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Review

On the Method of Differential Invariants for Solving Higher Order Ordinary Differential Equations

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Abstract: There are many routines developed for solving ordinary differential Equations (ODEs) of different types. In the case of an nth-order ODE that admits an r-parameter Lie group $(3 \leq r \leq n)$, there is a powerful method of Lie symmetry analysis by which the ODE is reduced to an $(n-r)$th-order ODE plus r quadratures provided that the Lie algebra formed by the infinitesimal generators of the group is solvable. It would seem this method is not widely appreciated and/or used as it is not mentioned in many related articles centred around integration of higher order ODEs. In the interest of mainstreaming the method, we describe the method in detail and provide four illustrative examples. We use the case of a third-order ODE that admits a three-dimensional solvable Lie algebra to present the gist of the integration algorithm.

Keywords: ordinary differential equation; lie symmetry analysis; solvable lie algebra; differential invariant; reduction of order

MSC: 34A05; 34C14; 34C20

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1. Introduction

The study of ODEs poses significant challenges, especially in cases involving equations of higher order that are nonlinear. As a result, various methods have been proposed for investigating different types of ODEs. Chandrasekar et al. [1], for example, propose a method that unifies and generalises known linearising transformations for finding general solutions of third-order nonlinear ODEs. Related work is done by Mohanasubha et al. [2] who propose a method of solution that involves deriving linearising transformations for a class of second-order nonlinear ordinary differential equations. In [3], conditions are provided for the linearisation of third-order ODEs by tangent transformations (see also the references in [3] for related work on the problem of transforming a given differential equation into a linear equation). It turns out that "symmetry properties", which are central in Lie symmetry analysis of differential equations, by and large, provide a basis for systematically solving the majority of ordinary differential equations for which exact solutions can be found [3–13].

There are several ways in which the symmetry group associated with a differential equation can be used to analyse the equation. For a given differential equation, the symmetry group may be used to derive new solutions of the equation from old ones [5,7], to reduce the order of the equation [5,7,8] or to establish whether or not the equation can be linearised, and to construct explicit linearisations when such exist [14–16]. Other uses include the derivation of conserved quantities [7].

Many symmetry-based approaches for solving ODEs involve reduction of order, whereby for a given ODE of order n, the problem is reduced to that of solving one or more ODEs of order at most $n-1$. Lie symmetry analysis has well-established algorithms for solution methods based on reduction of order. It is well known, in particular, that if an

nth-order ODE admits a one-parameter Lie symmetry group, then the order of the equation can be reduced by one. The method of differential invariants extends this in that an ODE of order n is reduced to an ODE of order $n-1$ plus r quadratures (where $3 \leq r \leq n$) provided that the ODE is invariant under an r-parameter Lie group whose infinitesimal generators form an r-dimensional solvable Lie algebra [5,12,17]. The method is essentially a general integration procedure for solving (or, at least, reduction of order of) any higher order ODE that admits a solvable lie algebra of the right dimension. It consists of a number of successive iterations that reduce the problem to integration of a number of first-order ODEs each of which has an admitted Lie point symmetry. Therefore, each of the first-order ODEs may be integrated routinely using the admitted Lie point symmetry [4–9]. It seems that the method of differential invariants has not been used widely to study higher order ODEs as we could not find many applications in the literature.

In this paper, we describe the method of differential invariants and provide four instructive examples involving nonlinear third-order ODEs that arise in different contexts.

The rest of the article is organised as follows: In Section 2, we present the algorithm of the method of differential invariants in the case where a third-order ODE admits a three-dimensional solvable Lie algebra. In Section 3, we provide four illustrative examples. We give concluding remarks in Section 4.

2. Reduction Algorithm for an nth-Order ODE ($n \geq 3$) with a Solvable Lie Algebra

Let us assume that an nth-order ODE admits an r-parameter Lie group of transformations. There is a reduction algorithm [5] by means of which the ODE can be reduced to an $(n-1)$th-order ODE plus r quadratures provided that the infinitesimal generators of the admitted Lie group form an r-dimensional solvable Lie algebra. We present the reduction algorithm in the simplified case involving a third-order ODE that admits a 3-parameter solvable Lie algebra. In this case, the reduction algorithm results in the general solution of the ODE.

