axioms

MDPI

Article

Fractional Nonlinearity for the Wave Equation with Friction and Viscoelastic Damping

Abdelhamid Mohammed Djaouti * and Muhammad Amer Latif

Preparatory Year Deanship, King Faisal University, Hofuf 31982, Saudi Arabia
* Correspondence: adjaout@kfu.edu.sa

Abstract: In this paper we consider a fractional nonlinearity for the wave equation with friction and viscoelastic damping. Using Fixed point theorem a global in time existence of small data solutions to the Cauchy problem is investigated in this research. Our main interest is to show the influence of the fractional nonlinearity parameter to the admissible range of exponent ς comparing with power nonlinearity and also the generating of loss of decay.

Keywords: Riemann-Liouville integral operator; global in time existence; Generalized Gagliardo-Nirenberg inequality; power nonlinearity; small data

MSC: 35L05; 35L71

Citation: Mohammed Djaouti, A.; Latif, M.A. Fractional Nonlinearity for the Wave Equation with Friction and Viscoelastic Damping. *Axioms* **2022**, *11*, 524. https://doi.org/10.3390/axioms11100524

Academic Editor: Maria Michaela Porzio

Received: 31 August 2022
Accepted: 28 September 2022
Published: 1 October 2022

1. Introduction and Tools

The left and right Riemann-Liouville fractional derivatives of order $a > 0$ for a function u are defined as:

Definition 1. *Let $u \in \mathcal{H}^1(0, \varrho_2)$, $\varrho_2 > \varrho_1$, $a > 0$ then, the definition of the left and right Riemann-Liouville fractional derivatives are:*

$$^{\mathcal{RL}}\mathcal{D}^a_{\varrho_1} u(z) = \frac{1}{\Gamma(n-a)} \frac{d^n}{dz^n} \int_{\varrho_1}^{z} (z-\lambda) u(\lambda) d\lambda \tag{1}$$

and

$$^{\mathcal{RL}}\mathcal{D}^a_{\varrho_2} u(z) = \frac{(-1)^n}{\Gamma(n-a)} \frac{d^n}{dz^n} \int_{z}^{\varrho_2} (\lambda - z) u(\lambda) d\lambda, \tag{2}$$

where n is an integer which satisfies $n - 1 \leq a < n$ and $\Gamma(\cdot)$ is the Euler's gamma function.

The left and right Riemann-Liouville fractional integrals of order $a > 0$ for a function u are defined as:

Definition 2. *Let $u \in \mathcal{H}^1(0, \varrho_2)$, $\varrho_2 > \varrho_1$, $a > 0$ then, the definition of the left and right Riemann-Liouville fractional integrals are:*

$$^{\mathcal{RL}}\mathcal{I}^a_{\varrho_1} u(z) = \frac{1}{\Gamma(a)} \int_{\varrho_1}^{z} (z-\lambda) u(\lambda) d\lambda \tag{3}$$

and

$$^{\mathcal{RL}}\mathcal{I}^a_{\varrho_2} u(z) = \frac{1}{\Gamma(a)} \int_{z}^{\varrho_2} (\lambda - z) u(\lambda) d\lambda, \tag{4}$$

where $\Gamma(\cdot)$ is the Euler's gamma function.

An important concept, not long ago, that has emerged is the Caputo-Fabrizio integral operator, which has been established in the last few years. This is how it is defined:

Definition 3 ([1])**.** *Let $u \in \mathcal{H}^1(0, \varrho_2)$, $\varrho_2 > \varrho_1$, $a \in [0,1]$ then, the definition of the new Caputo fractional derivative is:*

$$^{CF}\mathcal{D}^a u(z) = \frac{\mathcal{M}(a)}{1-a} \int_{\varrho_1}^{z} u^{'}(\lambda) \exp\left[-\frac{a(z-\lambda)}{1-a}\right] d\lambda, \tag{5}$$

where $\mathcal{M}(a)$ is normalization function.

The integral formula for the Caputo-Fabrizio fractional derivative is as follows.

Definition 4 ([2])**.** *Let $u \in \mathcal{H}^1(0, \varrho_2)$, $\varrho_2 > \varrho_1$, $a \in [0,1]$ then, the definition of the left and right side of Caputo-Fabrizio fractional integrals are:*

$$\left({}^{CF}_{\varrho_1}\mathcal{I}^a\right) u(z) = \frac{1-a}{\mathcal{B}(a)} u(z) + \frac{a}{\mathcal{B}(a)} \int_{\varrho_1}^{z} u(\lambda) d\lambda \tag{6}$$

and

$$\left({}^{CF}\mathcal{I}^a_{\varrho_2}\right) u(z) = \frac{1-a}{\mathcal{B}(a)} u(z) + \frac{a}{\mathcal{B}(a)} \int_{z}^{\varrho_2} u(\lambda) d\lambda, \tag{7}$$

where $\mathcal{B}(a)$ is normalization function.

Atangana-Baleanu [3] has found a solution to the problem of the Caputo-Fabrizio operator not being reduced to the original function in a special case, despite the fact that the operator is an effective tool in the solution of many systems of differential equations. The features of the Caputo-Fabrizio operator are present in the normalization function.

Some fractional order derivative and integral operators include the power law in their kernel. Nature does not usually exhibit power law behavior. This novel derivative and integral operator incorporates the Mittag-Leffler function [3]. The Mittag-Leffler function is required to model nature. This improved the Atangana-Baleanu operator and piqued researchers' interest. That the work uses the Atangana-Baleanu operator for Hermite-Hadamard inequalities is unusual. When the parameter is set to zero, the Atangana-Baleanu original function can be derived and compared to the Caputo-Fabrizio results.

Definition 5 ([3])**.** *Let $u \in \mathcal{H}^1(0, \varrho_2)$, $\varrho_2 > \varrho_1$, $a \in [0,1]$ then, the definition of the new fractional derivative is given below*

$$^{\mathcal{ABC}}_{\varrho_1}\mathcal{D}^a_z[u(z)] = \frac{\mathcal{B}(a)}{1-a} \int_{\varrho_1}^{z} u^{'}(\lambda) E_a\left[-\frac{a(z-\lambda)^a}{1-a}\right] d\lambda. \tag{8}$$

Definition 6 ([3])**.** *Let $u \in \mathcal{H}^1(0, \varrho_2)$, $\varrho_2 > \varrho_1$, $a \in [0,1]$ then, the definition of the new fractional derivative is given below:*

$$^{\mathcal{ABC}}_{\varrho_1}\mathcal{D}^a_z[u(z)] = \frac{\mathcal{B}(a)}{1-a} \frac{d}{dz} \int_{\varrho_1}^{z} u(\lambda) E_a\left[-\frac{a(z-\lambda)^a}{1-a}\right] d\lambda. \tag{9}$$

Equations (8) and (9) have a non-local kernel. Also in (9) when the function is constant we get zero. Associated integral operator for Atangana-Baleanu fractional derivative has been defined as follows.

Definition 7 ([3])**.** *The fractional integral associate to the new fractional derivative with non-local kernel of a function $u \in \mathcal{H}^1(0, \varrho_2)$ as defined:*

$$\left({}^{\mathcal{AB}}_{\varrho_1}\mathcal{I}^a\right)\{u(z)\} = \frac{1-a}{\mathcal{B}(a)} u(z) + \frac{a}{\mathcal{B}(a)\Gamma(a)} \int_{\varrho_1}^{z} u(\lambda)(z-\lambda)^{a-1} d\lambda, \tag{10}$$

where $\varrho_2 > \varrho_1$, $a \in [0,1]$.

The authors of [4] described the integral operator's right hand side as follows:

$$^{\mathcal{AB}}\mathcal{I}^{a}_{\varrho_2}\{u(z)\} = \frac{1-a}{\mathcal{B}(a)}u(z) + \frac{a}{\mathcal{B}(a)\Gamma(a)}\int_z^{\varrho_2} u(\lambda)(\lambda - z)^{a-1}d\lambda, \tag{11}$$

where $\varrho_2 > \varrho_1$, $a \in [0,1]$. Differential equations of arbitrary real order $\gamma > 0$ are used to model various physical models arising in many branches of science and engineering. Applications of such mathematical models can be seen from statistical mechanics and Brownian motion, via visco-elasticity problems, to continuum and quantum mechanics, biosciences, chemical engineering, and control theory, see for instance [5]. Nonlinearity is very important in studying wave equations, evolution equations, damped equations etc. and in some problems fractional nonlinearity is considered as the Riemann-Liouville fractional operator to get a nonlinear memory term also know as nonlinearity. In this paper, we study the global (in time) existence of small data solutions to

$$\begin{cases} \zeta_{\lambda\lambda} - \Delta\zeta + \zeta_\lambda - \Delta\zeta_\lambda = \mathcal{I}^\gamma(|\zeta|^\varsigma) & \text{for} \quad (\lambda, z) \in (0,\infty)\times\mathbb{R}^r, \\ \zeta(0,z) = \zeta_0(z), \zeta_\lambda(0,z) = \zeta_1(z) & \text{for} \quad z \in \mathbb{R}^r, \end{cases} \tag{12}$$

where

$$\mathcal{I}^\gamma(\zeta) \approx \int_0^\lambda (\lambda - s)^{-\gamma}\zeta(s)ds$$

