* $[0, 1]$ *as* $\varrho(\tau) = \frac{1}{\alpha_2-\alpha_1}$, $\tau \in [\alpha_1, \alpha_2]$.

A number of mathematicians have come up with important results that accurately describe the properties of the above mappings and inequalities that improve to the inequalities (1) and (2), we refer to the interested reader the research of Dragomir et al. [24] that provided the refinements of (1). Dragomir [26] proved inequalities which connect the mappings H, F and L. Teseng et al. [31] obtained a weighted generalization of the inequalities of a result of [24] using the mappings I and N. Dragomir et al. [5] established Hermite–Hadamard-type inequalities that connect the mappings H, G and L. Tseng et al. [32,34] gave a result related to Fejér's result which gives a weighted generalization of the inequalities of a result proved in [24]. Teseng et al. [33] further investigated some Fejér-type and Hermite–Hadamard-type inequalities related to the functions H, F, L, H_p,

L_p, M, S_p and K as defined above and a result obtain the weighted generalizations of some results in [24,32].

Let us recall one of the generalizations of the convex functions is harmonically convex functions:

Definition 1 ([13]). *Define $I \subseteq \mathbb{R}\backslash\{0\}$ as an interval of real numbers. A function $\nu : I \to \mathbb{R}$ to the real numbers is considered to be harmonically convex, if*

$$\nu\left(\frac{\tau\lambda}{\sigma\tau + (1-\sigma)\lambda}\right) \leq \sigma\nu(\lambda) + (1-\sigma)\nu(\tau) \tag{3}$$

for all $\tau, \lambda \in I$ and $\sigma \in [0,1]$. The function ν is defined to be harmonically concave if the inequality in (3) is reversed.

İşcan used harmonically convex functions to develop inequalities of the Hermite–Hadamard type.

Theorem 1 ([13]). *Let $\nu : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $\alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$. If $\nu \in L([\alpha_1, \alpha_2])$ then the following inequalities hold:*

$$\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2}\right) \leq \frac{\alpha_1\alpha_2}{\alpha_2 - \alpha_1}\int_{\alpha_2}^{\alpha_1} \frac{\nu(\tau)}{\tau^2} d\tau \leq \frac{\nu(\alpha_1) + \nu(\alpha_2)}{2}. \tag{4}$$

The notion of a harmonically symmetric function is given in the definition below.

Definition 2 ([16]). *A function $\varrho : [\alpha_1, \alpha_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is harmonically symmetric with respect to $\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}$ if*

$$\varrho(\tau) = \varrho\left(\frac{1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} - \frac{1}{\tau}}\right)$$

holds for all $\tau \in [\alpha_1, \alpha_2]$.

Fejér-type inequalities using harmonically convex functions and the notion of harmonically symmetric functions were presented in Chan and Wu [2].

Theorem 2 ([2]). *Let $\nu : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $\alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$. If $\nu \in L([\alpha_1, \alpha_2])$ and $\varrho : [\alpha_1, \alpha_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}$, then*

$$\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2}\right)\int_{\alpha_2}^{\alpha_1} \frac{\varrho(\tau)}{\tau^2} d\tau \leq \int_{\alpha_2}^{\alpha_1} \frac{\nu(\tau)\varrho(\tau)}{\tau^2} d\tau \leq \frac{\nu(\alpha_1) + \nu(\alpha_2)}{2}\int_{\alpha_2}^{\alpha_1} \frac{\varrho(\tau)}{\tau^2} d\tau. \tag{5}$$

Some important facts which relate harmonically convex and convex functions are given in the results below.

Theorem 3 ([6,7]). *If $[\alpha_1, \alpha_2] \subset I \subset (0, \infty)$ and if we consider the function $g : \left[\frac{1}{\alpha_2}, \frac{1}{\alpha_1}\right] \to \mathbb{R}$ defined by $g(\sigma) = \nu\left(\frac{1}{\sigma}\right)$, then ν is harmonically convex on $[\alpha_1, \alpha_2]$ if and only if g is convex in the usual sense on $\left[\frac{1}{\alpha_2}, \frac{1}{\alpha_1}\right]$.*

Theorem 4 ([6,7]). *If $I \subset (0, \infty)$ and ν is convex and nondecreasing function then ν is HA-convex and if ν is HA-convex and nonincreasing function then ν is convex.*

The main objectives of this study are to define some mappings on $[0,1]$ using a harmonically convex function $\nu : [\alpha_1, \alpha_2] \subset I \subset (0, \infty) \to \mathbb{R}$ and a non-negative integrable symmetric function $\varrho : [\alpha_1, \alpha_2] \to \mathbb{R}$ about $\tau = \frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}$ related to the inequalities (4) and (5) and to prove variants of inequalities that have been proven in [33]. We also discuss some properties of the mappings corresponding to the mapping $\mathcal{K}$ in the next section and establish a variant of lemma ([14] Lemma on page 65) for harmonically convex functions.

