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Complete Invariant $\star$-Metrics on Semigroups and Groups

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Abstract: In this paper, we study the complete $\star$-metric semigroups and groups and the Raĭkov completion of invariant $\star$-metric groups. We obtain the following. (1) Let $(X, d^\star)$ be a complete $\star$-metric space containing a semigroup (group) G that is a dense subset of X. If the restriction of $d^\star$ on G is invariant, then X can become a semigroup (group) containing G as a subgroup, and $d^\star$ is invariant on X. (2) Let $(G, d^\star)$ be a $\star$-metric group such that $d^\star$ is invariant on G. Then, $(G, d^\star)$ is complete if and only if $(G, \tau_{d^\star})$ is Raĭkov complete.

Keywords: $\star$-metric; topological group; topological semigroup; Raĭkov completion

MSC: 22A05; 54H11; 54D30; 54G20

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Citation: He, S.-Y.; Wei, J.-C.; Xie, L.-H. Complete Invariant $\star$-Metrics on Semigroups and Groups. *Axioms* **2022**, *11*, 546. https://doi.org/10.3390/axioms11100546

Academic Editor: Salvador Hernández

Received: 25 August 2022
Accepted: 21 September 2022
Published: 11 October 2022

1. Introduction

The combination of topological structure and algebraic structure is a very useful tool in modern mathematics research. This has stimulated the research enthusiasm of scholars (see [1,2]). In 1975, Kramosil and Michalek introduced a notion of metric fuzziness [3], which quickly became a hot topic of scholars (see [4–6]). In 2001, Romaguera and Sanchis introduced and studied the concept of the fuzzy metric group, extended the classical theorem of the metric group to the fuzzy metric group, and studied the properties of quotient subgroups of a fuzzy metric group [7]. Then, Sánchez and Sanchis found a sufficient condition for topological algebraic structures to become stronger topological structures [8].

Recently, Sánchez and Sanchis studied complete invariant fuzzy metrics on groups [9]. They proved that:

Theorem 1. *If $(G, M, *)$ is a fuzzy metric group such that $(M, *)$ is invariant, then a fuzzy metric completion $(\tilde{G}, \tilde{M}, *)$ of $(G, M, *)$ is a fuzzy metric group and $(\tilde{M}, *)$ is invariant.*

Theorem 2. *If $(M, *)$ is a complete fuzzy metric on (G, τ), then every compatible left invariant (or right invariant) fuzzy metric on G is complete.*

At the same time, J.J. Tu and L.H. Xie further conducted further research on [9] and found that Theorem 1 is also valid for fuzzy metric semigroups [10]. In addition, they also found:

Theorem 3. *Let $(G, M, *)$ be a fuzzy metric group such that $(M, *)$ is invariant on G. Then, $(G, M, *)$ is complete if and only if (G, τ_M) is Raĭkov complete.*

In 2020, Khatami and Mirzavaziri proposed the concept of the $\star$-metric and gave an example to illustrate that $\star$-metrics are not metrics ([11] (Example 2.4)). Some notations and definitions are as follows:

Recall that a *t-definer* is a function $\star$: $[0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying the following conditions for each $a, b, c \in [0, \infty)$: (T1) $a \star b = b \star a$; (T2) $a \star (b \star c) = (a \star b) \star c$; (T3) if

$a \leqslant b$, then $a \star c \leqslant b \star c$; (T4) $a \star 0 = a$; (T5) $\star$ is continuous in its first component with respect to the Euclidean topology. In addition, $\star$ is continuous by [11].

Definition 1 ([11] (Definition 2.2))**.** *Let X be a nonempty set and $\star$ is a t-definer. If, for every $x, y, z \in X$, a function $d^\star : X \times X \to [0, \infty)$ satisfies the following conditions:*
(M1) $d^\star(x, y) = 0$ if and only if $x = y$;
(M2) $d^\star(x, y) = d^\star(y, x)$;
(M3) $d^\star(x, y) \leqslant d^\star(x, z) \star d^\star(z, y)$,
then $d^\star$ is called a $\star$-metric on X. The set X with a $\star$-metric is called $\star$-metric space, denoted by $(X, d^\star)$.