denote the fractional Riemann-Liouville integral operator of order γ for some $\gamma \in (0,1)$. We also derive decay estimates for solutions to (12) and show the influence of fractional integral parameter γ. This problem important to the researchers working in field of differential equations and mathematical modeling. Over the last decade several papers have been devoted to the study of semilinear evolution model with the nonlinear term of memory type as in (13). In the pioneering paper [6] the authors determine the critical exponent for the semilinear heat equation with nonlinear memory term. Afterwards, this kind of result has been generalized for fractional (either in space or in time) heat equations [7–9] and for weakly coupled system of heat equations [8,10,11]. Mezadek et al. [12] considered the Cauchy problem for the semilinear wave equation with friction damping, visco-elastic damping and power nonlinearity

$$\begin{cases} \zeta_{\lambda\lambda} - \Delta\zeta + \zeta_\lambda - \Delta\zeta_\lambda = |\zeta|^\varsigma & \text{for} \quad (\lambda, z) \in (0,\infty)\times\mathbb{R}^r, \\ \zeta(0,z) = \zeta_0(z), \zeta_\lambda(0,z) = \zeta_1(z) & \text{for} \quad z \in \mathbb{R}^r, \end{cases} \tag{13}$$

where the data ζ_0 and ζ_1 are known as Cauchy data. Mezadek et al. [12] defined a generalized diffusion phenomena and demonstrated that the long time asymptotic of solutions is a mixture of diffusion and wave equation solutions. A more general case also treated recently in [13].

For the semilinear classical damped wave equation with friction damping and no viscoelastic damping, many mathematicians have attempted to solve the following Cauchy problem:

$$\begin{cases} \zeta_{\lambda\lambda} - \Delta\zeta + \zeta_\lambda = |\zeta|^\varsigma & \text{for} \quad (\lambda, z) \in (0,\infty)\times\mathbb{R}^r, \\ \zeta(0,z) = \zeta_0(z), \zeta_\lambda(0,z) = \zeta_1(z) & \text{for} \quad z \in \mathbb{R}^r. \end{cases} \tag{14}$$

Energy solutions for compactly supported data $(\zeta_0, \zeta_1) \in \mathcal{H}^1 \times \mathcal{L}^2$ and $\varsigma > 1$ (and $\varsigma \leq 1 + \frac{2}{r-2}$ if $r \geq 3$) were found to exist locally (in time) by Nakao and Ono [14]. Todorova and Yordanov [15] established the global (in time) existence for $\varsigma > 1 + \frac{2}{r}$, where the data $(\zeta_0, \zeta_1) \in \mathcal{H}^1 \times \mathcal{L}^2$ and that in terms of nonlinearities, the Fujita exponent is the vital exponent. $\left\{|\zeta|^\varsigma\right\}_{\varsigma>1}$ as can be seen from the results of Ikeda et al. [16] and Ikeda et al. [16]. In this case a blow-up result is proved for $\varsigma \in (1, 1 + \frac{2}{r}]$, even for small data from $(\mathcal{H}^1 \cap \mathcal{L}^1) \times (\mathcal{L}^2 \cap \mathcal{L}^1)$, (resp. $(\mathcal{H}^{\sigma,0} \cap \mathcal{H}^{0,\rho}) \times (\mathcal{H}^{\sigma-1,0} \cap \mathcal{H}^{0,\rho})$, where ρ satisfies $r\left(\frac{1}{a} - \frac{1}{2}\right) < \rho < \frac{2}{\varsigma-1} - \frac{r}{2}$ and $a \in [1,2]$. Ikehata and Ohta [17] studied (14) under the assumption $(\zeta_0, \zeta_1) \in (\mathcal{H}^1 \cap \mathcal{L}^k) \times (\mathcal{L}^2 \cap \mathcal{L}^k)$ for additional regularity $\mathcal{L}^k$, $k \in [1,2)$ for the data. Ikehata and Ohta obtained a new critical exponent $\varsigma_{cri\lambda} = 1 + \frac{2k}{r}$ for small data Sobolev solutions from both the global (in time) and blow-up perspectives.

Semilinear viscoelastic damped wave equations with viscoelastic damping and without friction have a large body of literature

$$\zeta_{\lambda\lambda} - \Delta\zeta - \Delta\zeta_\lambda = |\zeta|^\varsigma, \zeta(0,z) = \zeta_0(z), \zeta_\lambda(0,z) = \zeta_1(z). \tag{15}$$

Ikehata et al. [18] addressed the related linear Cauchy problem with vanishing right-hand side and demonstrated global well posedness. Shibata [19] provided $\mathcal{L}^\varsigma - \mathcal{L}^\beta$ estimations for Sobolev solutions and examined the diffusion phenomenon. Ikehata [20] investigated the asymptotic characteristic of Sobolev solutions as $\lambda \to \infty$, assuming that $(\zeta_0, \zeta_1) \in (\mathcal{H}^1 \cap \mathcal{L}^{1,1}) \times (\mathcal{L}^2 \cap \mathcal{L}^{1,1})$, where $\mathcal{L}^{1,1}$ represents a weighted $\mathcal{L}^1$ space. D'Abbicco and Reissig [21] devoted their findings to determining the crucial exponent ς^*. The critical exponent denotes the existence of global (in time) Sobolev solutions for little data for $\varsigma > \varsigma^*$ and simply the existence of local (in time) Sobolev solutions for large data for $1 < \varsigma \leq \varsigma^*$. Thus, a blow-up behavior can be expected in general. Since D'Abbicco and Reissig were unable to verify such a critical exponent so the critical exponent remains an open problem. It should be noted that the results of Theorem 2 in D'Abbicco and Reissig [21] are based on the usage of higher order regularity for the data, specifically, second order in space, and on the blending of multiple regularities for the data. The data, in particular, belong to the classical energy space.

Thus, it is feasible to investigate (13) using both friction and viscoelastic damping terms. The subject of the qualitative features of solutions to (13) emerges. It is an interesting point to understand the relationship between friction and viscoelastic damping. As a result, Ikehata and Sawada [22] demonstrated that the frictional damping effect is more prominent than the viscoelastic damping effect for the asymptotic profile as $\lambda \to \infty$. D'Abbicco [23] recently addressed the Cauchy problem (13), in which the data (ζ_0, ζ_1) are assumed to belong to the energy space with an additional $\mathcal{L}^1$ regularity, namely to $(\mathcal{H}^1 \cap \mathcal{L}^1) \times (\mathcal{L}^2 \cap \mathcal{L}^1)$. The author established that small data energy solutions exist globally (in time) for admissible exponents $\varsigma \in \left(1 + \frac{2}{r}, 1 + \frac{2}{r-2}\right]$ for $r > 1$. Ikehata and Takeda [24] investigated the identical Cauchy problem (13) with the following data assumptions $(\zeta_0, \zeta_1) \in (W^{\frac{r}{2}+\varepsilon,1} \cap W^{\frac{r}{2}+\varepsilon,\infty}) \times (\mathcal{L}^1 \cap \mathcal{L}^\infty)$, where $\varepsilon > 0$. The authors established a result regarding the global (in time) existence of small data Sobolev solutions exclusively for $r = 1, 2, 3$. They obtained this result for the range of permissible exponents ς meeting the condition $\varsigma > 1 + \frac{2}{r}$. Mezadek et al. [12] examined the Cauchy problem (13) under particular data and dimension r assumptions and investigated the effect of the regularity parameters $\sigma_1, \sigma_2 \in \mathbb{R}^+$, and the additional regularity parameter $k \in [1, 2)$ on the data $(\zeta_0, \zeta_1) \in (\mathcal{H}^{\sigma_1} \cap \mathcal{L}^k) \times (\mathcal{H}^{\sigma_2} \cap \mathcal{L}^k)$ on the acceptable ranges of exponents ς, which enables the global (in time) existence of small data Sobolev or energy solutions with a sufficient degree of regularity. Additionally, the authors investigated the effect of σ_1, σ_2 and k on solution regularity. Mezadek et al. [12] demonstrated the global (in time) existence of small data solutions to the semilinear Cauchy problem (13) in any space dimension $r \geq 1$ by applying estimates of linear Cauchy problem solutions to the semilinear Cauchy problem with power nonlinearity $|\zeta|^\varsigma$.

Small data solutions to the Cauchy problem for a semilinear wave equation with friction, viscoelastic damping and a fractional nonlinearity are the main goal of this paper. We want to find out if these solutions are globally available in time for small data. The main objectives of this paper is to show that fractional nonlinearity has an effect on the range of exponent ς that can be used and it also causes the decay rate to slow down with respect to the solution to the corresponding linear problem.