Consider a third-order

$$f(x, y, y', y'', y''') = 0 \tag{1}$$

that admits a 3-parameter Lie group of point transformations, and for which the associated infinitesimal generators Y_1, Y_2, Y_3 form a solvable Lie algebra. Without loss of generality, we can assume that the infinitesimal generators have the following commutation relations:

$$[Y_i, Y_j] = \sum_{k=1}^{j-1} C_{ij}^k Y_k, \quad 1 \leq i < j, \quad j = 2, 3. \tag{2}$$

for some real structure constants C_{ij}^k [5].

Let $r_1(x, y)$, $v_1(x, y, y')$ be such that

$$Y_1 r_1 = 0, \quad Y_1^{(1)} v_1 = 0,$$

so that

$$w_1 = \frac{dv_1}{dr_1} \tag{3}$$

is a differential invariant, i.e., $Y_1^{(2)} w_1 = 0$. In terms of the invariants r_1 and v_1, and the differential invariant w_1, (1) is reduced to a second-order ODE

$$w_1 = \psi^1(r_1, v_1), \tag{4}$$

for some function ψ^1. Writing $Y_2^{(1)}$ in terms of r_1 and v_1, we obtain

$$Y_2^{(1)} = \alpha_1(r_1)\frac{\partial}{\partial r_1} + \beta_1(r_1, v_1)\frac{\partial}{\partial v_1}, \tag{5}$$

with the first extension given by

$$Y_2^{(2)} = Y_2^{(1)} + \gamma_1(r_1, v_1, w_1)\frac{\partial}{\partial w_1}, \tag{6}$$

where

$$\alpha_1(r_1) = Y_2 r_1, \quad \beta_1(r_1, v_1) = Y_2^{(1)} v_1, \quad \gamma_1(r_1, v_1, w_1) = Y_2^{(2)} w_1,$$

for some functions α_1, β_1 and γ_1. It is noteworthy that (5) is admitted by Equation (4).

Let $r_2(r_1, v_1)$, $v_2(r_1, v_1, w_1)$ be such that

$$Y_2^{(1)} r_2 = 0, \quad Y_2^{(2)} v_2 = 0,$$

so that

$$w_2 = \frac{dv_2}{dr_2} \tag{7}$$

is a differential invariant, i.e., $Y_2^{(3)} w_2 = 0$. In terms of the invariants r_2, v_2 and w_2, the ODE (1) reduces to a first-order ODE

$$w_2 = \psi^2(r_2, v_2), \tag{8}$$

for some function ψ^2. Writing $Y_3^{(2)}$ in terms of r_2 and v_2, we obtain

$$Y_3^{(2)} = \alpha_2(r_2)\frac{\partial}{\partial r_2} + \beta_2(r_2, v_2)\frac{\partial}{\partial v_2}, \tag{9}$$

with the first extension given by

$$Y_3^{(3)} = Y_3^{(2)} + \gamma_2(r_2, v_2, w_2)\frac{\partial}{\partial w_2}, \tag{10}$$

where

$$\alpha_2(r_2) = Y_3^{(1)} r_2, \quad \beta_2(r_2, v_2) = Y_3^{(2)} v_2, \quad \gamma_2(r_2, v_2, w_2) = Y_3^{(3)} w_2,$$

for some functions α_2, β_2 and γ_2. Here also (9) is admitted by Equation (8).

In light of the admitted symmetry (10), the first-order Equation (8) can be integrated routinely to give a solution of the form

$$v_2 = \omega^2(r_2) \tag{11}$$

for some function ω^2. Expressing (11) in terms of v_1 and r_1, we obtain a first-order ODE

$$\frac{dv_1}{dr_1} = \psi^1(v_1, r_1), \tag{12}$$

i.e., we determine the hitherto unknown function ψ^1 in (4). Solving Equation (12), we obtain a solution of the form

$$v_1 = \omega^1(r_1) \tag{13}$$

for some function ω^1. Again, the solution (13) can be expressed in terms of x and y to obtain the last first-order ODE in the form

$$\frac{dy}{dx} = \psi^0(x, y), \tag{14}$$

for some function ψ^0. Equation (14) admits Y_1 and, when solved, provides the general solution of Equation (1).

