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Fekete-Szegö Inequalities for Some Certain Subclass of Analytic Functions Defined with Ruscheweyh Derivative Operator

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Abstract: In our present investigation, we introduce and study some new subclasses of analytic functions associated with Ruscheweyh differential operator D^r. We obtain a Fekete–Szegö inequality for certain normalized analytic function defined on the open unit disk for which $\left[(D^r l)'(z)\right]^{\vartheta}\left(\frac{z(D^r l)'(z)}{(D^r l)(z)}\right)^{1-\vartheta} \prec e^z$ $(0 \leq \vartheta \leq 1)$ lies in a starlike region with respect to 1 and symmetric with respect to the real axis. As a special case of this result, the Fekete–Szegö inequality for a class of functions defined through Poisson distribution series is obtained.

Keywords: Fekete–Szegö problem; analytic functions; starlike and convex functions; subordination; Ruscheweyh differential operator; Poisson distribution series

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1. Introduction

Let $\mathcal{A}$ denote the class of functions l of the form:

$$l(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions that are univalent in $\mathfrak{D}$. If l and h are analytic in $\mathfrak{D}$, we say that l is subordinate to h, written as $l \prec h$ in $\mathfrak{D}$ or $l(z) \prec h(z)$ $(z \in \mathfrak{D})$, if there exists a Schwarz function $\omega(z)$ that is analytic in $\mathfrak{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathfrak{D})$ such that $l(z) = h(\omega(z))$ $(z \in \mathfrak{D})$. In particular, if the function $h(z)$ is univalent in $\mathfrak{D}$, then the following equivalence holds (see [1]):

$$l(z) \prec h(z) \Longleftrightarrow l(0) = h(0) \text{ and } l(\mathfrak{D}) \subset h(\mathfrak{D}).$$

For a constant $0 \leqslant \alpha < 1$, a function l in $\mathcal{A}$ is called starlike of order α if

$$\Re\left(\frac{zl'(z)}{l(z)}\right) > \alpha$$

for $z \in \mathfrak{D}$, denoted by $\mathcal{S}^*(\alpha)$. Note that the class $\mathcal{S}^*(0) = \mathcal{S}^*$ is known to consist of starlike functions in $\mathcal{A}$.

For a constant $0 \leqslant \alpha < 1$, a function l in $\mathcal{A}$ is called convex of order α if

$$\Re\left(1 + \frac{zl''(z)}{l'(z)}\right) > \alpha$$

for $z \in \mathfrak{D}$ denoted by $\mathcal{C}(\alpha)$. Note that the class $\mathcal{C}(0) = \mathcal{C}$ is known to consist of convex functions in $\mathcal{A}$.

By definition, it is obvious that for $0 \leqslant \alpha < 1$,

$$\mathcal{C}(\alpha) \subset \mathcal{C} \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}.$$

Nasr and Aouf (see [2]), Wiatrowski (see [3]), and Nasr and Aouf (see [4]) investigated some properties of $\alpha-$starlikeness and $\alpha-$convexity.

The familiar coefficient conjecture for the functions $l \in \mathcal{S}$ having the series form (1), was given by Bieberbach in 1916 and it was later proved by Louis de-Branges [5] in 1985. It was one of the most celebrated conjectures in classical analysis, one that has stood as a challenge to mathematicians for a very long time. Numerous mathematicians studied to calculate this conjecture, and due to this, they were able to derive coefficient bounds for various subfamilies of the class $\mathcal{S}$ of univalent functions.

Ma and Minda [6] established two classes of analytical functions;

$$\mathcal{S}^*(\varphi) = \left\{ l \in \mathcal{A}\colon \frac{zl'(z)}{l(z)} \prec \varphi(z),\ (z \in \mathfrak{D}) \right\}$$

and

$$\mathcal{C}(\varphi) = \left\{ l \in \mathcal{A}\colon 1 + \frac{zl''(z)}{l'(z)} \prec \varphi(z),\ (z \in \mathfrak{D}) \right\},$$

where the function φ is an analytic univalent function that maps $\mathfrak{D}$ onto a region that is starlike with respect to 1 and symmetric with respect to the real axis, and $\Re(\varphi(z)) > 0$ in $\mathfrak{D}$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$. By choosing $\vartheta = 0$, $\vartheta = 1$, $r = 0$ and changing the function φ several well-known classes can be obtained as the following:

1. For $\varphi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, we obtain the class $\mathcal{S}^*(A, B)$, for more information see [7].
2. $\mathcal{S}^*(\alpha) = \mathcal{S}^*(1 - 2\alpha, -1)$ is displayed in [8] for various values of A and B.
3. For $\varphi(z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, the class was described and investigated in [9].
4. For $\varphi(z) = \sqrt{1+z}$, the class is denoted by $\mathcal{S}^*_L$. Further research on this class can be found in [10,11].
5. For $\varphi(z) = z + \sqrt{1+z^2}$ the class is indicated by $\mathcal{S}^*_l$, for further information see [12].
6. If $\varphi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, then this class, denoted by $\mathcal{S}^*_C$, was first presented by [13] and was later researched by [14].
7. For $\varphi(z) = e^z$, the class $\mathcal{S}^*_e$ was defined and researched in [15,16].
8. For $\varphi(z) = \cos z$ the class is denoted by $\mathcal{S}^*_{\cosh}$, for more information see [17].
9. The class is indicated for $\varphi(z) = 1 + \sin z$ by $\mathcal{S}^*_{\sin}$ see [18]. For further information and additional research, see [19].

Recently in [14,20–23] by selecting a specific function for φ as described above, inequalities relating to the coefficient bounds of several subclasses of univalent functions have been thoroughly addressed. One of the inequalities Fekete and Szegö (1933) discovered for the coefficients of univalent analytic functions and connected to the Bieberbach conjecture is the Fekete–Szegö inequality.

The Fekete–Szegö functional is also known as the functional $a_3 - a_2^2$, and it is typical to discuss the more generalized functional $a_3 - \eta a_2^2$ where η is a real number (see [24]). The Fekete–Szegö problem is the estimation of $\left|a_3 - \eta a_2^2\right|$'s upper bound.

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} 3 - 4\eta & \text{if} \quad \eta \leqslant 0, \\ 1 + 2\exp\left(\frac{-2\eta}{1-\eta}\right) & \text{if} \quad 0 < \eta < 1, \\ 4\eta - 3 & \text{if} \quad \eta \geqslant 1. \end{cases}$$

It is well known that $\left|a_3 - a_2^2\right| \leqslant 1$ for $l \in \mathcal{S}$ given by (1). This is known as classic Fekete–Szegö's theorem (see [24]) and the inequality is sharp. Pfluger (see [25]) has since taken into account the complex values of η and given

$$\left|a_3 - \eta a_2^2\right| \leqslant 1 + 2\left|\exp\left(\frac{-2\eta}{1-\eta}\right)\right|.$$

For the classes of starlike and convex functions, the Fekete–Szegö problem was resolved in 1969 by Keogh and Merkes [26]. The publication by Orhan et al. [27] contains special cases of the Fekete–Szegö problem for the classes of starlike functions of order η and convex functions of order η.

In fact, a number of writers have studied the Fekete–Szegö problem for various subclasses of $\mathcal{A}$, for example, the upper bound for $\left|a_3 - \eta a_2^2\right|$ has been studied by a number of authors (see [27–31]).

Then the Hadamard product (or convolution) $l(z) * h(z)$ of $l(z)$ and $h(z)$ is defined by

$$(l * h)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (h * l)(z) \quad (z \in \mathfrak{D}),$$

where the function $h(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$ is also analytic in $\mathfrak{D}$.

For a function $l \in \mathcal{A}$ defined by (1), the Ruscheweyh derivative operator $D^r : \mathcal{A} \to \mathcal{A}$ (see [32]) is defined as follows:

$$(D^r l)(z) = \frac{z(z^{r-1} l(z))^{(r)}}{r!} = \frac{z}{(1-z)^{r+1}} * l(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(r+k)}{\Gamma(r+1)(k-1)!} a_k z^k (r > -1). \tag{2}$$

Let us start with the definition that follows.

Definition 1 ([33]). *Let $0 \leq \vartheta \leq 1$. A function $l \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*_{\exp}(r, \vartheta)$ if it satisfies the following subordination condition.*

$$\left[(D^r l)'(z)\right]^{\vartheta} \left(\frac{z(D^r l)'(z)}{(D^r l)(z)}\right)^{1-\vartheta} \prec e^z$$

where $(D^r l)(z)$ is defined by (2).

Note that,

$$\mathcal{S}^*_{\exp}(r, 0) = \mathcal{S}^*_{\exp}(r; e^z) = \left\{ l \in \mathcal{A} \colon \left(\frac{z(D^r l)'(z)}{(D^r l)(z)}\right) \prec e^z \right\}$$

and

$$\mathcal{S}^*_{\exp}(r, 1) = \mathcal{S}_{\exp}(r) = \left\{ l \in \mathcal{A} \colon \left((D^r l)'(z)\right) \prec e^z \right\}.$$

2. Main Results

As is usually the case, we let p be the family of functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ regular and with $p(0) = 1$ and $\Re p(z) > 0$ for z in $\mathfrak{D}$. We denote by the symbol $\mathcal{P}$ the family of all functions p, analytic in $\mathfrak{D}$. The following lemmas allow us to prove our next theorem.