2. Main Results

2.1. Strategies

For the Cauchy problem (12), we will prove several results here. Our primary goal is to demonstrate the worldwide viability of small data solutions in the near future. As a

result, we can assume that the zero solution is stable right away.
We introduce for $\sigma_1 \geq \sigma_2 \geq 0$ and $k \in [1,2)$ the function space

$$\mathbb{A}_k^{\sigma_1,\sigma_2} := (\mathcal{H}^{\sigma_1} \cap \mathcal{L}^k) \times (\mathcal{H}^{\sigma_2} \cap \mathcal{L}^k), \qquad \mathbb{A}^{\sigma_1,\sigma_2} = \mathbb{A}_1^{\sigma_1,\sigma_2}$$

with the norm

$$\|(\zeta,v)\|_{\mathbb{A}_k^{\sigma_1,\sigma_2}} := \|\zeta\|_{\mathcal{H}^{\sigma_1}} + \|\zeta\|_{\mathcal{L}^k} + \|v\|_{\mathcal{H}^{\sigma_2}} + \|v\|_{\mathcal{L}^k}.$$

In the first instance, we suppose that the low regular data come from $\mathbb{A}^{\sigma_1,0}$. We will demonstrate that Sobolev solutions exist globally in time.

$$\mathbf{C}([0,\infty),\mathcal{H}^{\sigma_1}) \cap \mathbf{C}^1([0,\infty),\mathcal{L}^2)$$

where for large dimensions, an upper bound on the power nonlinearity exponent ς is necessary because it is larger than the modified Fujita exponent.

Case two outcomes are related to the highly regular data, which implies they belong to $\mathbb{A}_k^{\sigma_1,\sigma_2}$ and have a $\sigma_2 > 0$. We use tools from harmonic analysis from [25] to prove the existence of a global (in time) solution (see Appendix A). All these results require the condition $\varsigma > \lceil \sigma_2 \rceil + 1$ if $\sigma_2 \in (0, \frac{r}{2}]$. Here, we denote by $\lceil \sigma_2 \rceil := \min\{a \in \mathbb{Z} : \sigma_2 \leq a\}$ the ceiling function in σ_2.

Finally, if $\sigma_2 > \frac{r}{2}$, then using fractional powers the last condition $\varsigma > \lceil \sigma_2 \rceil$ will be weakened to $\varsigma > \sigma_2$.

2.2. Low Regular Data

2.2.1. Low Dimension

Theorem 1. *Let $r < 2(2-\sigma_1)$ and $\sigma_1 \in (0,2)$ be a real number and the data (ζ_0,ζ_1) are in $\mathbb{A}^{\sigma_1,0}$. Suppose that the exponent ς satisfies*

$$\varsigma > \frac{2+r}{r+2(\gamma-1)} \tag{16}$$

and

$$\begin{cases} 2 \leq \varsigma & \text{if} \quad \sigma_1 \geq \frac{r}{2}, \\ 2 \leq \varsigma \leq \varsigma_{GN,\sigma_1}(r) := \frac{r}{r-2\sigma_1} & \text{if} \quad \sigma_1 < \frac{r}{2}, \end{cases} \tag{17}$$

thus, there exists a small ϵ such that if

$$\|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}} \leq \epsilon,$$

then there exists a uniquely determined globally (in time) energy solution to (12) in

$$\mathbf{C}([0,\infty),\mathcal{H}^{\sigma_1}) \cap \mathbf{C}^1([0,\infty),\mathcal{L}^2).$$

According to estimates, the solution satisfies

$$\begin{aligned} \|\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{4}+(1-\gamma)}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}, \\ \||\mathcal{D}|^{\sigma_1}\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{4}-\frac{\sigma_1}{2}+(1-\gamma)}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}, \\ \|\zeta_\lambda(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{4}-1+(2-\gamma)}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}. \end{aligned}$$

Example 1. *In order to determine the permissible range for the fractional integral parameter γ, we look at the conditions (16) and (17) in some particular situations of low dimension that are dependent on the parameters r, k, σ and γ.*

r	Regularity σ_1	Admissible range for ς
$r=1$	$\sigma_1 \in \left[\frac{1}{2}, \frac{3}{2}\right)$	$\frac{3}{2\gamma-1} < \varsigma < \infty$.
	$\sigma_1 \in \left(\frac{2-\gamma}{3}, \frac{1}{2}\right)$	$\frac{3}{2\gamma-1} < \varsigma \leq \frac{1}{1-2\sigma_1}$.
$r=2$	$\sigma_1 \in [1,2)$	$\frac{2}{\gamma} < \varsigma < \infty$
	$\sigma_1 \in (1-\frac{\gamma}{2}, 1)$	$\frac{2}{\gamma} < \varsigma \leq \frac{1}{1-\sigma_1}$.

Example 2. *In this example we treat the case of third dimension $r = 3$ for fixed $\gamma = \frac{1}{2}$. The model that we have is given by*

$$\zeta_{\lambda\lambda} - \Delta\zeta + \zeta_\lambda - \Delta\zeta_\lambda = \mathcal{I}^{\frac{1}{2}}(|\zeta|^\varsigma) \text{ for } (\lambda, z) \in (0,\infty) \times \mathbb{R}^3,$$
$$\zeta(0,z) = \zeta_0(z), \zeta_\lambda(0,z) = \zeta_1(z) \text{ for } z \in (0,\infty) \times \mathbb{R}^3.$$

Then, the admissible range for ς can be described as follows:

$$\frac{5}{2} < \varsigma \quad \text{if} \quad \sigma_1 \in \left[\frac{3}{2}, 2\right),$$
$$\frac{5}{2} < \varsigma \leq \frac{3}{3-2\sigma_1} \quad \text{if} \quad \sigma_1 \in \left(\frac{9}{10}, \frac{3}{2}\right).$$

2.2.2. Higher Dimension

Theorem 2. *Let $r > 4$ and $\sigma_1 \geq 2$ be a real number and the data (ζ_0, ζ_1) are in $\mathbb{A}^{\sigma_1,0}$, where $k \in [1,2)$. Suppose that the exponent ς satisfies*

$$\varsigma > \frac{1}{\gamma} \tag{18}$$

and

$$\begin{cases} 2 \leq \varsigma & \text{if} \quad \sigma_1 \geq \frac{r}{2}, \\ 2 \leq \varsigma \leq \varsigma_{GN,\sigma_1}(r) & \text{if} \quad \sigma_1 < \frac{r}{2}, \end{cases} \tag{19}$$

thus, there exists a small ϵ such that if

$$\|(\zeta_0, \zeta_1)\|_{\mathbb{A}^{\sigma_1,0}} \leq \epsilon,$$

then there exists a uniquely determined globally (in time) energy solution to (12) in

$$\mathbf{C}([0,\infty), \mathcal{H}^{\sigma_1}) \cap \mathbf{C}^1([0,\infty), \mathcal{L}^2).$$

According to estimates, the solution satisfies

$$\|\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} \lesssim (1+\lambda)^{-\frac{r}{4}+A_\zeta} \|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}},$$
$$\||\mathcal{D}|^{\sigma_1}\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} \lesssim (1+\lambda)^{-\frac{r}{4}-\frac{\sigma_1}{2}+A_{|\mathcal{D}|^{\sigma_1}v}} \|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}},$$
$$\|\zeta_\lambda(\lambda,\cdot)\|_{\mathcal{L}^2} \lesssim (1+\lambda)^{-\frac{r}{4}-1+A_{\zeta_\lambda}} \|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}},$$

where $A_\zeta \geq \frac{r}{4} - \gamma$, $A_{|\mathcal{D}|^{\sigma_1}\zeta} = A_\zeta + \frac{\sigma_1}{2}$ and $A_{\zeta_\lambda} = A_\zeta + 1$ define the loss of decay in comparison with the corresponding decay estimates for the solution ζ to the linear Cauchy problem with vanishing right-hand side.

In the following theorems we will take the additional regularity $\mathcal{L}^k$ since the goal is the global existence where there is no blow-up results exist in the literature and also use the last case of estimates (22) in Proposition 1.

2.3. Data from Sobolev Spaces with Suitable Regularity

Theorem 3. *Let $r < \frac{2k}{2-k}(2-\sigma_1)$ and the data (ζ_0, ζ_1) are supposed to belong to $\mathbb{A}_k^{\sigma_1,\sigma_2}$, where $k \in [1,2), \sigma_2 \leq \frac{r}{2}$. Suppose that for the exponent ς :*

$$\varsigma > \max\left\{\frac{1+\frac{r}{2k}}{\frac{r}{2k}-1+\gamma}; \lceil\sigma_2\rceil + 1\right\}, \tag{20}$$

and

$$\begin{cases} 1+\frac{2(\sigma_1-\sigma_2)}{r} \leq \varsigma & \text{if} \quad \sigma_1 \in \left[\frac{r}{2}, \frac{r}{2}+\sigma_2\right), \\ 1+\frac{2(\sigma_1-\sigma_2)}{r} \leq \varsigma \leq 1+\min\left\{\frac{2}{r-2\sigma_1}; \frac{2(\sigma_1-\sigma_2)}{r-2\sigma_1}\right\} & \text{if} \quad \sigma_1 \in \left(0, \frac{r}{2}\right). \end{cases} \tag{21}$$

there exists a small ϵ such that, if

$$\|(\zeta_0, \zeta_1)\|_{\mathbb{A}_k^{\sigma_1,\sigma_2}} \leq \epsilon,$$

and there exists a uniquely determined globally (in time) energy solution to (12) in

$$\mathbf{C}([0,\infty), \mathcal{H}^{\sigma_1}) \cap \mathbf{C}^1([0,\infty), \mathcal{H}^{\sigma_2}).$$

Furthermore, the solution satisfies:

$$\begin{aligned}
\|\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1-\gamma}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,0}}, \\
\||\mathcal{D}|^{\sigma_1}\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-\frac{\sigma_1}{2}+1-\gamma}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,0}}, \\
\|\zeta_\lambda(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1-\gamma}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,0}}, \\
\||\mathcal{D}^{\sigma_2}\zeta_\lambda(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-\frac{\sigma_2}{2}+1-\gamma}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,0}}.
\end{aligned}$$