Assume that $(X, d^\star)$ is a $\star$-metric space. For any $a \in X$ and $r > 0$, denote

$$B_{d^\star}(a, r) = \{x \in X : d^\star(a, x) < r\} \tag{1}$$

and

$$\tau_{d^\star} = \{U \subseteq X : \text{ for each } a \in U \text{ there is } r > 0 \text{ such that } B_{d^\star}(a, r) \subseteq U\}. \tag{2}$$

Let $\mathscr{B} = \{B_{d^\star}(x, \frac{1}{n}) \mid x \in X, \epsilon > 0\}$ be a family of open balls on a $\star$-metric space $(X, d^\star)$.

Lemma 1 ([11] (Definition 3.1 and Lemma 3.3))**.** *Let $(X, d^\star)$ be a $\star$-metric space; then, $\tau_{d^\star}$ is a topology on X and $\mathscr{B}$ constitutes a topological basis of $\tau_{d^\star}$*

According to the $\star$-metric proposed by Khatami and Mirzavaziri [11] and the $\star$-(quasi)-pseudometric semigroup proposed by S.Y. He, Y.Y. Jin, and L.H. Xie [12], we propose $\star$-metric semigroups and $\star$-metric groups. Referring to the research method in Refs. [9,10], we consider the following questions. (1) Is Theorem 1 still valid under $\star$-metric semigroups and $\star$-metric groups? (2) Let $(G, d^\star)$ be a $\star$-metric group. Then, what is the relationship between $(G, d^\star)$ completeness and $(G, \tau_{d^\star})$ Raĭkov completeness?

With this in mind, in Section 2, we prove that the topology induced by a invariant $\star$-metric can make an abstract semigroup become a topological semigroup, and give a characterization of invariant $\star$-metrics. Then, we obtain the following. Let $(X, d^\star)$ be a complete $\star$-metric space containing a semigroup G that is a dense subset of X. If the restriction of $d^\star$ on G is invariant, then X can become a semigroup containing G as a subgroup and $d^\star$ is invariant on X. In Section 3, we discuss the Raĭkov completeness of $\star$-metrics and find the following. Let $(G, d^\star)$ be a $\star$-metric group such that $d^\star$ is invariant on G. Then, $(G, d^\star)$ is complete if and only if $(G, \tau_{d^\star})$ is Raĭkov complete.

2. Complete Invariant $\star$-Metrics on Semigroups and Groups

In this chapter, we first give the characterization of invariant $\star$-metrics on $\star$-metric groups. Then, using Sánchez's method, we prove that Theorem 1 is still valid under $\star$-metric semigroups and $\star$-metric groups.

A *topological semigroup* (G, τ) is an algebraic semigroup G with a topology τ that makes the multiplication in G jointly continuous. A *paratopological group* G is a topological semigroup such that G is an algebraic group. We say that a paratopological group (G, τ) is a *topological group* if the inverse is continuous.

A $\star$-metric $d^\star$ on a group G is *left-invariant* (respectively, *right-invariant*) if $d^\star(x, y) = d^\star(ax, ay)$ (respectively, $d^\star(x, y) = d^\star(xa, ya)$) whenever $a, x, y \in G$. We say that $d^\star$ is *invariant* if it is both left-invariant and right-invariant.

The notions and concepts of topological groups such as "group", "semigroup", "continuous function", and so forth are defined as usual (e.g., see [13]).

Proposition 1. *Let S be an abstract semigroup and $d^\star$ be a $\star$-metric on S. If $d^\star$ is invariant, then $(S, \tau_{d^\star})$ is a topological semigroup.*

Proof. According to Lemma 1, $\tau_{d^\star}$ is a topology on S. We will prove that the multiplication of semigroup S is joint continuous with respect to $\tau_{d^\star}$.

For every $y, z \in S$ and an open set U containing yz, according to the definition of $\tau_{d^\star}$, there exists an $\epsilon > 0$ such that $yz \in B_{d^\star}(yz, \epsilon) \subseteq U$. Since the $\star$ is continuous, for the ϵ above, we can find a $r > 0$ such that $r \star r < \epsilon$.

Now, we will prove that $B_{d^\star}(y, r)B_{d^\star}(z, r) \subseteq B_{d^\star}(yz, \epsilon)$.

In fact, for every $a \in B_{d^\star}(y, r)$ and $b \in B_{d^\star}(z, r)$, $ab \in B_{d^\star}(y, r)B_{d^\star}(z, r)$. Thus, we have

$$d^\star(yz, ab) \leqslant d^\star(yz, yb) \star d^\star(yb, ab) = d^\star(z, b) \star d^\star(y, a) < r \star r < \epsilon. \tag{3}$$

We have proved that $ab \in B_{d^\star}(yz, \epsilon)$. Therefore, $(S, \tau_{d^\star})$ is a topological semigroup.