Lemma 1 ([34]). *Let $p(z) \in \mathcal{P}$, then*

$$|c_k| \leqslant 2, \quad \textit{for } k \geqslant 1.$$

If $|c_1| = 2$ *then* $p(z) \equiv p_1(z) = (1+\gamma_1 z)/(1-\gamma_1 z)$ *with* $\gamma_1 = c_1/2$. *Conversely, if* $p(z) \equiv p_1(z)$ *for some* $|\gamma_1| = 1$, *then* $c_1 = 2\gamma_1$ *and* $|c_1| = 2$. *Furthermore, we have*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leqslant 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ *and* $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|^2}{2}$, *then* $p(z) \equiv p_2(z)$, *where*

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z + \gamma_1}{1+\bar{\gamma}_1\gamma_2 z}}{1 - z\frac{\gamma_2 z + \gamma_1}{1+\bar{\gamma}_1\gamma_2 z}},$$

and $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4-|c_1|^2}$. *Inversely, if* $p(z) \equiv p_2(z)$, *then* $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4-|c_1|^2}$ *and* $\left|c_2 - \frac{c_1^2}{2}\right| \leqslant 2 - \frac{|c_1|^2}{2}$ *for some* $|\gamma_1| < 1$ *and* $|\gamma_2| = 1$.

Lemma 2 ([35]). *Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, *then for* $v \in \mathbb{C}$

$$\left|c_2 - v c_1^2\right| \leqslant 2\max\{1, |2v-1|\},$$

and for the functions provided by, the conclusion is sharp

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 3 ([6]). *Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, *then*

$$\left|c_2 - v c_1^2\right| \leqslant \begin{cases} -4v+2, & \text{if} \quad v \le 0, \\ 2, & \text{if} \quad 0 \le v \le 1, \\ 4v-2, & \text{if} \quad v \ge 1. \end{cases}$$

If $v < 0$ *or* $v > 1$, *the equality holds if and only if* $p(z)$ *is* $(1+z)/(1-z)$ *or one of its rotations. If* $0 < v < 1$, *then equality holds if and only if* $p(z)$ *is* $\left(1+z^2\right)/\left(1-z^2\right)$ *or one of its rotations. If and only if* $v = 0$, *or one of its rotations, the equality holds true.*

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \le \lambda \le 1).$$

Only when p *is the reciprocal of one of the functions that guarantee the equality when* $v = 0$ *does the equality hold if* $v = 1$ *and only in that case.*

We begin with the following result.

3. Coefficient Bounds and the Fekete-Szegö Inequality for $l \in \mathcal{S}^*_{\exp}(r, \vartheta)$

We will establish the bounds on the coefficients for the function class $\mathcal{S}^*_{\exp}(r, \vartheta)$ in the first theorem.

Theorem 1. *If* $l \in \mathcal{S}^*_{\exp}(r, \vartheta)$ *and* l *is defined by* (1), *then*

$$|a_2| \leqslant \frac{1}{(1+\vartheta)(r+1)}, \tag{3}$$

$$|a_3| \leqslant \frac{2}{(2+\vartheta)(r+1)(r+2)} \max\left\{1, \left|\frac{\vartheta+3}{2(1+\vartheta)^2}\right|\right\} \tag{4}$$

and

$$\left|a_3-\eta a_2^2\right| \leq \frac{2}{(2+\vartheta)(r+1)(r+2)} \max\left\{1, \left|\frac{(\vartheta+3)(r+1)-\eta(2+\vartheta)(r+2)}{2(1+\vartheta)^2(r+1)}\right|\right\}, \quad (5)$$

where $\eta \in \mathbb{C}$.