3. Illustrative Examples

In this section, we use the method of differential invariants to find general solutions of four third-order ODEs, each of which admits a symmetry Lie algebra of order greater than three. In each case, we identify a three-dimensional solvable subalgebra and use it to perform complete integration of the ODE.

Example 1. *Consider the ODE*

$$(y')^2 y'' - 2y(y'')^2 + yy'y''' = 0, \tag{15}$$

which arises in the context of group classification of the $1+1$ *Fokker–Planck diffusion-convection equation [18]*

$$\theta_t = [D(\theta)\theta_z]_z - K'(\theta)\theta_z, \tag{16}$$

where t *is time,* z *is the depth,* $\theta(t,z)$ *is the volumetric soil water content,* $D(\theta)$ *is the soil water diffusivity and* $K(\theta)$ *is the hydraulic conductivity, with* $K'(\theta) = \frac{dK}{d\theta} \neq 0$.

Besides the translation symmetries

$$X_1 = \frac{\partial}{\partial z} \quad \text{and} \quad X_2 = \frac{\partial}{\partial t}, \tag{17}$$

which are clearly admitted by (16), additional symmetries are possible only if D *solves this third-order nonlinear ODE [19]*

$$D'(\theta)^2 D''(\theta) - 2D(\theta)D''(\theta)^2 + D(\theta)D'(\theta)D'''(\theta) = 0, \tag{18}$$

which is Equation (15) with θ *and* D *replaced with* x *and* y*, respectively.*

Equation (15) admits a four-dimensional symmetry Lie algebra spanned by the operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = y\frac{\partial}{\partial y}, \quad X_4 = y\ln y\frac{\partial}{\partial y}. \tag{19}$$

We use the solvable algebra $\langle X_1, X_3, X_4\rangle$*, for which*

$$[X_3, X_4] = X_3 \tag{20}$$

is the only nonzero Lie bracket. We relabel the symmetries as follows:

$$X_3 \to Y_1, \quad X_4 \to Y_2, \quad X_1 \to Y_3,$$

to ensure that the commutation relations of the operators Y_1, Y_2 *and* Y_3 *satisfy (2).*

To carry out the reduction algorithm, we first need the following extended infinitesimal generators:

$$\left.\begin{array}{rcl} Y_1^{(1)} & = & y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} \\ Y_2^{(2)} & = & y\ln y\frac{\partial}{\partial y} + y'(1+\ln y)\frac{\partial}{\partial y'} + \left(\frac{y'^2}{y} + y'' + y''\ln y\right)\frac{\partial}{\partial y''} \\ Y_3^{(3)} & = & \frac{\partial}{\partial x}. \end{array}\right\} \tag{21}$$

Starting with $Y_1^{(1)}$*, we solve the corresponding characteristic equations*

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dy'}{y'} \tag{22}$$

to obtain invariants

$$r_1 = x, \quad v_1 = \frac{y'}{y}, \tag{23}$$

and derive the differential invariant

$$w_1 = \frac{dv_1}{dr_1} = \frac{yy'' - (y')^2}{y^2}. \tag{24}$$

Writing $Y_2^{(2)}$ in terms of r_1, v_1 and w_1, we obtain

$$Y_2^{(2)} = v_1 \frac{\partial}{\partial v_1} + w_1 \frac{\partial}{\partial w_1}. \tag{25}$$

From the corresponding characteristic equation

$$\frac{dr_1}{0} = \frac{dv_1}{v_1} = \frac{dw_1}{w_1}, \tag{26}$$

we obtain invariants

$$r_2 = r_1 \quad \text{and} \quad v_2 = \frac{w_1}{v_1}, \tag{27}$$

which, in view of (23), can be written in terms of x, y, y' and y'' as follows:

$$r_2 = x \quad \text{and} \quad v_2 = \frac{yy'' - (y')^2}{yy'}. \tag{28}$$

From (28) we derive the differential invariant

$$w_2 = \frac{dv_2}{dr_2} = \frac{(y')^2}{y^2} - \frac{y''}{y} + \frac{y'y''' - (y'')^2}{(y')^2}. \tag{29}$$