2. Main Results

Let $\nu : [\alpha_1, \alpha_2] \subset (0, \infty) \to \mathbb{R}$ be a harmonically convex mapping and consider the following mappings defined on $[0, 1]$ to $\mathbb{R}$ by

$$\mathcal{G}_1(\sigma) = \frac{1}{2}\left[\nu\left(\frac{2\alpha_1\alpha_2}{2\alpha_2\sigma + (1-\sigma)(\alpha_1+\alpha_2)}\right) + \nu\left(\frac{2\alpha_1\alpha_2}{2\alpha_1\sigma + (1-\sigma)(\alpha_1+\alpha_2)}\right)\right],$$

$$\mathcal{F}(\sigma) = \left(\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\right)^2 \int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2} \frac{1}{\tau^2\lambda^2}\nu\left(\frac{\tau\lambda}{\sigma\lambda + (1-\sigma)\tau}\right)d\tau d\lambda$$

$$\mathcal{U}(\sigma) = \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \frac{1}{\tau^2}\nu\left(\frac{2\alpha_1\alpha_2\tau}{2\alpha_1\alpha_2\sigma + (1-\sigma)\tau(\alpha_1+\alpha_2)}\right)d\tau$$

$$\mathcal{U}_\varrho(\sigma) = \int_{\alpha_1}^{\alpha_2} \nu\left(\frac{2\alpha_1\alpha_2\tau}{2\alpha_1\alpha_2\sigma + (1-\sigma)\tau(\alpha_1+\alpha_2)}\right)\frac{\varrho(\tau)}{\tau^2}d\tau,$$

$$\begin{aligned}\mathcal{K}(\sigma) = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\Bigg[&\nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma(\alpha_1+\tau)\alpha_2 + (1-\sigma)(\alpha_1+\alpha_2)\tau}\right)\\ &+\nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma(\alpha_1+\tau)\alpha_2 + (1-\sigma)(\alpha_1+\alpha_2)\tau}\right)\Bigg]\frac{\varrho(\tau)}{\tau^2}d\tau,\end{aligned}$$

$$\begin{aligned}\check{\mathcal{K}}(\sigma) = \frac{1}{4}\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2}\Bigg[&\nu\left(\frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau) + (1-\sigma)\tau(\alpha_2+\lambda)}\right)\\ &+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau) + (1-\sigma)\tau(\alpha_1+\lambda)}\right) + \nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda(\alpha_1+\tau) + (1-\sigma)\tau(\alpha_2+\lambda)}\right)\\ &+\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau) + (1-\sigma)\tau(\alpha_1+\lambda)}\right)\Bigg]\frac{\varrho(\tau)\varrho(\lambda)}{\tau^2\lambda^2}d\tau d\lambda,\end{aligned}$$

$$\mathcal{N}(\sigma) = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_1\tau}{2\sigma\tau + (1-\sigma)(\alpha_1+\tau)}\right) + \nu\left(\frac{2\alpha_2\tau}{2\sigma\tau + (1-\sigma)(\alpha_2+\tau)}\right)\right]\frac{\varrho(\tau)}{\tau^2}d\tau,$$

$$\mathcal{L}(\sigma) = \frac{\alpha_1\alpha_2}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{\alpha_1\tau}{\sigma\tau + (1-\sigma)\alpha_1}\right) + \nu\left(\frac{\alpha_2\tau}{\sigma\tau + (1-\sigma)\alpha_2}\right)\right]\frac{d\tau}{\tau^2},$$

$$\mathcal{L}_\varrho(\sigma) = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{\alpha_1\tau}{\sigma\tau + (1-\sigma)\alpha_1}\right) + \nu\left(\frac{\alpha_2\tau}{\sigma\tau + (1-\sigma)\alpha_2}\right)\right]\frac{\varrho(\tau)}{\tau^2}d\tau$$

and

$$\begin{aligned}\mathcal{S}_\varrho(\sigma) = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\Bigg[&\nu\left(\frac{2\alpha_1\tau}{2\tau\sigma + (1-\sigma)(\alpha_1+\tau)}\right) + \nu\left(\frac{2\alpha_1\alpha_2\tau}{2\alpha_2\tau\sigma + (1-\sigma)(\tau+\alpha_2)}\right)\\ &+\nu\left(\frac{2\alpha_1\alpha_2\tau}{2\alpha_1\tau\sigma + (1-\sigma)(\alpha_1+\tau)\alpha_2}\right) + \nu\left(\frac{2\alpha_2\tau}{2\tau\sigma + (1-\sigma)(\tau+\alpha_2)}\right)\Bigg]\frac{\varrho(\tau)}{\tau^2}d\tau.\end{aligned}$$

Remark 2. *It should be noted that* $\mathcal{U} = \mathcal{U}_{\varrho} = \check{\mathcal{K}}$, $\mathcal{F} = \mathcal{K}$ *and* $\mathcal{L} = \mathcal{L}_{\varrho} = \mathcal{S}_{\varrho}$ *on* $[0,1]$ *for* $\varrho(\tau) = \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}$, $\tau \in [\alpha_1, \alpha_2]$.

The author in [28] obtained the following refinement inequalities for (4) related to the mapping $\mathcal{U}$.

Theorem 5 ([28])**.** *Let* $\nu : [\alpha_1, \alpha_2] \subset (0, \infty) \to \mathbb{R}$ *be a harmonically convex function on* $[\alpha_1, \alpha_2]$. *Then*

(i) $\mathcal{U}$ *is harmonically convex* $(0,1]$ *and increases monotonically on* $[0,1]$.