Proof. Given that $l \in \mathcal{S}^*_{\exp}(r,\vartheta)$ in accordance with the subordination relationship, a Schwarz function $\omega(z)$ with $\omega(0)=0$ and $|\omega(z)|<1$ exists, satisfying

$$\left[(D^r l)'(z)\right]^{\vartheta}\left(\frac{z(D^r l)'(z)}{(D^r l)(z)}\right)^{1-\vartheta} = e^{\omega(z)}.$$

Here,

$$\begin{aligned}
&\left[(D^r l)'(z)\right]^{\vartheta}\left(\frac{z(D^r l)'(z)}{(D^r l)(z)}\right)^{1-\vartheta} \\
= \;& 1+(1+\vartheta)(r+1)a_2 z \\
&+\frac{(2+\vartheta)}{2}\left[(r+1)(r+2)a_3-(1-\vartheta)(r+1)^2a_2^2\right]z^2 \\
&+\frac{(3+\vartheta)}{6}\left[(1-\vartheta)(2-\vartheta)(r+1)^3a_2^3-3(1-\vartheta)(r+1)^2(r+2)a_2a_3\right. \\
&\left.+(r+1)(r+2)(r+3)a_4\right]z^3+\ldots
\end{aligned} \quad (6)$$

Now, we define a function

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+c_1z+c_2z^2+c_3z^3+\ldots.$$

It is obvious that $p(z)\in\mathcal{P}$ and

$$\omega(z)=\frac{p(z)-1}{p(z)+1}=\frac{c_1}{2}z+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)z^2+\left(\frac{c_1^3}{8}+\frac{c_3}{2}-\frac{c_1c_2}{2}\right)z^3+\ldots \quad (7)$$

On the other hand,

$$\begin{aligned}
e^{\omega(z)} \;=\;& 1+\frac{c_1}{2}z+\left(\frac{c_2}{2}-\frac{c_1^2}{8}\right)z^2+\left(\frac{c_1^3}{48}+\frac{c_3}{2}-\frac{c_1c_2}{4}\right)z^3 \\
&+\left(\frac{c_1^4}{384}+\frac{c_4}{2}-\frac{c_2^2}{8}+\frac{c_1^2c_2}{16}-\frac{c_1c_3}{4}\right)z^4+\ldots
\end{aligned} \quad (8)$$

Comparing the coefficients of z, z^2, z^3 between the Equations (6) and (8), we obtain

$$a_2=\frac{c_1}{2(1+\vartheta)(r+1)}, \quad (9)$$

$$(2+\vartheta)\left[(r+1)(r+2)a_3-(1-\vartheta)(r+1)^2a_2^2\right]=c_2-\frac{c_1^2}{4}, \quad (10)$$

$$\begin{aligned}
&\frac{(3+\vartheta)}{3}\left[(1-\vartheta)(2-\vartheta)(r+1)^3a_2^3-3(1-\vartheta)(r+1)^2(r+2)a_2a_3\right. \\
&\left.+(r+1)(r+2)(r+3)a_4\right] \\
=\;& \frac{c_1^3}{24}+c_3-\frac{c_1c_2}{2}
\end{aligned} \quad (11)$$

Applying Lemma 1, we easily obtain

$$|a_2| \leqslant \frac{1}{(1+\vartheta)(r+1)},$$

$$\begin{aligned} a_3 &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left[c_2 - c_1^2\left(\frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}\right)\right] \qquad (12) \\ |a_3| &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left|c_2 - c_1^2\left(\frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}\right)\right| \\ &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left|c_2 - vc_1^2\right|, \end{aligned}$$

where $v = \frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}$. Now, by applying Lemma 2, we obtain

$$|a_3| \leqslant \frac{2}{(2+\vartheta)(r+1)(r+2)}\max\left\{1, \left|\frac{\vartheta+3}{2(1+\vartheta)^2}\right|\right\}.$$

From (9) and (12), we have

$$\begin{aligned} & a_3 - \eta a_2^2 \\ &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left[c_2 - c_1^2\left(\frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}\right)\right] - \eta c_1^2\frac{1}{4(1+\vartheta)^2(r+1)^2} \\ &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left[c_2 - c_1^2\left(\frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}\right) - \eta c_1^2\left(\frac{(2+\vartheta)(r+2)}{4(1+\vartheta)^2(r+1)}\right)\right] \qquad (13) \\ &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left[c_2 - c_1^2\left(\frac{(2\vartheta^2+3\vartheta-1)(r+1)+\eta(2+\vartheta)(r+2)}{4(1+\vartheta)^2(r+1)}\right)\right] \\ &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left\{c_2 - vc_1^2\right\}, \end{aligned}$$

where

$$v = \frac{(2\vartheta^2+3\vartheta-1)(r+1)+\eta(2+\vartheta)(r+2)}{4(1+\vartheta)^2(r+1)}.$$