Equation (15) can now be reduced into a first-order ODE of the form

$$\frac{dv_2}{dr_2} = \psi^2(r_2, v_2)$$

for some function ψ^2. To find ψ^2, we express Equation (15) as

$$y''' = \frac{2y(y'')^2 - (y')^2 y''}{yy'}. \tag{30}$$

and replace y''' in (29) by the right hand-side of (30). We obtain

$$\frac{dv_2}{dr_2} = \left[\frac{yy'' - (y')^2}{yy'}\right]^2 = v_2^2, \tag{31}$$

which is a first-order ODE that admits $Y_3^{(2)}$ written in terms of r_2 and v_2, i.e.,

$$Y_3^{(2)} = \frac{\partial}{\partial r_2}. \tag{32}$$

Solving (31) we obtain

$$v_2 = -\frac{1}{r_2 + \kappa_1}, \tag{33}$$

where κ_1 is an arbitrary constant. In terms of r_1 and v_1, Equation (33) is transformed, via (27), into another first-order ODE,

$$\frac{dv_1}{dr_1} = -\frac{v_1}{r_1 + \kappa_1}, \tag{34}$$

which admits symmetry (25). Equation (34) is another simple ODE, the solution of which is

$$v_1 = \frac{\kappa_2}{r_1 + \kappa_1}, \tag{35}$$

where κ_2 is another arbitrary constant. Using (23), we write (35) as a first-order ODE in the variables x and y, namely

$$y' = \frac{\kappa_2 y}{x + \kappa_1}, \tag{36}$$

which admits symmetry Y_1 from (21). Equation (36) is the last first-order ODE in the series of iterations and is also a simple variables-separable equation. The solution of (36) is

$$y = \kappa_3 (x + \kappa_1)^{\kappa_2}, \tag{37}$$

where κ_3 is a further arbitrary constant. This is in fact the general solution of Equation (15).

Example 2. *Consider the nonlinear ODE*

$$y''' = \frac{3}{2}\frac{y''2}{y'}, \tag{38}$$

which is the canonical form of every third ODE that admits a transitive fiber-preserving six-dimensional point symmetry group [20].

Equation (38) admits a six-dimensional symmetry Lie algebra L_6 spanned by the operators

$$\left.\begin{array}{lll} X_1 = \frac{\partial}{\partial x} & X_2 = x\frac{\partial}{\partial x} & X_3 = x^2\frac{\partial}{\partial x} \\ X_4 = \frac{\partial}{\partial y} & X_5 = y\frac{\partial}{\partial y} & X_6 = y^2\frac{\partial}{\partial y}. \end{array}\right\} \tag{39}$$

The symmetries X_2, X_3 and X_4 span a solvable Lie algebra which has

$$[X_2, X_3] = X_3 \tag{40}$$

as the only nonzero Lie bracket. With relabelling

$$X_3 \to Y_1, \quad X_2 \to Y_2, \quad X_4 \to Y_3,$$

the commutation relations of the operators Y_1, Y_2 and Y_3 satisfy (2).

We extend the identified infinitesimal generators:

$$\left.\begin{array}{lll} Y_1^{(1)} & = & x^2\frac{\partial}{\partial x} - 2xy'\frac{\partial}{\partial y'} \\ Y_2^{(2)} & = & x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''} \\ Y_3^{(3)} & = & \frac{\partial}{\partial y}. \end{array}\right\} \tag{41}$$

Solving the characteristic equations

$$\frac{dx}{x^2} = \frac{dy}{0} = \frac{dy'}{-2xy'} \tag{42}$$

arising from $Y_1^{(1)}$, we obtain invariants

$$r_1 = y, \quad v_1 = x^2 y', \tag{43}$$

and derive the differential invariant

$$w_1 = \frac{dv_1}{dr_1} = x\left(\frac{xy''}{y'} + 2\right). \tag{44}$$

In terms of r_1, v_1 and w_1, $Y_2^{(2)}$ becomes

$$Y_2^{(2)} = v_1 \frac{\partial}{\partial v_1} + w_1 \frac{\partial}{\partial w_1}. \tag{45}$$

From the corresponding characteristic equation

$$\frac{dr_1}{0} = \frac{dv_1}{v_1} = \frac{dw_1}{w_1}, \tag{46}$$

we obtain the next set of invariants

$$r_2 = r_1 \quad \text{and} \quad v_2 = \frac{w_1}{v_1}, \tag{47}$$

which, in view of (43), can be written in terms of x, y, y' and y'' as follows:

$$r_2 = y \quad \text{and} \quad v_2 = \frac{2y' + xy''}{x(y')^2}. \tag{48}$$

From (48) we derive the differential invariant

$$w_2 = \frac{dv_2}{dr_2} = \frac{y'''}{(y')^3} - \frac{2(y'')^2}{(y')^4} - \frac{2y''}{x(y')^3} - \frac{2}{x^2(y')^2}. \tag{49}$$