(ii) The following hold:

$$\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right) = \mathcal{U}(0) \le \mathcal{U}(\sigma) \le \mathcal{U}(1) = \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{\nu(\tau)}{\tau^2}d\tau. \tag{6}$$

Theorem 6 ([28])**.** *Let* $\nu : [\alpha_1, \alpha_2] \subset (0, \infty) \to \mathbb{R}$ *be a harmonically convex function on* $[\alpha_1, \alpha_2]$. *Then*

(i) The following identities hold:

$$\mathcal{F}\left(\sigma + \frac{1}{2}\right) = \mathcal{F}\left(\frac{1}{2} - \sigma\right) \text{ for all } \sigma \in \left[0, \frac{1}{2}\right].$$

(ii) $\mathcal{F}$ *is harmonically convex on* $(0,1]$.

(iii) The following identities hold:

$$\inf_{\sigma\in[0,1]} \mathcal{F}(\sigma) = \mathcal{F}\left(\frac{1}{2}\right) = \left(\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\right)^2 \int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2} \frac{1}{\tau^2\lambda^2}\nu\left(\frac{2\tau\lambda}{\tau+\lambda}\right)d\tau d\lambda$$

and

$$\sup_{\sigma\in[0,1]} \mathcal{F}(\sigma) = \mathcal{F}(0) = \mathcal{F}(1) = \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{\nu(\tau)}{\tau^2}d\tau.$$

(iv) The following inequality is valid

$$\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right) \le \mathcal{F}\left(\frac{1}{2}\right).$$

(v) $\succeq$ *increases monotonically on* $\left[\frac{1}{2}, 1\right]$ *and decreases monotonically on* $\left[0, \frac{1}{2}\right]$.

(vi) $\mathcal{U}(\sigma) \le \mathcal{F}(\sigma)$ *for all* $\sigma \in [0,1]$.

Here we point out the following lemma which is very important to prove the results in the current study.

Lemma 1 ([29])**.** *Let* $\nu : [\alpha_1, \alpha_2] \subset (0, \infty) \to \mathbb{R}$ *be a harmonically convex function and let* $\alpha_1 \le \lambda_1 \le \tau_1 \le \tau_2 \le \lambda_2 \le \alpha_2$ *with* $\frac{\tau_1\tau_2}{\tau_1+\tau_2} = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}$. *Then*

$$\nu(\tau_1) + \nu(\tau_2) \le \nu(\lambda_1) + \nu(\lambda_2).$$

Chan and Wu [2] also defined some mappings related to (5) and discussed important properties of those mappings.

The author proved Fejér-type inequalities in [29] which extend the inequalities given in Theorem 5 and Theorem 6 for the mappings related to (5) which in turn provide refinements of the inequalities (5). The author used the Lemma 1 to obtain those refinements for (5). One of the results from [29] is mentioned below to be used in the continuation of the paper.

Theorem 7 ([29])**.** *Let ν, $\mathcal{K}$, $\mathcal{N}$ and ϱ be as defined above, then*

$$\begin{aligned}\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \le \mathcal{K}(0) \le \mathcal{K}(\sigma) \le \mathcal{K}(1)\\ = \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_1\tau}{\alpha_1+\tau}\right)+\nu\left(\frac{2\alpha_2\tau}{\tau+\alpha_2}\right)\right]\frac{\varrho(\tau)}{\tau^2}d\tau\\ = \mathcal{N}(0) \le \mathcal{N}(\sigma) \le \mathcal{N}(1) = \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau. \quad (7)\end{aligned}$$

Some further Fejér-type inequalities were also obtained in [30] by the author that relate the mappings $\mathcal{G}_1$, $\mathcal{K}$, $\mathcal{S}_\varrho$ and $\mathcal{L}_\varrho$.

Theorem 8 ([30])**.** *Let ν, ϱ, $\mathcal{G}_1$, $\mathcal{S}_\varrho$, $\mathcal{L}_\varrho$ be defined as above. Then we have the following results:*

(i) $\mathcal{L}_\varrho$ *is harmonically convex on* $(0,1]$.

(ii) *The following inequalities hold for all* $\sigma \in [0,1]$:

$$\begin{aligned}\mathcal{U}_\varrho(\sigma) \le \mathcal{G}_1(\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \le \mathcal{L}_\varrho(\sigma) \le (1-\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\nu(\tau)\varrho(\tau)}{\tau^2}d\tau\\ +\sigma\cdot\frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \le \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau, \quad (8)\end{aligned}$$

$$\mathcal{S}_\varrho(1-\sigma) \le \mathcal{L}_\varrho(\sigma) \quad (9)$$

and

$$\frac{\mathcal{S}_\varrho(\sigma)+\mathcal{S}_\varrho(1-\sigma)}{2} \le \mathcal{L}_\varrho(\sigma). \quad (10)$$

(iii) *The following bound is true:*

$$\sup_{\sigma\in[0,1]}\mathcal{L}_\varrho(\sigma) = \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau. \quad (11)$$

Theorem 9 ([30])**.** *Let ν, ϱ, $\mathcal{G}_1$, $\mathcal{K}$, $\mathcal{S}_\varrho$ be defined as above. Then*

(i) $\mathcal{S}_\varrho$ *is convex on* $[0,1]$.