This completes the proof. □

Proposition 2. *Let G be an abstract group and $d^\star$ be a $\star$-metric on G. If $d^\star$ is invariant, then $(G, \tau_{d^\star})$ is a topological group.*

Proof. Similar to the proof of Proposition 1, we see that the multiplication of group G is joint continuous with respect to $\tau_{d^\star}$. Therefore, $(G, \tau_{d^\star})$ is a paratopological group.

Now, we shall prove that the inverse mapping is continuous. To complete the proof, it is enough to show that $B_{d^\star}(x, \epsilon_0)^{-1} \subseteq B_{d^\star}(x^{-1}, \epsilon)$ for $x \in G$. In fact, let $\epsilon_0 = \epsilon$, and take $a \in B_{d^\star}(x, \epsilon_0)$. Since $d^\star$ is invariant, we have

$$d^\star(x^{-1}, a^{-1}) = d^\star(x^{-1}aa^{-1}, x^{-1}xa^{-1}) = d^\star(a, x) < \epsilon. \tag{4}$$

Therefore, we have proven that the inverse mapping is continuous.

Thus, $(G, \tau_{d^\star})$ is a topological group. □

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a $\star$-metric space $(X, d^\star)$ is said to be a *Cauchy sequence* provided that, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d^\star(x_n, x_m) < \epsilon$ for every $n, m \geqslant n_0$. A $\star$-metric space is *complete* if every Cauchy sequence is convergent. If $(X, d^\star)$ is a complete $\star$-metric space, then the $\star$-metric $d^\star$ is said to be a *complete $\star$-metric*.

A topological space (X, τ) is called *completely $\star$-metrizable* if there exists a completely $\star$-metric $d^\star$ on X such that the topology $\tau_{d^\star}$ induced by $d^\star$ coincides with τ; such a completely $\star$-metric is said to be *compatible* with τ.

By a *$\star$-metric group* (resp., *$\star$-metric semigroup*), we mean a pair $(G, d^\star)$ such that $(G, d^\star)$ is a $\star$-metric space and $(G, \tau_{d^\star})$ is a topological group (resp., topological semigroup).

Now, we give the characterization of invariant $\star$-metrics.

Lemma 2. *Let $(G, d^\star)$ be a $\star$-metric group. Then, $d^\star$ is invariant if and only if $d^\star(ab, cd) \leqslant d^\star(a, c) \star d^\star(b, d)$ for each $a, b, c, d \in G$.*

Proof. If $d^\star$ is invariant, then

$$d^\star(ab, cd) \leqslant d^\star(ab, cb) \star d^\star(cb, cd) = d^\star(a, c) \star d^\star(b, d). \tag{5}$$

For the converse, if $a, b, g \in G$, then we have

$$d^\star(ga, gb) \leqslant d^\star(g, g) \star d^\star(a, b) = 0 \star d^\star(a, b) = d^\star(a, b). \tag{6}$$

Similarly,

$$d^\star(a, b) = d^\star(g^{-1}ga, g^{-1}gb) \leqslant d^\star(g^{-1}, g^{-1}) \star d^\star(ga, gb) = d^\star(ga, gb), \tag{7}$$

which means $d^\star$ is left invariant. With a similar argument, we can show that $d^\star$ is right invariant. Therefore, $d^\star$ is invariant. □

Remark 1. *Let $(G, d^\star)$ be a $\star$-metric semigroup such that $d^\star$ is invariant. Then, $d^\star(ab, cd) \leqslant d^\star(a, c) \star d^\star(b, d)$ for each $a, b, c, d \in G$.*

Theorem 4. *Let $(X, d^\star)$ be a complete $\star$-metric space containing a semigroup S that is a dense subset of X. If the restriction of $d^\star$ on S is invariant, then X can become a semigroup containing S as a subsemigroup and $d^\star$ is invariant on X.*

Proof. Take $a, b \in X$. Let $(a_n)_n$, $(b_n)_n \subseteq S$ be Cauchy sequences such that $(a_n)_n$ converges to a and $(b_n)_n$ converges to b.