2.4. Large Regular Data

Theorem 4. *Let $r < \frac{2k}{2-k}(2-\sigma_1)$ and the data (ζ_0, ζ_1) are supposed to belong to $\mathbb{A}_k^{\sigma_1,\sigma_2}$, where $k \in [1,2), \sigma_2 > \frac{r}{2}$. Suppose that for the exponent ς :, we have*

$$\varsigma > \max\left\{\frac{1+\frac{r}{2k}}{\frac{r}{2k}-1+\gamma}; \sigma_2; 2\right\}.$$

Then for small ϵ, we have

$$\|(\zeta_0, \zeta_1)\|_{\mathbb{A}_k^{\sigma_1,\sigma_2}} \leq \epsilon,$$

and there exists a uniquely determined globally (in time) energy solution to (12) in

$$\mathbf{C}([0,\infty), \mathcal{H}^{\sigma_1}) \cap \mathbf{C}^1([0,\infty), \mathcal{H}^{\sigma_2}).$$

Moreover, the solution meets the estimates:

$$\begin{aligned}
\|\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1-\gamma}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,0}}, \\
\||\mathcal{D}|^{\sigma_1}\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-\frac{\sigma_1}{2}+1-\gamma}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,0}}, \\
\|\zeta_\lambda(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1-\gamma}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,0}}, \\
\||\mathcal{D}|^{\sigma_2}\zeta_\lambda(\lambda,\cdot)\|_{\mathcal{L}^2} &\lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-\frac{\sigma_2}{2}+1-\gamma}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,0}}.
\end{aligned}$$

Remark 1. *If we take in Theorems 3 and 4 the data belongs to $\mathbb{A}^{\sigma_1,\sigma_2}$, then we have to take $r > 2(2-\sigma_1)$ and the decay of the solution and their derivative will be define by $-\gamma$ as power similarly to Theorem 2.*

3. Philosophy of Our Approach and Proofs

In order to proceed to our proofs, we first introduce some tools and previous results.

Proposition 1. *Let $a \in \mathbb{R}, b > 1$ and $\gamma \in (0,1)$. Then,*

$$\int_0^\lambda (1+\lambda-\tau)^{-a} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-b} ds d\tau \lesssim \begin{cases} (1+\lambda)^{-\gamma} & \text{if} \quad a > 1, \\ (1+\lambda)^{-\gamma}\log(2+\lambda) & \text{if} \quad a = 1, \\ (1+\lambda)^{-\gamma+1-a} & \text{if} \quad a < 1. \end{cases} \tag{22}$$

Proposition 2. *Let $a \in \mathbb{R}$ and $b, \gamma \in (0,1)$. Then,*

$$\int_0^\lambda (1+\lambda-\tau)^{-a} \int_0^\tau (\tau-s)^{-\gamma}(1+s)^{-b} ds d\tau \lesssim \begin{cases} (1+\lambda)^{-\gamma+1-b} & \text{if} \quad a > 1, \\ (1+\lambda)^{-\gamma+1-b}\log(2+\lambda) & \text{if} \quad a = 1, \\ (1+\lambda)^{-\gamma+2-a-b} & \text{if} \quad a < 1. \end{cases} \tag{23}$$

Reference [26] contains the evidence supporting Propositions 1 and 2.

Proposition 3. *Let $(\zeta_0, \zeta_1) \in (\mathcal{H}^{\sigma_1} \cap \mathcal{L}^1) \times (\mathcal{H}^{\sigma_2} \cap \mathcal{L}^1)$ with $\sigma_2 + 2 \geq \sigma_1 \geq \sigma_2 \geq 0, k \in [1,2)$. Then the solution to the Cauchy problem*

$$\zeta_{\lambda\lambda} - \Delta\zeta - \zeta_\lambda - \Delta\zeta_\lambda = o, \qquad \zeta(0,z) = \zeta_0(z), \qquad \zeta_\lambda(0,z) = \zeta_1(z)$$

satisfies the decay estimates

$$\|\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} \lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)} \|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}, \tag{24}$$

$$\||\mathcal{D}|^{\sigma_2}\zeta_\lambda(\lambda,\cdot)\|_{\mathcal{L}^2} \lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-\frac{\sigma_2}{2}-1} \|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}, \tag{25}$$

$$\|\zeta_\lambda(\lambda,\cdot)\|_{\mathcal{L}^2} \lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-1} \|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}, \tag{26}$$

$$\||\mathcal{D}|^{\sigma_1}\zeta(\lambda,\cdot)\|_{\mathcal{L}^2} \lesssim (1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-\frac{\sigma_1}{2}} \|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}. \tag{27}$$

Proof of Proposition 3 can be found in [12].

We define the space of solutions $\chi(\lambda)$ by

$$\chi(\lambda) = \mathbf{C}([0,\lambda], \mathcal{H}^{\sigma_1}) \cap \mathbf{C}^1([0,\lambda], \mathcal{H}^{\sigma_2}),$$

where the norm of $\chi(\lambda)$ will be proposed separately for each theorem. We introduce the operator $\mathbf{N}$ by

$$\mathbf{N} : \zeta \in \chi(\lambda) \to \mathbf{N}\zeta = \mathbf{N}\zeta(\lambda,z) := \zeta^{ln}(\lambda,z) + \zeta^{nl}(\lambda,z). \tag{28}$$

We denote by $E_0(\lambda,0,z)$ and $E_1(\lambda,0,z)$ the fundamental solutions to the linear equation, namely

$$\zeta^{ln}(\lambda,z) := E_0(\lambda,0,z) *_{(z)} \zeta_0(z) + E_1(\lambda,0,z) *_{(z)} \zeta_1(z)$$

is a solution to the Cauchy problem

$$\zeta_{\lambda\lambda} - \Delta\zeta - \zeta_\lambda - \Delta\zeta_\lambda = 0, \;\; \zeta(0,z) = \zeta_0(z), \;\; \zeta_\lambda(0,z) = \zeta_1(z),$$

and

$$\zeta^{nl}(\lambda,z) := \int_0^\lambda E_1(\lambda,\tau,z) *_{(z)} \int_0^\tau (\tau-s)^{-\gamma} |\zeta(s,\chi)|^{\varsigma} ds d\tau$$

is a solution to the Cauchy problem

$$\zeta_{\lambda\lambda} - \Delta\zeta - \zeta_\lambda - \Delta\zeta_\lambda = \mathcal{I}^\gamma(|\zeta|^\varsigma), \;\; \zeta(0,z) = 0, \;\; \zeta_\lambda(0,z) = 0.$$

As a result of Proposition A4, we may prove the following inequalities:

$$\|\mathbf{N}\zeta\|_{\chi(\lambda)} \le C_0(\lambda)\|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,\sigma_2}} + C_1(\lambda)\|\zeta\|^{\varsigma}_{\chi(\lambda)}, \tag{29}$$

$$\|\mathbf{N}\zeta - \mathbf{N}\zeta\|_{\chi(\lambda)} \le C_2(\lambda)\|\zeta - \zeta\|_{\chi(\lambda)}\big(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)} + \|\zeta\|^{\varsigma-1}_{\chi(\lambda)}\big), \tag{30}$$

where $C_1(\lambda), C_2(\lambda) \longrightarrow 0$ for $\lambda \longrightarrow 0$ and $C_1(\lambda), C_2(\lambda) \le C$ for all $\lambda \in [0,\infty)$.

3.1. Proof of Theorem 1

Let

$$\|\zeta\|_{\chi(\lambda)} = \sup_{\tau\in[0,\lambda]}\left\{\sum_{|\eta|+\mathcal{J}\le 1}(1+\tau)^{\frac{r}{4}+\frac{|\eta|}{2}\sigma_1+\mathcal{J}-(1+\mathcal{J}-\gamma)}\|\partial_z^\eta\partial_\lambda^{\mathcal{J}}\zeta(\tau,\cdot)\|_{\mathcal{L}^2}\right\},$$

where $\mathcal{J} + |\eta| = 0,1$. We remark that if $\zeta \in \chi(\lambda)$, then $\|\zeta\|_{\chi(\tau)} \le \|\zeta\|_{\chi(\lambda)}$ for any $0 \le \tau \le \lambda$.