Our result now follows by an application of Lemma 2 to get

$$\left|a_3 - \eta a_2^2\right| \leq \frac{2}{(2+\vartheta)(r+1)(r+2)}\max\left\{1, \left|\frac{(\vartheta+3)(r+1)-\eta(2+\vartheta)(r+2)}{2(1+\vartheta)^2(r+1)}\right|\right\}. \qquad (14)$$

This completes the proof of Theorem 1. □

Remark 1. *By taking $\eta = 1$ in Theorem 1, we have*

$$\left|a_3 - a_2^2\right| \leq \frac{2}{(2+\vartheta)(r+1)(r+2)}\max\left\{1, \left|\frac{(\vartheta+3)(r+1)-(2+\vartheta)(r+2)}{2(1+\vartheta)^2(r+1)}\right|\right\}.$$

Remark 2. *If $\eta = 1$, $\vartheta = 0$ in Theorem 1 and $l \in \mathcal{S}^*_{\exp}(r)$, then we obtain*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{(r+1)(r+2)}\max\left\{1, \left|\frac{3(r+1)-2(r+2)}{2(r+1)}\right|\right\}$$

and if $\eta = 1$, $\vartheta = 1$ *in Theorem 1 and* $l \in \mathcal{S}_{\exp}(r)$, *we have*

$$\left|a_3 - a_2^2\right| \leq \frac{2}{3(r+1)(r+2)} \max\left\{1, \left|\frac{4(r+1) - 3(r+2)}{8(r+1)}\right|\right\}.$$

Corollary 1. *If* $r = 0$ *in Remark 2 and* $l \in \mathcal{S}^*_{\exp}$, *then we obtain*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{2}$$

and if $r = 0$ *in Remark 2 and* $l \in \mathcal{S}_{\exp}$, *then we have*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{3}.$$

Theorem 2. *If the function* $l \in \mathcal{S}^*_{\exp}(r, \vartheta)$ *and is of the form (1), then for* $\eta \in \mathbb{R}$,

$$\left|a_3 - \eta a_2^2\right| \leqslant \begin{cases} -\frac{1}{(2+\vartheta)(r+1)(r+2)}\left(\frac{-(\vartheta+3)(r+1)}{(1+\vartheta)^2(r+1)} + \frac{\eta(2+\vartheta)(r+2)}{(1+\vartheta)^2(r+1)}\right), & \text{if} \quad \eta < \rho_1, \\ \frac{2}{(2+\vartheta)(r+1)(r+2)}, & \text{if} \quad \rho_1 \leq \eta \leq \rho_2, \\ \frac{1}{(2+\vartheta)(r+1)(r+2)}\left(\frac{-(\vartheta+3)(r+1)}{(1+\vartheta)^2(r+1)} + \frac{\eta(2+\vartheta)(r+2)}{(1+\vartheta)^2(r+1)}\right), & \text{if} \quad \eta > \rho_2, \end{cases},$$

where

$$\rho_1 = \frac{-\left(2\vartheta^2 + 3\vartheta - 1\right)(r+1)}{(2+\vartheta)(r+2)} \quad \text{and} \quad \rho_2 = \frac{\left(2\vartheta^2 + 5\vartheta + 5\right)(r+1)}{(2+\vartheta)(r+2)}.$$

Proof. From (14), we have

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left[c_2 - c_1^2\left(\frac{\left(2\vartheta^2 + 3\vartheta - 1\right)(r+1) + \eta(2+\vartheta)(r+2)}{4(1+\vartheta)^2(r+1)}\right)\right] \\ &= \frac{1}{(2+\vartheta)(r+1)(r+2)}\left(c_2 - v c_1^2\right) \end{aligned}$$

where

$$v = \frac{\left(2\vartheta^2 + 3\vartheta - 1\right)(r+1) + \eta(2+\vartheta)(r+2)}{4(1+\vartheta)^2(r+1)}.$$

By an application of Lemma 3, the conclusion of Theorem 2 follows. Thus, the proof of Theorem 2 is finished. □

4. Coefficient Inequalities for l^{-1}

Theorem 3. *In the event that* $l \in \mathcal{S}^*_{\exp}(r, \vartheta)$, *which is given by (1) and* $l^{-1}(w) = w + \sum_{k=2}^{\infty} d_k w^k$ *of the inverse function of* l *with* $|w| < r_0$, *where* $r_0 > \frac{1}{4}$ *is the radius of the Koebe domain, which is the analytic continuation to* $\mathfrak{D}$, *then for any* $\eta \in \mathbb{C}$, *we obtain*

$$|d_2| \leq \frac{1}{(1+\vartheta)(r+1)}, \tag{15}$$

$$|d_3| \leq \frac{2}{(2+\vartheta)(r+1)(r+2)} \max\left\{1, \left|\frac{(\vartheta+3)(r+1) - 2(2+\vartheta)(r+2)}{2(1+\vartheta)^2(r+1)}\right|\right\} \tag{16}$$

and

$$\left|d_3 - \eta d_2^2\right| \leqslant \frac{2}{(2+\vartheta)(r+1)(r+2)} \max\left\{1, \left|\frac{(\vartheta+3)(r+1) + (\eta+2)(2+\vartheta)(r+2)}{2(1+\vartheta)^2(r+1)}\right|\right\}. \tag{17}$$