We claim that $(a_n b_n)_n$ is a Cauchy sequence in $(X, d^\star)$. For every $\epsilon > 0$, since $\star$ is a continuous t-definer, there is $r > 0$ such that $r \star r < \epsilon$. In addition, $(a_n)_n$, $(b_n)_n$ are Cauchy sequences in S, so there exists $n_0 \in \mathbb{N}$ such that $d^\star(a_n, a_m) < r$ and $d^\star(b_n, b_m) < r$ whenever $n, m \geqslant n_0$. Since $d^\star$ is invariant in S, by Remark 1, we have

$$\begin{aligned} d^\star(a_n b_n, a_m b_m) &\leqslant d^\star(a_n b_n, a_m b_n) \star d^\star(a_m b_n, a_m b_m) \\ &\leqslant d^\star(a_n, a_m) \star d^\star(b_n, b_m) \leqslant r \star r < \epsilon. \end{aligned} \tag{8}$$

This proves the claim.

Now we define a binary operation $\cdot$ on X as follows. Given two elements a, b of X and two sequences $(a_n)_n$, $(b_n)_n \subseteq S$, as above, $a \cdot b = x$ where x is the limit of $(a_n b_n)_n$ with respect to $\tau_{d^\star}$. Let us show that $\cdot$ is well defined.

In fact, choose two sequences $(c_n)_n$ and $(d_n)_n$ in S converging to a and b, respectively. We shall show that $(c_n d_n)_n$ converges to x. Take $\epsilon > 0$; there exists $r > 0$ such that $r \star r \star r < \epsilon$. According to assumptions, there exists $n_0 \in \mathbb{N}$ such that $d^\star(x, a_n b_n) < r$, $d^\star(a_n, c_n) < r$, $d^\star(b_n, d_n) < r$, for $n \geqslant n_0$. Hence, for every $n \geqslant n_0$, Remark 1 implies that

$$\begin{aligned} d^\star(x, c_n d_n) &\leqslant d^\star(x, a_n b_n) \star d^\star(a_n b_n, c_n d_n) \\ &< r \star d^\star(a_n, c_n) \star d^\star(b_n, d_n) \\ &< r \star r \star r < \epsilon. \end{aligned} \tag{9}$$

Thus, the binary operation $\cdot$ is well defined.

Let us show that $(X, \cdot)$ is a semigroup. Since

$$(a \cdot b) \cdot c = \lim_{n \to \infty} (a_n b_n) c_n = \lim_{n \to \infty} a_n (b_n c_n) = a \cdot (b \cdot c),$$

it follows that the operation $\cdot$ is associative.

Now, we will prove that $d^\star$ is invariant on X. Fix $a, x, y \in X$ and $\epsilon > 0$. We can choose $r > 0$ such that $r \star r \star r \star r < \epsilon$. Take sequences $(a_n)_n$, $(x_n)_n$, $(y_n)_n$ in S converging to a, x, and y, respectively. We can find $N \in \mathbb{N}$ such that $d^\star(ax, a_N x_N) < r$, $d^\star(a_N y_N, ay) < r$, $d^\star(x_N, x) < r$ and $d^\star(y, y_N) < r$. Since, $d^\star$ is left invariant on S, we have

$$\begin{aligned} d^\star(ax, ay) &\leqslant d^\star(ax, a_N x_N) \star d^\star(a_N x_N, a_N y_N) \star d^\star(a_N y_N, ay) \\ &< r \star d^\star(x_N, y_N) \star r \\ &< r \star r \star d^\star(x_N, x) \star d^\star(x, y) \star d^\star(y, y_N) \\ &< r \star r \star r \star r \star d^\star(x, y) \\ &< \epsilon \star d^\star(x, y). \end{aligned} \tag{10}$$

We have thus proven that $d^\star(ax, ay) < \epsilon \star d^\star(x, y)$. Since the $\star$ is continuous, we have $d^\star(ax, ay) \leqslant d^\star(x, y)$. In a similar way, $d^\star(ax, ay) \geqslant d^\star(x, y)$. Hence, $d^\star$ is left invariant on X. Using a similar argument and the fact that $d^\star$ is right invariant on S, we can prove that $d^\star$ is right invariant on X. Therefore, $d^\star$ is invariant on X.