We begin the proof of (29). From the estimates (24) to (27) of Theorem 3 and the definition of the norm of solutions space $\chi(\lambda)$ we have

$$\begin{aligned}
\|\zeta^{ln}\|_{\chi(\lambda)} = \sup_{\tau\in[0,\lambda]}&\Big\{(1+\tau)^{\frac{r}{4}-1+\gamma}\|\zeta^{ln}(\tau,\cdot)\|_{\mathcal{L}^2}\\
&+(1+\tau)^{\frac{r}{4}+\frac{\sigma_1}{2}-1+\gamma}\||\mathcal{D}|^{\sigma_1}\zeta^{ln}(\tau,\cdot)\|_{\mathcal{L}^2(\mathbb{R}^r)} + (1+\tau)^{\frac{r}{4}-1+\gamma}\|\zeta^{ln}_\lambda(\tau,\cdot)\|_{\mathcal{L}^2(\mathbb{R}^r)}\Big\}\\
\lesssim \sup_{\tau\in[0,\lambda]}&\Big\{(1+\tau)^{\frac{r}{4}-1+\gamma}(1+\tau)^{-\frac{r}{4}}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}\\
&+(1+\tau)^{\frac{r}{4}+\frac{\sigma_1}{2}-1+\gamma}(1+\tau)^{-\frac{r}{4}-\frac{\sigma_1}{2}}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}\\
&+(1+\tau)^{\frac{r}{4}-1+\gamma}(1+\tau)^{-\frac{r}{4}-1}\|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}\Big\}\\
\lesssim \|(\zeta_0,\zeta_1)&\|_{\mathbb{A}^{\sigma_1,0}}.
\end{aligned}$$

Consequently,

$$\|\zeta^{ln}\|_{\chi(\lambda)} \lesssim \|(\zeta_0,\zeta_1)\|_{\mathbb{A}^{\sigma_1,0}}. \tag{31}$$

Simply proving (29), it suffices to prove:

$$\|\zeta^{nl}\|_{\chi(\lambda)} \lesssim \|\zeta\|^{\varsigma}_{\chi(\lambda)}. \tag{32}$$

For the term $\|\zeta^{nl}\|_{\mathcal{L}^2}$ we have

$$\|\zeta^{nl}\|_{\mathcal{L}^2} \lesssim \int_0^\lambda (1+\lambda-\tau)^{-\frac{r}{4}}\int_0^\tau(\tau-s)^{-\gamma}\||\zeta(\tau,\cdot)|^\varsigma]\|_{\mathcal{L}^1\cap\mathcal{L}^2}dsd\tau.$$

Thus, we have

$$\||\zeta(\tau,\cdot)|^\varsigma\|_{\mathcal{L}^1(\mathbb{R}^r)\cap\mathcal{L}^2(\mathbb{R}^r)} \lesssim \||\zeta(\tau,z)|^\varsigma\|_{\mathcal{L}^1} + \||\zeta(\tau,z)|^\varsigma\|_{\mathcal{L}^2}.$$

Proposition A1 and Gagliardo-Nirenberg inequality can be used to estimate both of the right-hand side terms. For the first term, we are able to get

$$\begin{aligned}\||\zeta(\tau,\cdot)|^\varsigma\|_{\mathcal{L}^1} &= \Big(\int_{\mathbb{R}^r}|\zeta(\tau,z)|^\varsigma d\chi\Big)^{\frac{1}{\varsigma}\varsigma}\\ &= \|\zeta(\tau,\cdot)\|^\varsigma_{\mathcal{L}^\varsigma(\mathbb{R}^r)}\\ &\lesssim \|\zeta(\tau,\cdot)\|^{(1-\theta)\varsigma}_{\mathcal{L}^2(\mathbb{R}^r)}\,\||\mathcal{D}|^\sigma\zeta(\tau,\cdot)\|^{\theta\varsigma}_{\mathcal{L}^2},\end{aligned}$$

where

$$\theta = \frac{r}{s}\Big(\frac{1}{2}-\frac{1}{\varsigma}\Big)\in[0,1]$$

which is due to the condition (17) for ς.

By using the norm of solutions space $\chi(\lambda)$ for $0\le\tau\le\lambda$, we get

$$\|\zeta(\tau,\cdot)\|^\varsigma_{\mathcal{L}^{k\varsigma}(\mathbb{R}^r)} \lesssim (1+\tau)^{(1-\theta)\varsigma\left(-\frac{r}{4}+1-\gamma\right)+\theta\varsigma\left(-\frac{r}{4}-\frac{s}{2}+1-\gamma\right)}\|\zeta\|^\varsigma_{\chi(\lambda)}.$$

Then

$$\|\zeta(\tau,\cdot)\|^\varsigma_{\mathcal{L}^{k\varsigma}(\mathbb{R}^r)} \lesssim (1+\tau)^{-\frac{r}{2}\varsigma+\frac{r}{2}+(1-\gamma)\varsigma}\|\zeta\|^\varsigma_{\chi(\lambda)}. \tag{33}$$

Estimating $\||\zeta(\tau,\cdot)|^\varsigma\|_{\mathcal{L}^2}$ follows the same principles. This is done by employing the Gagliardo-Nirenberg inequalities and the definition of the norm of the solution space $\chi(\lambda)$.

$$\begin{aligned}\||\zeta(\tau,\cdot)|^\varsigma\|_{\mathcal{L}^2} &= \Big(\int_{\mathbb{R}^r}|\zeta(\tau,z)|^{2\varsigma} d\chi\Big)^{\frac{1}{2\varsigma}\varsigma}\\ &= \|\zeta(\tau,\cdot)\|^\varsigma_{\mathcal{L}^{2\varsigma}(\mathbb{R}^r)}\\ &\lesssim \|\zeta(\tau,\cdot)\|^{\varsigma(1-\tilde\theta)}_{\mathcal{L}^2(\mathbb{R}^r)}\||\mathcal{D}|^\sigma\zeta(\tau,\cdot)\|^{\varsigma\tilde\theta}_{\mathcal{L}^2}\\ &\lesssim (1+\tau)^{(1-\tilde\theta)\varsigma\left(-\frac{r}{4}+1-\gamma\right)+\tilde\theta\varsigma\left(-\frac{r}{4}-\frac{\sigma}{2}+1-\gamma\right)}\|\zeta\|^\varsigma_{\chi(\lambda)}\\ &\lesssim (1+\tau)^{-\frac{r}{2}\varsigma+\frac{r}{4}+(1-\gamma)\varsigma}\|\zeta\|^\varsigma_{\chi(\lambda)},\end{aligned}$$

where

$$\tilde\theta = \frac{r}{\sigma}\Big(\frac{1}{2}-\frac{1}{2\varsigma}\Big)\in[0,1]$$

this is from condition (17) for ς. Hence, we may conclude for $0\le\tau\le\lambda$ the following estimate:

$$\|u(\tau,\cdot)\|^\varsigma_{\mathcal{L}^{2\varsigma}} \lesssim (1+\tau)^{-\frac{r}{2}\varsigma+\frac{r}{4}+(1-\gamma)\varsigma}\|\zeta\|^\varsigma_{\chi(\lambda)}. \tag{34}$$

All together leads to

$$\begin{aligned}\|\zeta^{nl}\|_{\mathcal{L}^2} &\lesssim \int_0^\lambda (1+\lambda-\tau)^{-\frac{r}{4}}\int_0^\tau(\tau-s)^{-\gamma}(1+\tau)^{-\frac{r}{2}\varsigma+\frac{r}{2}+(1-\gamma)\varsigma}\|\zeta\|^\varsigma_{\chi(\lambda)}dsd\tau\\ &\lesssim \|\zeta\|^\varsigma_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}+1-\gamma},\end{aligned}$$

Moreover, we have to assume the condition $-\frac{r}{2}\varsigma+\frac{r}{2}+(1-\gamma)\varsigma<-1$ which generate a Fujita-like upper bound (16). So, it follows the desired estimate

$$\|\zeta^{nl}\|_{\mathcal{L}^2} \lesssim \|\zeta\|^\varsigma_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}+1-\gamma}. \tag{35}$$

In the same way one can get

$$\||\mathcal{D}|^{\sigma_1}\zeta^{nl}\|_{\mathcal{L}^2} \lesssim \|\zeta\|^\varsigma_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}+\frac{\sigma_1}{2}+1-\gamma}, \tag{36}$$

and

$$\|\zeta^{nl}_\lambda\|_{\mathcal{L}^2} \lesssim \|\zeta\|^\varsigma_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}+1-\gamma}. \tag{37}$$

Taking into consideration (31) and (35) to (37) the estimate (29) is proved.

To prove (30) we assume that ζ and ξ belong to $\chi(\lambda)$. Then

$$\mathbf{N}\zeta - \mathbf{N}\xi = \int_0^\lambda E_1(\lambda,\tau,z) *_{(z)} \int_0^\tau (\tau-s)^{-\gamma}\big(|\zeta(\tau,z)|^\varsigma - |\xi(\tau,z)|^\varsigma\big)\, ds d\tau.$$

We control all norms appearing in $\|\mathbf{N}\zeta - \mathbf{N}\xi\|_{\chi(\lambda)}$. We have

$$\|\mathbf{N}\zeta - \mathbf{N}\xi\|_{\mathcal{L}^2} \lesssim \int_0^\lambda (1+\lambda-\tau)^{-\frac{r}{4}} \int_0^\tau (\tau-s)^{-\gamma} \||\zeta(\tau,z)|^\varsigma - |\xi(\tau,z)|^\varsigma\|_{\mathcal{L}^1\cap\mathcal{L}^2} ds d\tau. \quad (38)$$

Hölder's inequality implies

$$\begin{aligned}&\big\||\zeta(\tau,\cdot)|^\varsigma - |\xi(\tau,\cdot)|^\varsigma\big\|_{\mathcal{L}^2}\\ &\lesssim \big\|\zeta(\tau,\cdot) - \xi(\tau\cdot)\big\|_{\mathcal{L}^{2\varsigma}}\big(\|\zeta(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^{2\varsigma}} + \|\xi(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^{2\varsigma}}\big),\end{aligned} \quad (39)$$

and

$$\begin{aligned}&\big\||\zeta(\tau,\cdot)|^\varsigma - |\xi(\tau,z)|^\varsigma\big\|_{\mathcal{L}^1}\\ &\lesssim \big\|\zeta(\tau,\cdot) - \xi(\tau\cdot)\big\|_{\mathcal{L}^{\varsigma}}\big(\|\zeta(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^{\varsigma}} + \|\xi(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^{\varsigma}}\big).\end{aligned} \quad (40)$$