Equation (38) can now be reduced into a first-order ODE of the form

$$\frac{dv_2}{dr_2} = \psi^2(r_2, v_2)$$

for some function ψ^2. To find ψ^2, substitute out y''' from (49) using (38) and then use (48) to write the resulting equation in terms of r_2 and v_2. We obtain the first-order ODE

$$\frac{dv_2}{dr_2} = -\frac{v_2^2}{2}, \tag{50}$$

which admits $Y_3^{(2)}$, written in terms of r_2 and v_2, i.e.,

$$Y_3^{(2)} = \frac{\partial}{\partial r_2}. \tag{51}$$

The solution of (50) is

$$v_2 = \frac{2}{r_2 - \kappa_1}, \tag{52}$$

where κ_1 is an arbitrary constant. In terms of r_1 and v_1, Equation (52) is transformed, using (47), into the next first-order ODE

$$\frac{dv_1}{dr_1} = \frac{2v_1}{r_1 - \kappa_1}, \tag{53}$$

which admits symmetry (45). Equation (53) is solved easily. We obtain

$$v_1 = \kappa_2(\kappa_1 - r_1)^2, \tag{54}$$

where κ_2 is another arbitrary constant. Using (43) we write (54) as a first-order ODE in the variables x and y, namely

$$y' = \frac{\kappa_2(y - \kappa_1)^2}{x^2}. \tag{55}$$

Equation (55) admits Y_1, i.e., the symmetry X_4 from (39) and is the last ODE in the series of iterations. Furthermore, it is a variables-separable ODE, the solution of which is

$$y = \frac{x}{\kappa_2 - \kappa_3 x} + \kappa_1, \tag{56}$$

where κ_3 is another arbitrary constant. This is the general solution of Equation (38).

Example 3. *Consider the nonlinear ODE*

$$y''' + x(y'')^2 + \tfrac{1}{x} y'' = 0, \tag{57}$$

an example of third-order ODEs that are equivalent to linear second-order ODEs via tangent transformations [3]. Equation (57) admits a four-dimensional symmetry Lie algebra spanned by the operators

$$X_1 = x^2 \tfrac{\partial}{\partial x} + x(y + \ln x - 1)\tfrac{\partial}{\partial y},\ X_2 = x\tfrac{\partial}{\partial x},\ X_3 = \tfrac{\partial}{\partial y},\ X_4 = x\tfrac{\partial}{\partial y}. \tag{58}$$

The commutator relations of X_2, X_3 and X_4 are such that

$$[X_2, X_4] = X_4 \tag{59}$$

is the only nonzero Lie bracket. This means that X_1, X_2 and X_4 span a solvable Lie algebra and satisfy (2), with the following labelling:

$$X_4 \to Y_1, \quad X_2 \to Y_2, \quad X_3 \to Y_3.$$

The extensions of the identified infinitesimal generators are:

$$\left.\begin{array}{rcl} Y_1^{(1)} & = & x\frac{\partial}{\partial y} + \frac{\partial}{\partial y'} \\ Y_2^{(2)} & = & x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''} \\ Y_3^{(3)} & = & \frac{\partial}{\partial y}. \end{array}\right\} \tag{60}$$

We solve characteristic equations

$$\frac{dx}{0} = \frac{dy}{x} = \frac{dy'}{1} \tag{61}$$

associated with $Y_1^{(1)}$, we obtain invariants

$$r_1 = x, \quad v_1 = y' - \frac{y}{x}, \tag{62}$$

and derive the differential invariant

$$w_1 = \frac{dv_1}{dr_1} = \frac{y}{x^2} - \frac{y'}{x} + y''. \tag{63}$$

Writing $Y_2^{(2)}$ in terms of r_1, v_1 and w_1, we obtain

$$Y_2^{(2)} = r_1\frac{\partial}{\partial r_1} - v_1\frac{\partial}{\partial v_1} - 2w_1\frac{\partial}{\partial w_1}, \tag{64}$$