Proof. Since

$$l^{-1}(w) = w + \sum_{k=2}^{\infty} d_k w^k \tag{18}$$

is the inverse of l,

$$l^{-1}(l(z)) = l\left(l^{-1}(z)\right) = z. \tag{19}$$

From Equation (19), we have

$$l^{-1}\left(z + \sum_{k=2}^{\infty} a_k z^k\right) = z. \tag{20}$$

Thus, (19) and (20) yield

$$z + (a_2 + d_2)z^2 + (a_3 + 2a_2 d_2 + d_3)z^3 + \ldots = z, \tag{21}$$

Thus, equating the respective coefficients of z, it can be seen that

$$d_2 = -a_2, \tag{22}$$

$$d_3 = 2a_2^2 - a_3. \tag{23}$$

From relations (9), (12), (22) and (23)

$$d_2 = -\frac{c_1}{2(1+\vartheta)(r+1)}, \tag{24}$$

$$\begin{aligned} d_3 &= \frac{c_1^2}{2(1+\vartheta)^2(r+1)^2} \\ &\quad - \frac{1}{(2+\vartheta)(r+1)(r+2)}\left[c_2 - c_1^2\left(\frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}\right)\right] \\ &= -\frac{1}{(2+\vartheta)(r+1)(r+2)} \\ &\quad \times\left[c_2 - c_1^2\left(\frac{\left(2\vartheta^2+3\vartheta-1\right)(r+1) - 2(2+\vartheta)(r+2)}{4(1+\vartheta)^2(r+1)}\right)\right] \end{aligned} \tag{25}$$

We obtain (15) and (16) by using Lemma 2 and taking the modulus on both sides. Think about any complex number η.

$$\begin{aligned} d_3 - \eta d_2^2 &= \frac{-1}{(2+\vartheta)(r+1)(r+2)} \\ &\quad \times\left[c_2 - c_1^2\left(\frac{\left(2\vartheta^2+3\vartheta-1\right)(r+1) - 2(2+\vartheta)(r+2)}{4(1+\vartheta)^2(r+1)}\right.\right. \\ &\quad \left.\left. - \frac{\eta(2+\vartheta)(r+2)}{4(1+\vartheta)^2(r+1)}\right)\right] \end{aligned} \tag{26}$$

By applying Lemma 2 on the right side of (26) and taking the modulus on both sides, one can arrive at the same conclusion as in (17).

The proof is now complete. □

5. Functions Described by the Poisson Distribution

If a variable χ takes the values $0, 1, 2, 3, \ldots$ with probability, it is said to have a Poisson distribution $e^{-\xi}, \xi\frac{e^{-\xi}}{1!}, \xi^2\frac{e^{-\xi}}{2!}, \xi^3\frac{e^{-\xi}}{3!}, \ldots$ respectively, where ξ is called the parameter. Thus,

$$P(\chi = \tau) = \xi^{\tau}\frac{e^{-\xi}}{\tau!}, \quad \tau = 0, 1, 2, \ldots .$$

Porwal [36] introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{I}(\xi, z) = z + \sum_{k=2}^{\infty}\frac{\xi^{k-1}}{(k-1)!}e^{-\xi}z^k, \quad (z \in \mathfrak{D}),$$

where $\xi > 0$. We observe that the radius of convergence of the above series is infinite, as can be verified by the ratio test. Due to recent research on [36,37], let the linear operator

$$\mathcal{I}^{\xi}(z) : \mathcal{A} \longrightarrow \mathcal{A}$$

be given by

$$\begin{aligned}
\left(\mathcal{I}^{\xi}D^r l\right)(z) &= \mathcal{I}(\xi, z) * (D^r l)(z) \\
&= z + \sum_{k=2}^{\infty}\left[\frac{\xi^{k-1}}{(k-1)!}e^{-\xi}\frac{\Gamma(r+k)}{\Gamma(r+1)(k-1)!}\right]a_k z^k \\
&= z + \sum_{k=2}^{\infty}\Upsilon_k(\xi, r)a_k z^k,
\end{aligned}$$

where $\Upsilon_k(\xi, r) = \frac{\xi^{k-1}}{(k-1)!}e^{-\xi}\frac{\Gamma(r+k)}{\Gamma(r+1)(k-1)!}$ and $*$ stand for the Hadamard product or convolution of two series. In particular,

$$\Upsilon_2(\xi, r) = \xi e^{-\xi}(r+1), \quad \Upsilon_3(\xi, r) = \frac{1}{4}\xi^2 e^{-\xi}(r+1)(r+2). \tag{27}$$