By using the norm of solution space $\chi(\lambda)$ and after applying the classical Gagliardo-Nirenberg inequality as we did for (33) and (34) we obtain the following estimates for $0 \le \tau \le \lambda$:

$$\|\zeta(\tau,\cdot) - \xi(\tau\cdot)\|_{\mathcal{L}^{2\varsigma}} \lesssim (1+\tau)^{-\frac{r}{2}+\frac{r}{4\varsigma}}\|\zeta-\zeta\|_{\chi(\lambda)},$$

$$\|\zeta(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^{2\varsigma}} \lesssim (1+\tau)^{\left(-\frac{r}{2}+\frac{r}{4\varsigma}\right)(\varsigma-1)}\|\zeta\|^{\varsigma-1}_{\chi(\lambda)},$$

$$\|\xi(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^{2\varsigma}} \lesssim (1+\tau)^{\left(-\frac{r}{2}+\frac{r}{4\varsigma}\right)(\varsigma-1)}\|\xi\|^{\varsigma-1}_{\chi(\lambda)},$$

$$\|\zeta(\tau,\cdot) - \xi(\tau,\cdot)\|_{\mathcal{L}^{\varsigma}} \lesssim (1+\tau)^{-\frac{r}{2}+\frac{r}{2\varsigma}}\|\zeta-\xi\|_{\chi(\lambda)},$$

$$\|\zeta(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^{\varsigma}} \lesssim (1+\tau)^{\left(-\frac{r}{2}+\frac{r}{2\varsigma}\right)(\varsigma-1)}\|u\|^{\varsigma-1}_{\chi(\lambda)},$$

$$\|\xi(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^{\varsigma}} \lesssim (1+\tau)^{\left(-\frac{r}{2}+\frac{r}{2\varsigma}\right)(\varsigma-1)}\|\xi\|^{\varsigma-1}_{\chi(\lambda)}.$$

Then we get

$$\begin{aligned}&\big\||\zeta(\tau,\cdot)|^\varsigma - |\xi(\tau,z)|^\varsigma\big\|_{\mathcal{L}^2}\\ &\lesssim (1+\tau)^{-\frac{r}{2}\varsigma+\frac{r}{4}}\|\zeta-\xi\|_{\chi(\lambda)}\big(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)} + \|\xi\|^{\varsigma-1}_{\chi(\lambda)}\big),\end{aligned} \quad (41)$$

and

$$\begin{aligned}&\big\||\zeta(\tau,\cdot)|^\varsigma - |\xi(\tau,z)|^\varsigma\big\|_{\mathcal{L}^1}\\ &\lesssim (1+\tau)^{-\frac{r}{2}\varsigma+\frac{r}{2}}\|\zeta-\xi\|_{\chi(\lambda)}\big(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)} + \|\xi\|^{\varsigma-1}_{\chi(\lambda)}\big).\end{aligned} \quad (42)$$

Applying the same ideas as we did to estimate $\left\||\mathcal{D}|^{\sigma}\zeta^{nl}(\lambda,\cdot)\right\|_{\mathcal{L}^2}$, this means, after plugging (41) and (42) into (38) one can get after using (16) the following estimates:

$$\begin{aligned}
&\left\||\mathcal{D}|^{\sigma}(\mathbf{N}\zeta-\mathbf{N}\xi)(\lambda,\cdot)\right\|_{\mathcal{L}^2} \\
&\lesssim (1+\lambda)^{-\frac{r}{4}-\frac{\sigma}{2}}\|\zeta-\xi\|_{\chi(\lambda)}\left(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)}+\|\xi\|^{\varsigma-1}_{\chi(\lambda)}\right),
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
&\left\|(\mathbf{N}\zeta-\mathbf{N}\xi)(\lambda,\cdot)\right\|_{\mathcal{L}^2} \\
&\lesssim (1+\lambda)^{-\frac{r}{4}}\|\zeta-\xi\|_{\chi(\lambda)}\left(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)}+\|\xi\|^{\varsigma-1}_{\chi(\lambda)}\right).
\end{aligned} \tag{44}$$

Then from the definition of $\chi(\lambda)$, the proof of (30) is completed.

3.2. Proof of Theorem 2

$$\chi(\lambda)=\mathbf{C}([0,\lambda],\mathcal{H}^{\sigma_1})\cap\mathbf{C}^1([0,\lambda],\mathcal{L}^2),$$

with the norm

$$\|\zeta\|_{\chi(\lambda)}=\sup_{\tau\in[0,\lambda]}\left\{\sum_{|\eta|+\mathcal{J}\le 1}(1+\tau)^{\frac{r}{4}-A_\zeta+\frac{|\eta|}{2}\sigma_1+\mathcal{J}}\|\partial_z^{\eta}\partial_\lambda^{\mathcal{J}}\zeta(\tau,\cdot)\|_{\mathcal{L}^2}\right\},$$

where $\mathcal{J}+|\eta|=0,1$. As a result of the introduced norm for the solution space's elements $\chi(\lambda)$ and the estimates of solutions to the homogeneous damped wave equation, we can derive (31). By demonstrating the inequality, we complete the prior theorem's proofs.

$$\|\zeta^{nl}\|_{\chi(\lambda)}\lesssim\|\zeta\|^{\varsigma}_{\chi(\lambda)} \tag{45}$$

- For $\|\zeta^{nl}\|_{\mathcal{L}^2}$ we have

  $$\|\zeta^{nl}\|_{\mathcal{L}^2}\lesssim\int_0^\lambda(1+\lambda-\tau)^{-\frac{r}{r}}\int_0^\tau(\tau-s)^{-\gamma}\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{L}^1\cap\mathcal{L}^2}dsd\tau.$$

  Similarly to (33) and under the same conditions described in the theorem we get

  $$\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{L}^1\cap\mathcal{L}^2}\lesssim\|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\tau)^{-\frac{r}{2}\varsigma+r+A_\zeta\varsigma-\frac{r}{4}\varsigma}. \tag{46}$$

  Using the last estimate we obtain

  $$\begin{aligned}
  \|\zeta^{nl}\|_{\mathcal{L}^2}&\lesssim\|\zeta\|^{\varsigma}_{\chi(\lambda)}\int_0^\lambda(1+\lambda-\tau)^{-\frac{r}{4}}\int_0^\tau(\tau-s)^{-\gamma}(1+\tau)^{-\frac{r}{2}\varsigma+r+A_\zeta\varsigma-\frac{r}{4}\varsigma}dsd\tau \\
  &\lesssim\|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\lambda)^{-\gamma} \\
  &\lesssim\|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}+A_\zeta}.
  \end{aligned}$$

  Hence

  $$\|\zeta^{nl}\|_{\mathcal{L}^2}\lesssim\|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}+A_\zeta}. \tag{47}$$

- For $\left\||\mathcal{D}|^{\sigma_1}\zeta^{nl}\right\|_{\mathcal{L}^2}$ we have

  $$\||\mathcal{D}|^{\sigma_1}\zeta^{nl}\|_{\mathcal{L}^2}\lesssim\int_0^\lambda(1+\lambda-\tau)^{-\frac{r}{4}-\frac{\sigma_1}{2}}\int_0^\tau(\tau-s)^{-\gamma}\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{L}^1\cap\mathcal{L}^2}dsd\tau.$$

Using (46) we get

$$\begin{aligned}\||\mathcal{D}|^{\sigma_1}\zeta^{nl}\|_{\mathcal{L}^2} &\lesssim \|\zeta\|^{\varsigma}_{\chi(\lambda)}\int_0^{\lambda}(1+\lambda-\tau)^{-\frac{r}{4}-\frac{\sigma_1}{2}}\int_0^{\tau}(\tau-s)^{-\frac{r}{2}\varsigma+r+A_\zeta\varsigma-\frac{r}{4}\varsigma}dsd\tau\\ &\lesssim \|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\lambda)^{-\gamma}\\ &\lesssim \|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}+A_\zeta}\\ &\lesssim \|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}-\frac{\sigma_1}{2}+A_{|\mathcal{D}|^{\sigma_1}\zeta}}.\end{aligned}$$

Hence

$$\||\mathcal{D}|^{\sigma_1}\zeta^{nl}\|_{\mathcal{L}^2} \lesssim \|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}-\frac{\sigma_1}{2}+A_{|\mathcal{D}|^{\sigma_1}\zeta}}. \tag{48}$$

- For $\|\zeta^{nl}_{\lambda}\|_{\mathcal{L}^2}$ analogously to (47) and (48) we get

$$\|\zeta^{nl}_{\lambda}\|_{\mathcal{L}^2} \lesssim \|\zeta\|^{\varsigma}_{\chi(\lambda)}(1+\lambda)^{-\frac{r}{4}-1+A_{\zeta_\lambda}}. \tag{49}$$

From (47)–(49) the prove of (45) is completed.
To control all norms appearing in $\|\mathbf{N}\zeta-\mathbf{N}\xi\|_{\chi(\lambda)}$ we follow exactly the same steps used in the proof of previous theorem.