Finally, Proposition 1 permits us conclude that $(X, \tau_{d^\star})$ is a topological semigroup. □

Theorem 5. *Let $(X, d^\star)$ be a complete $\star$-metric space containing a group G that is a dense subset of X. If the restriction of $d^\star$ on G is invariant, then X can become a group containing G as a subgroup and $d^\star$ is invariant on X.*

Proof. Take $a, b \in X$. Let $(a_n)_n$, $(b_n)_n \subseteq G$ be Cauchy sequences such that $(a_n)_n$ converges to a and $(b_n)_n$ converges to b.

We claim that $(a_n b_n)_n$ is a Cauchy sequence in $(X, d^\star)$. For every $\epsilon > 0$, since $\star$ is a continuous t-definer, there is $r > 0$ such that $r \star r < \epsilon$. In addition, $(a_n)_n$, $(b_n)_n$ are Cauchy sequences in G, so there exists $n_0 \in \mathbb{N}$ such that $d^\star(a_n, a_m) < r$ and $d^\star(b_n, b_m) < r$ whenever $n, m \geqslant n_0$. Since $d^\star$ is invariant in G, by Lemma 2, we have

$$d^\star(a_n b_n, a_m b_m) \leqslant d^\star(a_n, a_m) \star d^\star(b_n, b_m) \leqslant r \star r < \epsilon. \tag{11}$$

This proves the claim.

Now we define a binary operation $\cdot$ on X as follows: given two elements a, b of X and two sequences $(a_n)_n$, $(b_n)_n \subseteq G$ as above, $a \cdot b = x$ where x is the limit of $(a_n b_n)_n$ with respect to $\tau_{d^\star}$. Similar to the proof of Theorem 4, the operation $\cdot$ is well defined.

Let us show that $(X, \cdot)$ is a group. First, notice that if e is the neutral element of G, then

$$a \cdot e = \lim_{n\to\infty} a_n e = a = \lim_{n\to\infty} e a_n = e \cdot a, \tag{12}$$

that is, e is the neutral element of $(X, \cdot)$. Second, since

$$(a \cdot b) \cdot c = \lim_{n\to\infty} (a_n b_n) c_n = \lim_{n\to\infty} a_n (b_n c_n) = a \cdot (b \cdot c), \tag{13}$$

it follows that the operation $\cdot$ is associative. In addition, since $d^\star$ is invariant on G, we have

$$d^\star(a_m, a_n) = d^\star(a_m a_n^{-1} a_n, a_m a_m^{-1} a_n) = d^\star(a_n^{-1}, a_m^{-1}), \tag{14}$$

and $e = \lim\limits_{n\to\infty} a_n a_n^{-1}$; the sequence $(a_n^{-1})_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore, $a^{-1} = \lim\limits_{n\to\infty} a_n^{-1}$ is the inverse element of a.

According to the proof of Theorem 4, we see that $d^\star$ is invariant on X.

Finally, Proposition 2 permits us conclude that $(X, \tau_{d^\star})$ is a topological group. □

Theorem 6. *Suppose that $(G, d^\star)$ is a $\star$-metric group such that $d^\star$ is invariant. If $d^\star$ is a complete $\star$-metric, then every compatible left invariant $\star$-metric on G is complete.*

Proof. Let $\bar{d}^\star$ be a compatible left invariant $\star$-metric on G. Take a Cauchy sequence $(x_n)_n$ in $(G, \bar{d}^\star)$. We claim that $(x_n)_n$ is a Cauchy sequence in $(G, d^\star)$ as well.

Choose $\epsilon > 0$. By the compatibility of $\bar{d}^\star$, we can find $r > 0$ such that $B_{\bar{d}^\star}(e, r) \subseteq B_{d^\star}(e, \epsilon)$, where e is the neutral element of G. Since $(x_n)_n$ is a Cauchy sequence in $(G, \bar{d}^\star)$, there exists $k \in \mathbb{N}$ such that $\bar{d}^\star(x_n, x_m) < r$ provided $n, m \geqslant k$, which means $x_m \in B_{\bar{d}^\star}(x_n, \epsilon)$. Since $\bar{d}^\star$ is left invariant, according to the proof of ([12] Theorem 3.5), we have $B_{\bar{d}^\star}(x_n, \epsilon) = x_n B_{\bar{d}^\star}(e, r)$, i.e., $x_n^{-1} x_m \in B_{\bar{d}^\star}(e, r)$ for each $n, m \geqslant k$. Therefore, $x_n^{-1} x_m \in B_{d^\star}(e, \epsilon)$; equivalently, $d^\star(x_n, x_m) < \epsilon$ for each $n, m \geqslant k$. We have thus proven our claim. Since $(G, d^\star)$ is complete, the sequence $(x_n)_n$ converges with respect to $\tau_{d^\star} = \tau_{\bar{d}^\star}$. Therefore, $\bar{d}^\star$ is complete as well. □

By Theorems 5 and 6, we easily see the following.