(ii) *The following inequalities hold for all* $\sigma \in [0,1]$:

$$\begin{aligned}\mathcal{K}(\sigma) \le \mathcal{G}_1(\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \le \mathcal{S}_\varrho(\sigma)\\ \le (1-\sigma)\cdot\frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\tau\alpha_2}{\tau+\alpha_2}\right)+\nu\left(\frac{2\alpha_1\tau}{\alpha_1+\tau}\right)\right]\frac{\varrho(\tau)}{\tau^2}d\tau\\ +\sigma\cdot\frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \le \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau, \quad (12)\end{aligned}$$

$$\mathcal{K}(1-\sigma) \le \mathcal{S}_\varrho(\sigma) \quad (13)$$

and

$$\frac{\mathcal{K}(\sigma)+\mathcal{K}(1-\sigma)}{2} \le \mathcal{S}_\varrho(\sigma). \quad (14)$$

(iii) *The following identity holds:*

$$\sup_{\sigma\in[0,1]}\mathcal{S}_\varrho(\sigma) = \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau. \quad (15)$$

Now we provide some Fejér-type inequalities related with the mappings $\check{\mathcal{K}}$, $\mathcal{K}$ and, as a consequence, we obtain a weighted generalization of Theorem 6.

Theorem 10. *Let ν, ϱ, $\breve{\mathcal{K}}$, $\mathcal{K}$ be defined as above. Then*

(i) $\breve{\mathcal{K}}$ *is harmonically convex on* $(0,1]$ *and symmetric about* $\frac{1}{2}$.

(ii) $\breve{\mathcal{K}}$ *is decreasing on* $\left[0,\frac{1}{2}\right]$ *and increasing on* $\left[\frac{1}{2},1\right]$,

$$\begin{aligned}\sup_{\sigma\in[0,1]}\breve{\mathcal{K}}(\sigma)&=\breve{\mathcal{K}}(0)=\breve{\mathcal{K}}(1)\\&=\left(\frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_1\tau}{\alpha_1+\tau}\right)+\nu\left(\frac{2\alpha_2\tau}{\tau+\alpha_2}\right)\right]\frac{\varrho(\tau)}{\tau^2}d\tau\right)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\end{aligned}\tag{16}$$

and

$$\begin{aligned}\inf_{\sigma\in[0,1]}\breve{\mathcal{K}}(\sigma)=\breve{\mathcal{K}}\left(\frac{1}{2}\right)=&\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{4\alpha_2\tau\lambda}{(\lambda+\tau)\alpha_2+2\tau\lambda}\right)+\nu\left(\frac{4\alpha_1\tau\lambda}{(\lambda+\tau)\alpha_1+2\tau\lambda}\right)\right.\\&\left.+2\nu\left(\frac{4\alpha_1\alpha_2\tau\lambda}{(\lambda+\tau)\alpha_1\alpha_2+\tau\lambda(\alpha_1+\alpha_2)}\right)\right]\frac{\varrho(\tau)\varrho(\lambda)}{\tau^2\lambda^2}d\tau d\lambda\end{aligned}\tag{17}$$

(iii) *The inequalities*

$$\mathcal{K}(\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\leq\breve{\mathcal{K}}(\sigma)\tag{18}$$

and

$$\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\left(\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right)^2\leq\breve{\mathcal{K}}\left(\frac{1}{2}\right).\tag{19}$$

Proof. (i) It is easily observed from the harmonically convexity of ν that $\breve{\mathcal{K}}$ is harmonically convex on $(0,1]$. By changing the variable, we have that

$$\breve{\mathcal{K}}(\sigma)=\breve{\mathcal{K}}(1-\sigma),\text{ for all }\sigma\in[0,1].$$

This proves that the mapping $\breve{\mathcal{K}}$ is symmetric about $\frac{1}{2}$.

(ii) Let $\sigma_1<\sigma_2$ in $\left[0,\frac{1}{2}\right]$. Using the symmetry of $\breve{\mathcal{K}}$, we have

$$\breve{\mathcal{K}}(\sigma_1)=\frac{1}{2}\left[\breve{\mathcal{K}}(\sigma_1)+\breve{\mathcal{K}}(1-\sigma_1)\right]$$

and

$$\breve{\mathcal{K}}(\sigma_2)=\frac{1}{2}\left[\breve{\mathcal{K}}(\sigma_2)+\breve{\mathcal{K}}(1-\sigma_2)\right].$$

Applying Lemma 1, we can prove that

$$\frac{1}{2}\left[\breve{\mathcal{K}}(\sigma_2)+\breve{\mathcal{K}}(1-\sigma_2)\right]\leq\frac{1}{2}\left[\breve{\mathcal{K}}(\sigma_1)+\breve{\mathcal{K}}(1-\sigma_1)\right].$$

This proves that $\breve{\mathcal{K}}$ decreases $\left[0,\frac{1}{2}\right]$. Since $\breve{\mathcal{K}}$ is symmetric about $\frac{1}{2}$ and $\breve{\mathcal{K}}$ is decreasing on $\left[0,\frac{1}{2}\right]$, we get that $\breve{\mathcal{K}}$ is increasing on $\left[\frac{1}{2},1\right]$. Using the symmetry and monotonicity of $\breve{\mathcal{K}}$, we derive (16) and (17).