for which the corresponding characteristic equations are

$$\frac{dr_1}{r_1} = \frac{dv_1}{-v_1} = \frac{dw_1}{-2w_1}. \tag{65}$$

We obtain from the solution of (65) invariants

$$r_2 = r_1 v_1 \quad \text{and} \quad v_2 = \frac{w_1}{v_1^2}, \tag{66}$$

which, in view of (62), can be written in terms of x, y, y' *and* y'' *as follows:*

$$r_2 = xy' - y \quad \text{and} \quad v_2 = \frac{y + x(xy'' - y')}{(y - xy')^2}. \tag{67}$$

From (67) we derive the differential invariant

$$w_2 = \frac{dv_2}{dr_2} = \frac{x(yy''' - 3y'y'') + 3yy'' + x^2\left(2(y'')^2 - y'y'''\right)}{y''(y - xy')^3}. \tag{68}$$

Equation (57) can now be reduced into a first-order ODE of the form

$$\frac{dv_2}{dr_2} = \psi^2(r_2, v_2)$$

for some function ψ^2. *To find* ψ^2, *we use (57) to substitute out* y''' *from (68) and then use (67) to write the resulting equation in terms of* r_2 *and* v_2. *We obtain the first-order ODE*

$$\frac{dv_2}{dr_2} = -\frac{(r_2 + 2)v_2 + 1}{r_2}, \tag{69}$$

that admits $Y_3^{(2)}$ *written in terms of* r_2 *and* v_2, *i.e.,*

$$Y_3^{(2)} = -\frac{\partial}{\partial r_2} + \frac{2r_2 v_2 + 1}{r_2^2}\frac{\partial}{\partial v_2}. \tag{70}$$

The solution of (69) is

$$v_2 = \frac{\kappa_1 e^{-r_2} - r_2 + 1}{r_2^2}, \tag{71}$$

where κ_1 *is an arbitrary constant. In terms of* r_1 *and* v_1, *Equation (71) is transformed, using (66), into another first-order ODE*

$$\frac{dv_1}{dr_1} = \frac{\kappa_1 e^{-r_1 v_1} - r_1 v_1 + 1}{r_1^2}, \tag{72}$$

which admits symmetry (64). The solution of (72) is

$$e^{r_1 v_1} = \kappa_2 r_1 - \kappa_1, \tag{73}$$

where κ_2 *is another arbitrary constant. Finally, we use (62) to write (73) as an ODE in the variables* x *and* y. *We obtain*

$$e^{xy' - y} = x\kappa_2 - \kappa_1, \tag{74}$$

which admits Y_1, *i.e., the symmetry* X_4 *from (58). The solution of (74), namely*

$$y = x \ln\left[\left(\frac{\kappa_1}{x} - \kappa_2\right)^{\kappa_2/\kappa_1} (\kappa_2 x - \kappa_1)^{-1/x}\right] + \kappa_3 x, \quad \kappa_1 \neq 0, \tag{75}$$

where κ_3 *is another arbitrary constant is the general solution of Equation (57).*

Example 4. *The equation we consider here*

$$y''' + \frac{3y'y''}{y} - 3y'' - \frac{3(y')^2}{y} + 2y' = 0, \tag{76}$$

drawn from [1] admits a seven-dimensional symmetry Lie algebra spanned by the operators

$$\left.\begin{array}{l} X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{1}{y}\frac{\partial}{\partial y}, \quad X_3 = 2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ X_4 = e^x\frac{\partial}{\partial x} + e^x\left(y + \dfrac{1}{y}\right)\frac{\partial}{\partial y}, \quad X_5 = e^{-x}\frac{\partial}{\partial x}, \quad X_6 = \frac{e^x}{y}\frac{\partial}{\partial y} \\ X_7 = \frac{e^{2x}}{y}\frac{\partial}{\partial y}. \end{array}\right\} \tag{77}$$

Using the solvable algebra $\langle X_1, X_3, X_7\rangle$, for which nonzero Lie brackets are

$$[X_1, X_7] = 2X_7 \quad \textit{and} \quad [X_3, X_7] = 2X_7, \tag{78}$$

we relabel the symmetries as follows:

$$X_7 \to Y_1, \quad X_3 \to Y_2, \quad X_1 \to Y_3,$$

to ensure that the commutation relations of Y_1, Y_2 and Y_3 satisfy (2).