Corollary 1. *Let $(X, d^\star)$ be a complete $\star$-metric space containing a group G that is a dense subset of X. If the restriction of $d^\star$ on G is invariant, then every left invariant $\star$-metric on the $(X, d^\star)$ is complete.*

3. The Raĭkov Completion of ⋆-Metrics Groups

A *filter* on a set X is a family η of non-empty subsets of X satisfying the next two conditions. (i) If U and V are in η, then $U \cap V$ is also in η. (ii) If $U \in \eta$ and $U \subseteq W \subseteq X$, then $W \in \eta$.

Let G be a topological group with the identity e. A filter η of a topological group G is said to be a *Cauchy filter* if for every open neighborhood V of e in G, there exist $a, b \in G$ and $A, B \in \eta$ such that $A \subseteq aV$ and $B \subseteq Vb$. In a topological space G, a filter η on G converges to a point x of G if every neighborhood of x belongs to η. A topological group G such that every Cauchy filter on G converges is called *Raĭkov complete*. Next, we shall investigate the Raĭkov completeness of the group topologies induced by invariant $\star$-metrics on groups.

Theorem 7. *Let $(G, d^\star)$ be a $\star$-metric group. If $(G, d^\star)$ is complete, then $(G, \tau_{d^\star})$ is Raĭkov complete.*

Proof. Suppose that $(G, d^\star)$ is complete. Take an arbitrary Cauchy filter $\eta \subseteq G$. Then, for each $n \in \mathbb{N}$, there are $F'_n \in \eta$ and $x_n \in G$ such that $F'_n \subseteq B_{d^\star}(x_n, \frac{1}{n})$. Put $F_n = \bigcap_{i=1}^n F'_i$. Clearly, $F_n \subseteq B_{d^\star}(x_n, \frac{1}{n})$ holds for each $n \in \mathbb{N}$. Take $y_n \in F_n$ for each $n \in \mathbb{N}$. Then, the sequence $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(G, d^\star)$. In fact, for each $n \in \mathbb{N}$, since $\star$ is continuous, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \star \frac{1}{n_0} < \frac{1}{n}$. Obviously, $y_i,\ y_j \in F_{n_0} \subseteq B_{d^\star}(x_{n_0}, \frac{1}{n_0})$, whenever $i,\ j \in \mathbb{N}$ and $i,\ j > n_0$. Thus,

$$d^\star(y_i, y_j) \leqslant d^\star(y_i, x_{n_0}) \star d^\star(x_{n_0}, y_j) < \frac{1}{n_0} \star \frac{1}{n_0} < \frac{1}{n} \tag{15}$$

whenever $i, j > n_0$. This shows that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(G, d^\star)$ is complete, $(y_n)_{n\in\mathbb{N}}$ converges to some $y \in G$. We shall show that the Cauchy filter η converges to y, which implies that $(G, \tau_{d^\star})$ is Raĭkov complete.

Take any open neighborhood V of y. Without loss of generality, we assume that $V = B_{d^\star}(y, \frac{1}{n})$. Since $\star$ is continuous, there is $r \in \mathbb{N}$ such that $\frac{1}{n_0} \star \frac{1}{n_0} < \frac{1}{n}$. Note that $\{y_n\}_{n\in\mathbb{N}}$ converges to y; then, there is $n' \in \mathbb{N}$ such that $\frac{1}{n'} \star \frac{1}{n'} < \frac{1}{n_0}$ and $d^\star(y, y_{n'}) < \frac{1}{n_0}$. Then, for each $x \in F_{n'}$, noting that $y_{n'} \in F_{n'} \subseteq B_{d^\star}(x_{n'}, \frac{1}{n'})$, we have

$$\begin{aligned} d^\star(y, x) &\leqslant d^\star(y, y_{n'}) \star d^\star(y_{n'}, x_{n'}) \star d^\star(x_{n'}, x) \\ &< \frac{1}{n_0} \star \frac{1}{n'} \star \frac{1}{n'} \\ &< \frac{1}{n_0} \star \frac{1}{n_0} < \frac{1}{n}. \end{aligned} \tag{16}$$