(iii) Using substitution rules for integration and the hypothesis of ϱ, we have the following identity

$$\breve{\mathcal{K}}(\sigma) = \frac{1}{4}\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau)+(1-\sigma)\tau(\alpha_2+\lambda)}\right)\right.$$
$$+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}\right)$$
$$+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}\right)$$
$$\left.+\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\lambda)}\right)\right]\frac{\varrho(\tau)\varrho(\lambda)}{\tau^2\lambda^2}d\tau d\lambda, \quad (20)$$

for all $\sigma \in [0,1]$.

By Lemma 1, the following inequalities hold for all $\sigma \in [0,1]$, $\tau \in [\alpha_1, \alpha_2]$ and $\lambda \in [\alpha_1, \alpha_2]$. The inequality

$$\frac{1}{2}\nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)}\right)$$
$$\leq \frac{1}{4}\left[\nu\left(\frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau)+(1-\sigma)\tau(\alpha_2+\lambda)}\right)\right.$$
$$\left.+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}\right)\right] \quad (21)$$

holds with the choices

$$\tau_1 = \tau_2 = \frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)},$$
$$\lambda_1 = \frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau)+(1-\sigma)\tau(\alpha_2+\lambda)}$$
$$\text{and } \lambda_2 = \frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}.$$

The inequality

$$\frac{1}{2}\nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)}\right)$$
$$\leq \frac{1}{4}\left[\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}\right)\right.$$
$$\left.+\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\lambda)}\right)\right] \quad (22)$$

holds with the choices

$$\tau_1 = \tau_2 = \frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)},$$
$$\lambda_1 = \frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}$$
$$\text{and } \lambda_2 = \frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\lambda)}.$$

The inequality

$$\nu\left(\frac{\alpha_1\alpha_2}{(\alpha_1+2\alpha_2)\tau-\alpha_1\alpha_2}\right) \leq \frac{1}{2}\left[\nu\left(\frac{\alpha_1\alpha_2\tau}{(1-\sigma)\alpha_2\tau+\sigma((\alpha_1+\alpha_2)\tau-\alpha_1\alpha_2)}\right)\right.$$
$$\left.+\nu\left(\frac{\alpha_1\alpha_2\tau}{\sigma\alpha_2\tau+(1-\sigma)((\alpha_1+\alpha_2)\tau-\alpha_1\alpha_2)}\right)\right] \quad (23)$$

holds with the choices

$$\tau_1=\tau_2=\frac{\alpha_1\alpha_2}{(\alpha_1+2\alpha_2)\tau-\alpha_1\alpha_2},\lambda_1=\frac{\alpha_1\alpha_2\tau}{\sigma\alpha_2\tau+(1-\sigma)((\alpha_1+\alpha_2)\tau-\alpha_1\alpha_2)}$$
$$\text{and } \lambda_2=\frac{\alpha_1\alpha_2\tau}{(1-\sigma)\alpha_2\tau+\sigma((\alpha_1+\alpha_2)\tau-\alpha_1\alpha_2)}.$$

Multiplying the inequalities (22) and (23) by $\frac{\varrho(\tau)\varrho(\lambda)}{\tau^2\lambda^2}$, integrating them over τ on $[\alpha_1,\alpha_2]$, over λ on $[\alpha_1,\alpha_2]$, we obtain

$$\begin{aligned}
&\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\lambda)}{\tau^2}d\lambda\cdot\frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)}\right)\right.\\
&\qquad\left.+\nu\left(\frac{\alpha_1\alpha_2}{(\alpha_1+2\alpha_2)\tau-\alpha_1\alpha_2}\right)\right]\frac{\varrho(\tau)\varrho(\lambda)}{\tau^2\lambda^2}d\tau d\lambda\\
&\le\frac{1}{4}\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\lambda)}\right)\right.\\
&\quad+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}\right)\\
&\quad+\nu\left(\frac{\alpha_1\alpha_2\tau}{(1-\sigma)\alpha_2\tau+\sigma((\alpha_1+\alpha_2)\tau-\alpha_1\alpha_2)}\right)\\
&\quad\left.+\nu\left(\frac{\alpha_1\alpha_2\tau}{\sigma\alpha_2\tau+(1-\sigma)((\alpha_1+\alpha_2)\tau-\alpha_1\alpha_2)}\right)\right]\frac{\varrho(\tau)\varrho(\lambda)}{\tau^2\lambda^2}d\tau d\lambda \qquad (24)
\end{aligned}$$

Now using identity (20) in (24), we derive the inequality (18).