According to the definition below, the class $\mathcal{S}^*_{\exp}(r, \vartheta; \Upsilon)$ is:

$$\mathcal{S}^*_{\exp}(r, \vartheta; \Upsilon) = \left\{l \in \mathcal{A}: \ \left(\mathcal{I}^{\xi}D^r l\right)(z) \in \mathcal{S}^*_{\exp}(r, \vartheta; \Upsilon)\right\}.$$

where by Definition 1 provides $\mathcal{S}^*_{\exp}(r, \vartheta; \Upsilon)$ and

$$\left(\mathcal{I}^{\xi}D^r l\right)(z) = z + \Upsilon_2(\xi, r)a_2 z^2 + \Upsilon_3(\xi, r)a_3 z^3 + \Upsilon_4(\xi, r)a_4 z^4 + \ldots .$$

The same method used in Theorems 1 and 2 can be used to obtain the coefficient bound for functions in $\mathcal{S}^*_{\exp}(r, \vartheta; \Upsilon)$ from the equivalent bounds for functions in $\mathcal{S}^*_{\exp}(r, \vartheta)$.

Theorem 4. *Let $0 \le \vartheta \le 1$ and $\left(\mathcal{I}^{\xi}D^r l\right)(z) = z + \Upsilon_2(\xi, r)a_2 z^2 + \Upsilon_3(\xi, r)a_3 z^3 + \Upsilon_4(\xi, r)a_4 z^4 + \ldots$. If $l \in \mathcal{S}^*_{\exp}(r, \vartheta; \Upsilon)$, then for $\eta \in \mathbb{C}$, we have*

$$\left|a_3 - \eta a_2^2\right| \leqslant \frac{1}{(2+\vartheta)\Upsilon_3(\xi, r)}\max\left\{1, \left|\frac{\eta(2+\vartheta)\Upsilon_3(\xi, r)}{(1+\vartheta)^2\Upsilon_2^2(\xi, r)} - \frac{\vartheta+3}{2(1+\vartheta)^2}\right|\right\}. \tag{28}$$

Proof. Since $l \in \mathcal{S}^*_{\exp}(r, \vartheta; \Upsilon)$, for $\left(\mathcal{I}^{\xi}D^r l\right)(z) = z + \Upsilon_2(\xi, r)a_2 z^2 + \Upsilon_3(\xi, r)a_3 z^3 + \Upsilon_4(\xi, r)a_4 z^4 + \ldots$, we have

$$\left[\left(\mathcal{I}^{\xi}D^r l\right)'(z)\right]^{\vartheta}\left(\frac{z\left(\mathcal{I}^{\xi}D^r l\right)'(z)}{\left(\mathcal{I}^{\xi}D^r l\right)(z)}\right)^{1-\vartheta} = e^{\omega(z)}$$

By (6), we can easily obtain

$$\begin{aligned}
&\left[\left(\mathcal{I}^{\xi}D^{r}l\right)'(z)\right]^{\vartheta}\left(\frac{z\left(\mathcal{I}^{\xi}D^{r}l\right)'(z)}{\left(\mathcal{I}^{\xi}D^{r}l\right)(z)}\right)^{1-\vartheta} \\
=\;& 1+(1+\vartheta)\Upsilon_2(\xi,r)a_2z+\frac{(2+\vartheta)}{2}\left[2\Upsilon_3(\xi,r)a_3-(1-\vartheta)\Upsilon_2^2(\xi,r)a_2^2\right]z^2 \\
&+\frac{(3+\vartheta)}{6}\Big[(1-\vartheta)(2-\vartheta)\Upsilon_2^3(\xi,r)a_2^3 \\
&-6(1-\vartheta)\Upsilon_2(\xi,r)\Upsilon_3(\xi,r)a_2a_3-6\Upsilon_4(\xi,r)a_4\Big]z^3+\ldots. \qquad (29)
\end{aligned}$$