3.3. *Proof of Theorem 3*

We define the space of solutions $\chi(\lambda)$ by

$$\chi(\lambda)=\mathbf{C}([0,\infty),\mathcal{H}^{\sigma_1})\cap\mathbf{C}^1([0,\infty),\mathcal{H}^{\sigma_2}).$$

Our goal is to prove (29) and (30). Let

$$\begin{aligned}\|\zeta\|_{\chi(\lambda)} = \sup_{\tau\in[0,\lambda]}\Big\{&(1+\tau)^{\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-(1-\gamma)}\|u(\tau,\cdot)\|_{\mathcal{L}^2}+(1+\tau)^{\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1-(2-\gamma)}\|\zeta_\lambda(\tau,\cdot)\|_{\mathcal{L}^2}\\ &+(1+\tau)^{\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1+\frac{\sigma_2}{2}-(2-\gamma)}\|\zeta_\lambda(\tau,\cdot)\|_{\mathcal{H}^{\sigma_2}}+(1+\tau)^{\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+\frac{\sigma_1}{2}-(1-\gamma)}\|\zeta(\tau,\cdot)\|_{\mathcal{H}^{\sigma_1}}\Big\}.\end{aligned}$$

From the defined norm and the estimates (24) to (27) we get

$$\|\zeta^{\mathcal{L}n}\|_{\chi(\lambda)} \lesssim \|(\zeta_0,\zeta_1)\|_{\mathbb{A}_k^{\sigma_1,\sigma_2}}.$$

Then, its remains to prove (32) to get (29).
For $\|\zeta^{n\mathcal{L}}\|_{\mathcal{L}^2}$ we have

$$\|\zeta^{n\mathcal{L}}\|_{\mathcal{L}^2} \lesssim \int_0^{\lambda}(1+\lambda-\tau)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)}\int_0^{\tau}(\tau-s)^{-\gamma}\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{L}^k\cap\mathcal{H}^{\sigma_2}}dsd\tau. \tag{50}$$

Similarly to (33) and (34) we obtain

$$\|\zeta(\tau,\cdot)\|^{\varsigma}_{\mathcal{L}^{k\varsigma}} \lesssim (1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{2k}+(1-\gamma)\varsigma}\|\zeta\|^{\varsigma}_{\chi(\lambda)} \tag{51}$$

and

$$\|\zeta(\tau,\cdot)\|^{\varsigma}_{\mathcal{L}^{2\varsigma}} \lesssim (1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{4}+(1-\gamma)\varsigma}\|\zeta\|^{\varsigma}_{\chi(\lambda)}, \tag{52}$$

provided that

$$\begin{aligned}\tfrac{2}{k}\le\varsigma \qquad &\text{if}\quad \sigma_1\ge\tfrac{r}{2},\\ \tfrac{2}{k}\le\varsigma\le\tfrac{r}{r-2\sigma_1}\qquad &\text{if}\quad \sigma_1<\tfrac{r}{2}.\end{aligned}$$

Still it remains to estimate the norm

$$\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{H}^{\sigma_2}}.$$

The fractional chain rule, which the reader can find in citation [25] or Appendix A, is therefore used as a harmonic analysis tool in this paper. We can estimate $\varsigma > \lceil \sigma_2 \rceil$ and $0 \le \tau \le \lambda$ by considering the Propositions A1 and A3, in particular Formula (A2).

$$\begin{aligned}
\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{H}^{\sigma_2}} &\lesssim \|\zeta(\tau,\cdot)\|_{\mathcal{L}^{q_1}}^{\varsigma-1}\||\mathcal{D}|^{\sigma_2}\zeta(\tau,\cdot)\|_{\mathcal{L}^{q_2}} \\
&\lesssim \|\zeta(\tau,\cdot)\|_{\mathcal{L}^2}^{(\varsigma-1)(1-\theta_1)}\||\mathcal{D}|^{\sigma_1}\zeta(\tau,\cdot)\|_{\mathcal{L}^2}^{(\varsigma-1)\theta_1}\|\zeta(\tau,\cdot)\|_{\mathcal{L}^2}^{1-\theta_2}\||\mathcal{D}|^{\sigma_1}\zeta(\tau,\cdot)\|_{\mathcal{L}^2}^{\theta_2} \\
&\lesssim (1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{4}-\frac{\sigma_2}{2}+(1-\gamma)\varsigma}\|\zeta\|_{\chi(\lambda)}^{\varsigma}.
\end{aligned}$$

Then

$$\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{H}^{\sigma_2}} \lesssim (1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{4}-\frac{\sigma_2}{2}+(1-\gamma)\varsigma}\|\zeta\|_{\chi(\lambda)}^{\varsigma}, \tag{53}$$

where

$$\frac{\varsigma-1}{q_1}+\frac{1}{q_2}=\frac{1}{2},\ \theta_1=\frac{r}{\sigma_1}\left(\frac{1}{2}-\frac{1}{q_1}\right)\in[0,1],\ \theta_2=\frac{r}{\sigma_1}\left(\frac{1}{2}-\frac{1}{q_2}\right)+\frac{\sigma_2}{\sigma_1}\in\left[\frac{\sigma_2}{\sigma_1},1\right].$$

Last conditions implies

$$\begin{array}{cccll}
2\le q_1 & \text{and} & 2\le q_2 & \text{if} & \sigma_1\in\left[\frac{r}{2}+\sigma_2,\infty\right), \\
2\le q_1 & \text{and} & 2\le q_2\le\frac{2r}{r-2(\sigma_1-\sigma_2)} & \text{if} & \sigma_1\in\left[\frac{r}{2},\frac{r}{2}+\sigma_2\right), \\
2\le q_1\le\frac{2r}{r-2\sigma_1} & \text{and} & 2\le q_2\le\frac{2r}{r-2(\sigma_1-\sigma_2)} & \text{if} & \sigma_1\in\left(0,\frac{r}{2}\right).
\end{array}$$

Then we get the following bounds:

$$\begin{array}{rll}
1+\frac{2(\sigma_1-\sigma_2)}{r}\le\varsigma & \text{if} & \sigma_1\in\left[\frac{r}{2},\frac{r}{2}+\sigma_2\right), \\
1+\frac{2(\sigma_1-\sigma_2)}{r}\le\varsigma\le1+\frac{2}{r-2\sigma_1} & \text{if} & \sigma_1\in\left(0,\frac{r}{2}\right).
\end{array}$$

Using (51)–(53) with the last estimates of (22) in (50), we get

$$\begin{aligned}
\|\zeta^{n\mathcal{L}}\|_{\mathcal{L}^2} &\lesssim \int_0^{\lambda}(1+\lambda-\tau)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)}\int_0^{\tau}(\tau-s)^{-\gamma}(1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{2k}+(1-\gamma)\varsigma}\|\zeta\|_{\chi(\lambda)}^{\varsigma}dsd\tau \\
&\lesssim \|\zeta\|_{\chi(\lambda)}^{\varsigma}(1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1-\gamma},
\end{aligned}$$

where $\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)<1$ from the restriction of dimension described in the theorem and using the conditions $-\frac{r}{2k}\varsigma+\frac{r}{2k}+(1-\gamma)\varsigma>1$ which generate a Fujita-like upper bound (16). Finally, we get

$$\|\zeta^{n\mathcal{L}}\|_{\mathcal{L}^2} \lesssim \|\zeta\|_{\chi(\lambda)}^{\varsigma}(1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1-\gamma}. \tag{54}$$

Similarly, we can get

$$\|\zeta_{\lambda}^{n\mathcal{L}}\|_{\mathcal{L}^2} \lesssim \|\zeta\|_{\chi(\lambda)}^{\varsigma}(1+\lambda)^{-\frac{n}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+1-\gamma}. \tag{55}$$

$$\||\mathcal{D}|^{\sigma_2}\zeta_{\lambda}^{n\mathcal{L}}\|_{\mathcal{L}^2} \lesssim \|\zeta\|_{\chi(\lambda)}^{\varsigma}(1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+\frac{\sigma_2}{2}+1-\gamma}. \tag{56}$$

$$\||\mathcal{D}|^{\sigma_1}\zeta^{n\mathcal{L}}\|_{\mathcal{L}^2} \lesssim \|\zeta\|_{\chi(\lambda)}^{\varsigma}(1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)+\frac{\sigma_1}{2}+1-\gamma}. \tag{57}$$

Last estimates complete the proof of (32). Now we prove the second inequality (30). Let ζ and ξ belong to $\chi(\lambda)$. Then

$$\|\mathbf{N}\zeta-\mathbf{N}\xi\|_{\mathcal{L}^2} \lesssim \int_0^{\lambda}(1+\lambda-\tau)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)}\int_0^{\tau}(\tau-s)^{-\gamma}\||\zeta(\tau,z)|^{\varsigma}-|\xi(\tau,z)|^{\varsigma}\|_{\mathcal{L}^k\cap\mathcal{H}^{\sigma_2}}dsd\tau.$$