From the inequality (18) and the monotonicity of $\breve{\mathcal{K}}$, we have

$$\begin{aligned}
\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\left(\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right)^2&=\breve{\mathcal{K}}(0)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\\
&\le\breve{\mathcal{K}}\left(\frac{1}{2}\right)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\le\breve{\mathcal{K}}\left(\frac{1}{2}\right). \qquad (25)
\end{aligned}$$

Hence the inequality (19) is proved. □

Remark 3. *If $\varrho(\tau)=\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}$, $\tau\in[\alpha_1,\alpha_2]$ in Theorem 10, then $\mathcal{K}(\sigma)=\mathcal{U}(\sigma)$, $\breve{\mathcal{K}}(\sigma)=\mathcal{F}(\sigma)$, $\sigma\in[0,1]$, Theorem 10 is identical with Theorem 6.*

Corollary 1. *From Theorem 7 and Theorem 10, we obtain the following Fejér-type inequality*

$$\begin{aligned}
\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\left(\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right)^2&\le\mathcal{K}(\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\le\breve{\mathcal{K}}(\sigma)\\
&\le\left(\frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_1\tau}{\alpha_1+\tau}\right)+\nu\left(\frac{2\alpha_2\tau}{\tau+\alpha_2}\right)\right]\frac{\varrho(\tau)}{\tau^2}d\tau\right)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau. \qquad (26)
\end{aligned}$$

Proof. From Theorem 7, we have

$$\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\le\mathcal{K}(0)\le\mathcal{K}(\sigma). \qquad (27)$$

Multiplying (26) by $\int_{\alpha_1}^{\alpha_2} \frac{\varrho(\tau)}{\tau^2} d\tau$ and using (18) and (16), we get

$$\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\left(\int_{\alpha_1}^{\alpha_2} \frac{\varrho(\tau)}{\tau^2} d\tau\right)^2 \leq \mathcal{K}(\sigma) \int_{\alpha_1}^{\alpha_2} \frac{\varrho(\tau)}{\tau^2} d\tau \leq \breve{\mathcal{K}}(\sigma)$$
$$\leq \left(\frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_1\tau}{\alpha_1+\tau}\right)+\nu\left(\frac{2\alpha_2\tau}{\tau+\alpha_2}\right)\right]\frac{\varrho(\tau)}{\tau^2} d\tau\right)\int_{\alpha_1}^{\alpha_2} \frac{\varrho(\tau)}{\tau^2} d\tau. \quad (28)$$

□

In the next theorem, we point out some inequalities for the functions ν, ϱ, $\mathcal{K}$, $\breve{\mathcal{K}}$ and $\mathcal{S}_\varrho$ considered above.

Theorem 11. *Let ν, ϱ, $\mathcal{K}$, $\breve{\mathcal{K}}$ and $\mathcal{S}_\varrho$ be defined as above. Then the following Fejér-type inequalities hold:*

$$0 \leq \breve{\mathcal{K}}(\sigma) - \mathcal{K}(\sigma)\int_{\alpha_1}^{\alpha_2} \frac{\varrho(\tau)}{\tau^2} d\tau \leq \mathcal{S}_\varrho(1-\sigma)\int_{\alpha_1}^{\alpha_2} \frac{\varrho(\tau)}{\tau^2} d\tau - \breve{\mathcal{K}}(\sigma), \quad (29)$$

hold for $\sigma \in [0,1]$.

Proof. Using substitution rules for integration and the hypothesis of ϱ, we have the following identity

$$\begin{aligned}
\breve{\mathcal{K}}(\sigma) = \frac{1}{4}\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2}\Bigg[&\nu\left(\frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau)+(1-\sigma)\tau(\alpha_2+\lambda)}\right)\\
&+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}\right)\\
&+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((\alpha_1+2\alpha_2)\lambda-\alpha_1\alpha_2)}\right)\\
&+\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\lambda)}\right)\Bigg]\frac{\varrho(\tau)\varrho(\lambda)}{\tau^2\lambda^2} d\tau d\lambda\\
= \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}}\Bigg[&\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+2(1-\sigma)\alpha_1\tau}\right)\\
&+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+2(1-\sigma)\tau((\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)}\right)\\
&+\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+2(1-\sigma)\tau((\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)}\right)\\
&+\nu\left(\frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau)+2(1-\sigma)\alpha_2\tau}\right)\Bigg]\frac{\varrho(\tau)\varrho\left(\frac{\alpha_2\lambda}{2\alpha_2-\lambda}\right)}{\tau^2\lambda^2} d\tau d\lambda
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\frac{4\alpha_1\alpha_2}{\alpha_1+3\alpha_2}}\Bigg[\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+2(1-\sigma)\tau((\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)}\right) \\
&\quad + \nu\left(\frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau)+2(1-\sigma)\alpha_2\tau}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((\alpha_2-\alpha_1)\lambda+2\alpha_1\alpha_2)}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((3\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+2(1-\sigma)\tau((\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((3\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((\alpha_2-\alpha_1)\lambda+2\alpha_1\alpha_2)}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+2(1-\sigma)\alpha_1\tau}\right)\Bigg]\frac{\varrho(\tau)\varrho\left(\frac{\alpha_2\lambda}{2\alpha_2-\lambda}\right)}{\tau^2\lambda^2}d\tau d\lambda \qquad (30)
\end{aligned}
$$

for $\sigma \in [0,1]$.