As in the previous examples, the following extensions of Y_1, Y_2 and Y_3 are needed in the calculations that follow:

$$\left.\begin{array}{lcl} Y_1^{(1)} & = & \frac{e^{2x}}{y}\frac{\partial}{\partial y} + e^{2x}\left(\frac{2}{y} - \frac{y'}{y^2}\right)\frac{\partial}{\partial y'} \\ Y_2^{(2)} & = & 2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} + y''\frac{\partial}{\partial y''} \\ Y_3^{(3)} & = & \frac{\partial}{\partial x}. \end{array}\right\} \tag{79}$$

We compute two invariants of $Y_1^{(1)}$,

$$r_1 = x, \quad v_1 = yy' - y^2, \tag{80}$$

from which we derive the differential invariant

$$w_1 = \frac{dv_1}{dr_1} = y(y'' - 2y') + (y')^2. \tag{81}$$

In terms of r_1, v_1 and w_1, $Y_2^{(2)}$ becomes

$$Y_2^{(2)} = \frac{\partial}{\partial r_1} + v_1\frac{\partial}{\partial v_1} + w_1\frac{\partial}{\partial w_1}. \tag{82}$$

Invariants of (82) are

$$r_2 = e^{-r_1}v_1 \quad \textit{and} \quad v_2 = \frac{w_1}{v_1}, \tag{83}$$

or, in terms of x, y and the derivatives,

$$r_2 = ye^{-x}(y' - y) \quad \textit{and} \quad v_2 = \frac{y(y'' - 2y') + (y')^2}{y(y' - y)}. \tag{84}$$

The differential invariant derived from (84) is

$$\begin{aligned} w_2 \quad = \quad & \frac{dv_2}{dr_2} = e^x\Big[y^3(2y'' - y''') - y^2\Big(2(y')^2 + y'(y'' - y''') + (y'')^2\Big) - (y')^4 \\ & + y(y')^2(2y' + y'')\Big]\Big[y^2(y - y')^2\Big(y^2 + y(y'' - 3y') + (y')^2\Big)\Big]^{-1}. \end{aligned} \tag{85}$$

We now use Equation (76) to substitute out y''' from (85) and then express the resulting equation in terms of r_2 and v_2 using (84). We obtain

$$\frac{dv_2}{dr_2} = -\frac{v_2}{r_2}, \tag{86}$$

a first-order ODE that admits $Y_3^{(2)}$ written in terms of r_2 and v_2, i.e.,

$$Y_3^{(2)} = r_2 \frac{\partial}{\partial r_2}. \tag{87}$$

The solution of (86) is

$$v_2 = \frac{\kappa_1}{r_2}, \tag{88}$$

where κ_1 is an arbitrary constant. We now use (83) to express (88) in terms of r_1 and v_1. We obtain

$$\frac{dv_1}{dr_1} = \kappa_1 e^{r_1}, \tag{89}$$

which admits symmetry (82). Upon solving (89), we obtain

$$v_1 = \kappa_1 e^{r_1} + \kappa_2, \tag{90}$$

where κ_2 is another arbitrary constant. Using (80) we write (90) an order ODE in the variables x and y,

$$y' = \frac{\kappa_1 e^x + \kappa_2 + y^2}{y}, \tag{91}$$

which admits Y_1, i.e., the symmetry X_7 from (77). Equation (91) is easily solved and we obtain

$$y = \left(\kappa_3 e^{2x} - 2\kappa_1 e^x - \kappa_2\right)^{1/2}, \tag{92}$$

where κ_3 is another arbitrary constant. This is in fact the general solution of Equation (76).

4. Concluding Remarks

In this paper, we have provided a clear exposition of the method of differential invariants for integrating (or, at least, reduction of order of) any higher order ODE that admits a solvable Lie algebra. We have included in the paper four illustrative examples that involve nonlinear ODEs of different classes and drawn from different contexts, each of which admits a three-dimensional solvable lie algebra. The presentation of the reduction algorithm in this paper is instructive in that the exposition is based on a third-order ODE, which makes the method easy to appreciate. In this connection, it is our hope that this paper will serve as an invitation to others to consider using the method of differential invariants on ODEs that they encounter.

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