This implies that $x \in B_{d^\star}(y, \frac{1}{n})$, i.e., $F_{n'} \subseteq B_{d^\star}(y, \frac{1}{n})$. Clearly, $F_{n'} \in \eta$; thus, we have proved = n that η converges to y. □

Theorem 8. *Let $(G, d^\star)$ be a $\star$-metric group with $d^\star$ being invariant. If $(G, \tau_{d^\star})$ is Raĭkov complete, then $(G, d^\star)$ is complete.*

Proof. Suppose that $(G, \tau_{d^\star})$ is Raĭkov complete. Take the arbitrary Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of $(G, d^\star)$. Put $\eta = \{A \subseteq G : F_n \subseteq A \text{ for some } F_n\}$, where $F_n = \{x_i : i \geq n\}$ for each $n \in \mathbb{N}$. Now, we shall prove that η is a Cauchy filter of G. Take any $B_{d^\star}(e, \frac{1}{n})$, where e is the identity of G. Since $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence, there is $n_0 \in \mathbb{N}$ such that $d^\star(x_k, x_m) < \frac{1}{n}$ whenever $k, m \geqslant n_0$. This implies $x_k \in B_{d^\star}(x_{n_0}, \frac{1}{n})$ whenever $k \geqslant n_0$. Hence, $F_{n_0} \subseteq B_{d^\star}(x_{n_0}, \frac{1}{n})$. Noting that $d^\star$ is invariant, $B_{d^\star}(x_{n_0}, \frac{1}{n}) = x_{n_0} B_{d^\star}(e, \frac{1}{n}) = B_{d^\star}(e, \frac{1}{n}) x_{n_0}$, so $F_{n_0} \subseteq x_{n_0} B_{d^\star}(e, \frac{1}{n})$ and $F_{n_0} \subseteq B_{d^\star}(e, \frac{1}{n}) x_{n_0}$. This implies that η is a Cauchy filter. Since $(G, \tau_{d^\star})$ is Raĭkov complete, the Cauchy filter η converges to a point g in G. Then, one can easily show that the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges to g. This implies that $(G, d^\star)$ is complete. □

By Theorems 7 and 8, we have the following:

Corollary 2. *Let $(G, d^{\star})$ be a $\star$-metric group such that $d^{\star}$ is invariant on G. Then, $(G, d^{\star})$ is complete if and only if $(G, \tau_{d^{\star}})$ is Raĭkov complete.*

4. Conclusions and Further Work

4.1. Conclusions

Our initial motivation was to study the invariant $\star$-metrics on semigroups and groups. In this paper, we first propose the concept of $\star$-metric semigroups (respectively, groups) and study some questions related to complete $\star$-metric topological semigroups. Invariant $\star$-metrics are characterized, which allow us to characterize this kind of $\star$-metric. We also study the $\star$-metric version of classical theorems in the framework of complete $\star$-metric groups. Our results fit in a long tradition of research in topological algebra. When applied to classical structures, they allow us to obtain new outcomes that generalize helpful results of the theory.

4.2. Limitations

In the Section 2, we discuss the conditions under which a complete metric space becomes a topological group. Since we have not yet proven that every $\star$-metric space has a completion, we first give a complete $\star$-metric space in Theorems 4 and 5. Otherwise, the expression of these theorems would be more concise and clearer.

4.3. Future Research Direction

It would be interesting to study the $\star$-metrics on the quotient groups. Future research could focus on the following topic. Giving a $\star$-metric group $(G, d^{\star})$ and a closed normal subgroup N, if one has information about two complete $\star$-metric groups $(G, d^{\star})$, $(N, d^{\star})$ and $(G/N, d^{\star})$, what can be said about the third group?

Author Contributions: All the authors have contributed in this article. Writing—original draft preparation, S.-Y.H.; supervision, L.-H.X.; writing—review and editing, S.-Y.H. and J.-C.W.; funding acquisition, L.H.X. All authors have read and agreed to the published version of the manuscript.

Funding: The project is supported by the Natural Science Foundation of Guangdong Province under Grant (No. 2021A1515010381) and the Innovation Project of Department of Education of Guangdong Province (No. 2022KTSCX145).

Data Availability Statement: No data has been used in this article.

Acknowledgments: The authors gratefully thank the reviewers for their constructive comments to improve the quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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