We can get the following inequalities as results of usage of Lemma 1 for all $\sigma \in [0,1]$, $\tau \in [\alpha_1, \alpha_2]$ and $\tau \in \left[\alpha_1, \frac{4\alpha_1\alpha_2}{\alpha_1+3\alpha_2}\right]$:

The inequality

$$
\begin{aligned}
&\nu\left(\frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau)+2(1-\sigma)\alpha_2\tau}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((3\alpha_1+\alpha_2)\lambda-2\alpha_1\alpha_2)}\right) \\
&\quad \le \nu\left(\frac{2\alpha_2\tau}{\sigma(\alpha_2+\tau)+2(1-\sigma)\tau}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_1(\alpha_2+\tau)+(1-\sigma)(\alpha_1+\alpha_2)\tau}\right) \qquad (31)
\end{aligned}
$$

holds for

$$
\begin{aligned}
\lambda_1 &= \frac{2\alpha_2\tau}{\sigma(\alpha_2+\tau)+2(1-\sigma)\tau}, \tau_2 = \frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+2(1-\sigma)\tau((\alpha_1+3\alpha_2)\lambda-\alpha_1\alpha_2)}, \\
\lambda_2 &= \frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_1(\alpha_2+\tau)+(1-\sigma)(\alpha_1+\alpha_2)}, \tau_1 = \frac{2\alpha_2\tau\lambda}{\sigma\lambda(\alpha_2+\tau)+2(1-\sigma)\alpha_2\tau}
\end{aligned}
$$

in Lemma 1.

The inequality

$$
\begin{aligned}
&\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((\alpha_2-\alpha_1)\lambda+2\alpha_1\alpha_2)}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+2(1-\sigma)\tau((\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)}\right) \\
&\quad \le \nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)}\right) \\
&\quad + \nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_1(\alpha_2+\tau)+2(1-\sigma)\alpha_2\tau}\right) \qquad (32)
\end{aligned}
$$

holds for

$$\tau_1 = \frac{2\alpha_1\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau((\alpha_2-\alpha_1)\lambda+2\alpha_1\alpha_2)},$$
$$\tau_2 = \frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_1(\alpha_2+\tau)+2(1-\sigma)\tau((\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)},$$
$$\lambda_1 = \frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_1(\alpha_2+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)} \text{ and } \lambda_2 = \frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_1(\alpha_2+\tau)+2(1-\sigma)\alpha_2\tau}$$

in Lemma 1.

The inequality

$$\begin{aligned}\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((3\alpha_1+\alpha_2)\lambda-2\alpha_1\alpha_2)}\right)&\\ +\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+2(1-\sigma)\alpha_1\tau}\right) &\le \nu\left(\frac{2\alpha_1\tau}{\sigma(\alpha_1+\tau)+2(1-\sigma)\tau}\right)\\ &+\nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+(1-\sigma)(\alpha_1+\alpha_2)\tau}\right)\end{aligned} \tag{33}$$

holds for

$$\tau_1 = \frac{2\alpha_1\tau\lambda}{\sigma\lambda(\alpha_1+\tau)+2(1-\sigma)\alpha_1\tau}, \lambda_2 = \frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+(1-\sigma)(\alpha_1+\alpha_2)\tau}$$
$$\tau_2 = \frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((3\alpha_1+\alpha_2)\lambda-2\alpha_1\alpha_2)}, \lambda_1 = \frac{2\alpha_1\tau}{\sigma(\alpha_1+\tau)+2(1-\sigma)\tau}$$

in Lemma 1.

The inequality

$$\begin{aligned}\nu\left(\frac{2\alpha_1\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((\alpha_2-\alpha_1)\lambda+2\alpha_1\alpha_2)}\right)&\\ +\nu\left(\frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+2(1-\sigma)\tau((\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)}\right)&\\ \le \nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+2(1-\sigma)\alpha_2\tau}\right)&\\ +\nu\left(\frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)}\right)&\end{aligned} \tag{34}$$

holds for

$$\tau_1 = \frac{2\alpha_1\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau((\alpha_2-\alpha_1)\lambda+2\alpha_1\alpha_2)},$$
$$\tau_2 = \frac{2\alpha_1\alpha_2\tau\lambda}{\sigma\lambda\alpha_2(\alpha_1+\tau)+2(1-\sigma)\tau((\alpha_1+\alpha_2)\lambda-\alpha_1\alpha_2)},$$
$$\lambda_1 = \frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+(1-\sigma)\tau(\alpha_1+\alpha_2)} \text{ and } \lambda_2 = \frac{2\alpha_1\alpha_2\tau}{\sigma\alpha_2(\alpha_1+\tau)+2(1-\sigma)\alpha_2\tau}$$

in Lemma 1.