Thus, by (29) and (8), we have

$$\begin{aligned}
&1+(1+\vartheta)\Upsilon_2(\xi,r)a_2z+\frac{(2+\vartheta)}{2}\left[2\Upsilon_3(\xi,r)a_3-(1-\vartheta)\Upsilon_2^2(\xi,r)a_2^2\right]z^2 \\
&+\frac{(3+\vartheta)}{6}\left[(1-\vartheta)(2-\vartheta)\Upsilon_2^3(\xi,r)a_2^3-6(1-\vartheta)\Upsilon_2(\xi,r)\Upsilon_3(\xi,r)a_2a_3-6\Upsilon_4(\xi,r)a_4\right]z^3+\ldots \\
=\;& 1+\frac{c_1}{2}z+\left(\frac{c_2}{2}-\frac{c_1^2}{8}\right)z^2+\left(\frac{c_1^3}{48}+\frac{c_3}{2}-\frac{c_1c_2}{4}\right)z^3+\ldots
\end{aligned}$$

Now, by equating corresponding coefficients of z, z^2 and proceeding as in Theorem 1,

$$a_2=\frac{c_1}{2(1+\vartheta)\Upsilon_2(\xi,r)}, \qquad (30)$$

$$a_3=\frac{1}{2(2+\vartheta)\Upsilon_3(\xi,r)}\left[c_2-c_1^2\left(\frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}\right)\right]. \qquad (31)$$

From (30) and (31), we obtain

$$\begin{aligned}
a_3-\eta a_2^2 &= \frac{1}{2(2+\vartheta)\Upsilon_3(\xi,r)}\left[c_2-c_1^2\left(\frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}\right)\right]-\frac{\eta c_1^2}{4(1+\vartheta)^2\Upsilon_2^2(\xi,r)} \\
&= \frac{1}{2(2+\vartheta)\Upsilon_3(\xi,r)}\left[c_2-c_1^2\left(\frac{2\vartheta^2+3\vartheta-1}{4(1+\vartheta)^2}+\frac{2\eta(2+\vartheta)\Upsilon_3(\xi,r)}{4(1+\vartheta)^2\Upsilon_2^2(\xi,r)}\right)\right]. \qquad (32)
\end{aligned}$$

By using Lemma 2, we achieve the desired result.
Consequently, the proof of Theorem 4 is finished. □

Theorem 5. *Let* $0\le\vartheta\le 1$ *and* $\left(\mathcal{I}^{\xi}D^{r}l\right)(z)=z+\Upsilon_2(\xi,r)a_2z^2+\Upsilon_3(\xi,r)a_3z^3+\Upsilon_4(\xi,r)a_4z^4+\ldots$, *with* $\eta\in\mathbb{R}$, *then*

$$\left|a_3-\eta a_2^2\right|\leqslant\begin{cases}-\frac{1}{2(2+\vartheta)\Upsilon_3(\xi,r)}\left(\frac{-(\vartheta+3)}{(1+\vartheta)^2}+\frac{2\eta(2+\vartheta)\Upsilon_3(\xi,r)}{(1+\vartheta)^2\Upsilon_2^2(\xi,r)}\right), & \text{if} \quad \eta<\rho_1,\\ \frac{1}{(2+\vartheta)\Upsilon_3(\xi,r)}, & \text{if} \quad \rho_1\le\eta\le\rho_2,\\ \frac{1}{2(2+\vartheta)\Upsilon_3(\xi,r)}\left(\frac{-(\vartheta+3)}{(1+\vartheta)^2}+\frac{2\eta(2+\vartheta)\Upsilon_3(\xi,r)}{(1+\vartheta)^2\Upsilon_2^2(\xi,r)}\right), & \text{if} \quad \eta>\rho_2,\end{cases}$$

where

$$\rho_1=\frac{-\left(2\vartheta^2+3\vartheta-1\right)\Upsilon_2^2(\xi,r)}{2(2+\vartheta)\Upsilon_3(\xi,r)} \quad \textit{and} \quad \rho_2=\frac{\left(2\vartheta^2+5\vartheta+5\right)\Upsilon_2^2(\xi,r)}{2(2+\vartheta)\Upsilon_3(\xi,r)}.$$

In particular, by using $\Upsilon_2(\xi,r)=\xi e^{-\xi}(r+1)$ and $\Upsilon_3(\xi,r)=\frac{1}{4}\xi^2e^{-\xi}(r+1)(r+2)$, we may readily assert the results above that are connected to Poisson distribution series.

We accomplish the desired result by applying Lemma 3 and Equation (32).

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