Multiplying the inequality (31)–(34) by $\frac{\varrho(\tau)\varrho\left(\frac{\alpha_2\lambda}{2\alpha_2-\lambda}\right)}{\tau^2\lambda^2}$, integrating both sides over τ on $[\alpha_1,\alpha_2]$ and over λ on $\left[\alpha_1,\frac{4\alpha_1\alpha_2}{\alpha_1+3\alpha_2}\right]$ using identity (30), we get

$$2\breve{\mathcal{K}}(\sigma) \le \left[\mathcal{K}(\sigma)+\mathcal{S}_\varrho(1-\sigma)\right]\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \tag{35}$$

for all $\sigma \in [0,1]$. From (18) and (35), we derive the first inequality of (29). $\square$

Remark 4. *If $\varrho(\tau) = \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}$, $\tau \in [\alpha_1, \alpha_2]$ in Theorem 11, then $\mathcal{K}(\sigma) = \mathcal{U}(\sigma)$, $\check{\mathcal{K}}(\sigma) = \mathcal{F}(\sigma)$ and $\mathcal{S}_\varrho(\sigma) = \mathcal{L}(\sigma)$, $\sigma \in [0,1]$ and Theorem 11 gives the inequality*

$$0 \le \mathcal{F}(\sigma) - \mathcal{U}(\sigma) \le \mathcal{L}(1-\sigma) - \mathcal{F}(\sigma) \tag{36}$$

which holds for all $\sigma \in [0,1]$.

The following two Fejér-type inequalities are natural consequences of Theorems 7–11 and we omit their proofs.

Theorem 12. *Let ν, ϱ, $\mathcal{G}_1$, $\mathcal{K}$, $\check{\mathcal{K}}$, $\mathcal{L}_\varrho$, $\mathcal{S}_\varrho$ be defined as above. Then, the following inequality holds for all $\sigma \in [0,1]$:*

$$\begin{aligned}
\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\left(\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right)^2 &\le \mathcal{K}(\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \\
&\le \check{\mathcal{K}}(\sigma) \le \frac{1}{2}[\mathcal{K}(\sigma)+\mathcal{S}_\varrho(1-\sigma)]\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \\
&\le \frac{1}{2}\left[\mathcal{G}_1(\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau+\mathcal{S}_\varrho(1-\sigma)\right]\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \\
&\le \frac{1}{2}[\mathcal{L}_\varrho(\sigma)+\mathcal{S}_\varrho(1-\sigma)]\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau+\frac{1}{2}\left[(1-\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\nu(\tau)\varrho(\tau)}{\tau^2}d\tau\right. \\
&\quad +\sigma\int_{\alpha_1}^{\alpha_2}\frac{1}{2}\left[\nu\left(\frac{2\alpha_1\tau}{\alpha_1+\tau}\right)+\nu\left(\frac{2\tau\alpha_2}{\tau+\alpha_2}\right)\right]\frac{\varrho(\tau)}{\tau^2}d\tau+\frac{\nu(\alpha_1)+\nu(\alpha_2)}{2} \\
&\quad \left.\times\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right]\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \le \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\left(\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right)^2
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\left(\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right)^2 &\le \mathcal{K}(\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \\
&\le \check{\mathcal{K}}(\sigma) \le \frac{1}{2}[\mathcal{K}(\sigma)+\mathcal{S}_\varrho(1-\sigma)]\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \\
&\le \frac{1}{2}\left[\mathcal{G}_1(\sigma)\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau+\mathcal{S}_\varrho(1-\sigma)\right]\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \\
&\le \frac{1}{2}[\mathcal{S}_\varrho(\sigma)+\mathcal{S}_\varrho(1-\sigma)]\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \\
&\le \frac{1}{2}\left[\frac{1}{2}\int_{\alpha_1}^{\alpha_2}\left[\nu\left(\frac{2\alpha_1\tau}{\alpha_1+\tau}\right)+\nu\left(\frac{2\tau\alpha_2}{\tau+\alpha_2}\right)\right]\frac{\varrho(\tau)}{\tau^2}d\tau+\frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right] \\
&\quad \times\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau \le \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\left(\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau\right)^2.
\end{aligned} \tag{38}$$

Corollary 2. *Let ν, $\mathcal{G}_1$, $\mathcal{U}$, $\mathcal{F}$, $\mathcal{L}$ be defined as above and $\varrho(\tau) = \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}$, $\tau \in [\alpha_1, \alpha_2]$, then we have from Theorem 12*

$$\begin{aligned}
\nu\left(\frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right) &\le \mathcal{U}(\sigma) \le \mathcal{F}(\sigma) \le \frac{1}{2}[\mathcal{U}(\sigma)+\mathcal{L}(1-\sigma)] \\
&\le \frac{1}{2}[\mathcal{G}_1(\sigma)+\mathcal{L}(1-\sigma)] \le \frac{1}{2}[\mathcal{L}(\sigma)+\mathcal{L}(1-\sigma)] \\
&\le \frac{1}{2}\left[\frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\frac{\varrho(\tau)}{\tau^2}d\tau+\frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}\right] \le \frac{\nu(\alpha_1)+\nu(\alpha_2)}{2}.
\end{aligned} \tag{39}$$

3. Conclusions

Researchers are using different generalizations of the convex sets and convex functions in the ever growing topic of mathematical inequalities and utilizing these generalizations of convex functions to prove new inequalities of Hermite-Hadamard and Fejér type. We employed harmonically convex functions to generalize results that have been proven for convex function. In this work, we defined new mappings over $[0, 1]$ and examined some interesting aspects of these mappings and refined Hermite–Hadamard and Fejér-type inequalities for harmonically convex functions. We hope the outcomes of this study will inspire mathematicians and young researchers to enter this subject.

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