Similar to (41) and (42), we get

$$\begin{aligned}&\big\||\zeta(\tau,\cdot)|^{\varsigma}-|\xi(\tau,z)|^{\varsigma}\big\|_{\mathcal{L}^2}\\&\lesssim(1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{4}}\|\zeta-\xi\|_{\chi(\lambda)}\big(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)}+\|\xi\|^{\varsigma-1}_{\chi(\lambda)}\big),\end{aligned}\tag{58}$$

$$\begin{aligned}&\big\||\zeta(\tau,\cdot)|^{\varsigma}-|\xi(\tau,z)|^{\varsigma}\big\|_{\mathcal{L}^k}\\&\lesssim(1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{2k}}\|\zeta-\xi\|_{\chi(\lambda)}\big(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)}+\|\xi\|^{\varsigma-1}_{\chi(\lambda)}\big).\end{aligned}\tag{59}$$

In the next step we may control $\big\||\zeta(\tau,\cdot)|^{\varsigma}-|\xi(\tau,z)|^{\varsigma}\big\|_{\mathcal{H}^{\sigma_2}(\mathbb{R}^r)}$. Indeed, using the fractional Leibniz rule from Proposition A2 we get

$$\begin{aligned}\||\zeta(\tau,\cdot)|^{\varsigma}-|\xi(\tau,\cdot)|^{\varsigma}\|_{\mathcal{H}^{\sigma_2}}&\lesssim\int_0^1\big\||\mathcal{D}|^{\sigma_2}\Big\{(\zeta-\xi)(\zeta-\iota(\zeta-\xi))|\zeta-\iota(\zeta-\xi)|^{\varsigma-2}\Big\}\big\|_{\mathcal{L}^2}dr\\&\lesssim\int_0^1\big\||\mathcal{D}|^{\sigma_2}(\zeta-\xi)\big\|_{\mathcal{L}^{q_1}}\big\|(\zeta-\iota(\zeta-\xi))|\zeta-\iota(\zeta-\xi)|^{\varsigma-2}\big\|_{\mathcal{L}^{q_2}}dr\\&+\int_0^1\|\zeta-\xi\|_{\mathcal{L}^{q_3}}\big\||\mathcal{D}|^{\sigma_2}\big[(\zeta-\iota(\zeta-\xi))|\zeta-\iota(\zeta-\xi)|^{\varsigma-2}\big]\big\|_{\mathcal{L}^{q_4}}dr,\end{aligned}$$

where

$$\frac{1}{2}=\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}+\frac{1}{q_4}.$$

For the first integral we use the classical Gagliardo-Nirenberg inequality and obtain for $0\le\tau\le\lambda$

$$\big\||\mathcal{D}|^{\sigma_2}(\zeta-\xi)\big\|_{\mathcal{L}^{q_1}}\lesssim\|\zeta-\xi\|^{1-\theta_1}_{\mathcal{L}^2}\big\||\mathcal{D}|^{\sigma_1}(\zeta-\xi)\big\|^{\theta_1}_{\mathcal{L}^2}\lesssim(1+\tau)^{-\frac{r}{2k}-\frac{\sigma_2}{2}+\frac{r}{2q_1}+(1-\gamma)}\|\zeta-\xi\|_{\chi(\lambda)},$$

and

$$\begin{aligned}\big\|(\zeta-\iota(\zeta-\xi))|\zeta-\iota(\zeta-\xi)|^{\varsigma-2}\big\|_{\mathcal{L}^{q_2}}&\lesssim\|\zeta-\iota(\zeta-\xi)\|^{(1-\theta_2)(\varsigma-1)}_{\mathcal{L}^2}\big\||\mathcal{D}|^{\sigma_1}(\zeta-\iota(\zeta-\xi))\big\|^{\theta_2(\varsigma-1)}_{\mathcal{L}^2}\\&\lesssim(1+\tau)^{-\frac{r}{2k}(\varsigma-1)+\frac{r}{2q_2}}\|\zeta-\iota(\zeta-\xi)\|^{\varsigma-1}_{\chi(\lambda)}\end{aligned}$$

for

$$\theta_1=\frac{r}{\sigma_1}\Big(\frac{1}{2}-\frac{1}{q_1}\Big)+\frac{\sigma_2}{\sigma_1}\in\Big[\frac{\sigma_2}{\sigma_1},1\Big],\ \ \theta_2=\frac{r}{\sigma_1}\Big(\frac{1}{2}-\frac{1}{q_2(\varsigma-1)}\Big)\in[0,1].$$

The last conditions implies that

$$\begin{array}{rclll}2\le q_1 & \text{and} & \frac{2}{\varsigma-1}\le q_2 & \text{if} & \sigma_1\in\big[\frac{r}{2}+\sigma_2,\infty\big),\\ 2\le q_1 & \text{and} & \frac{2}{\varsigma-1}\le q_2\le\frac{2r}{r-2(\sigma_1-\sigma_2)} & \text{if} & \sigma_1\in\big[\frac{r}{2},\frac{r}{2}+\sigma_2\big),\\ 2\le q_1\le\frac{2r}{r-2(\sigma_1-\sigma_2)} & \text{and} & \frac{2}{\varsigma-1}\le q_2\le\frac{2r}{r-2(\sigma_1-\sigma_2)} & \text{if} & \sigma_1\in\big(0,\frac{r}{2}\big).\end{array}$$

Thus we get the following bounds:

To estimate the first term in the second integral we use again the Gagliardo-Nirenberg inequality.

In this way we, obtain

$$\|\zeta-\xi\|_{\mathcal{L}^{q_3}}\lesssim\|\zeta-\xi\|^{1-\theta_3}_{\mathcal{L}^2}\big\||\mathcal{D}|^{\sigma_1}(\zeta-\xi)\big\|^{\theta_3}_{\mathcal{L}^2}\lesssim(1+\tau)^{-\frac{r}{2k}+\frac{r}{2q_3}+1-\gamma}\|\zeta-\xi\|_{\chi(\lambda)},$$

where

$$\theta_3=\frac{r}{\sigma_1}\Big(\frac{1}{2}-\frac{1}{q_3}\Big)\in[0,1].$$

To estimate the second term we use the fractional chain rule from Proposition A3.

Hence, we get

$$\left\||\mathcal{D}|^{\sigma_2}\left[(\zeta-\iota(\zeta-\xi))|\zeta-\iota(\zeta-\xi)|^{\varsigma-2}\right]\right\|_{\mathcal{L}^{q_4}} \lesssim \|\zeta-\iota(\zeta-\xi)\|^{\varsigma-2}_{\mathcal{L}^{q_5}}\||\mathcal{D}|^{\sigma_2}(\zeta-\iota(\zeta-\xi))\|_{\mathcal{L}^{q_6}},$$

where

$$\frac{1}{q_4}=\frac{\varsigma-2}{q_5}+\frac{1}{q_6}.$$

Using Gagliardo-Nirenberg inequality to estimate the last two norms, we get

$$\begin{aligned}\|\zeta-\iota(\zeta-\xi)\|^{\varsigma-2}_{\mathcal{L}^{q_5}} &\lesssim \|\zeta-\iota(\zeta-\xi)\|^{(p-2)(1-\theta_5)}_{\mathcal{L}^2}\||\mathcal{D}|^{\sigma_1}(\zeta-\iota(\zeta-\xi))\|^{(p-2)\theta_5}_{\mathcal{L}^2}\\ &\lesssim (1+\tau)^{\left(-\frac{r}{2k}+\frac{r}{2q_5}+1-\gamma\right)(p-2)}\|\zeta-\iota(\zeta-\xi)\|^{\varsigma-2}_{\chi(\lambda)},\end{aligned}$$

and

$$\begin{aligned}\||\mathcal{D}|^{\sigma_2}(\zeta-\iota(\zeta-\xi))\|_{\mathcal{L}^{q_6}} &\lesssim \|\zeta-\iota(\zeta-\xi)\|^{1-\theta_6}_{\mathcal{L}^2}\||\mathcal{D}|^{\sigma_1}(\zeta-\iota(\zeta-\xi))\|^{\theta_6}_{\mathcal{L}^2}\\ &\lesssim (1+\tau)^{-\frac{r}{2k}+\frac{r}{2q_6}-\frac{\sigma_2}{2}+1-\gamma}\|\zeta-\iota(\zeta-v)\|_{\chi(\lambda)}\end{aligned}$$

for

$$\theta_5=\frac{r}{\sigma_1}\left(\frac{1}{2}-\frac{1}{q_5}\right)\in[0,1],\ \ \theta_6=\frac{r}{\sigma_1}\left(\frac{1}{2}-\frac{1}{q_6}\right)+\frac{\sigma_2}{\sigma_1}\in\left[\frac{\sigma_2}{\sigma_1},1\right].$$

The last conditions imply that

$$\begin{array}{rcll} 2\le q_3,q_5 & \text{and} & 2\le q_6 & \text{if}\ \ \sigma_1\in\left[\frac{r}{2}+\sigma_2,\infty\right),\\ 2\le q_3,q_5 & \text{and} & 2\le q_6\le\frac{2r}{r-2(\sigma_1-\sigma_2)} & \text{if}\ \ \sigma_1\in\left[\frac{r}{2},\frac{r}{2}+\sigma_2\right),\\ 2\le q_3,q_5\le\frac{2r}{r-2\sigma_1} & \text{and} & 2\le q_6\le\frac{2r}{r-2(\sigma_1-\sigma_2)} & \text{if}\ \ \sigma_1\in\left(0,\frac{r}{2}\right).\end{array}$$

One possibility to choose the parameters q_3, q_4, q_5 and q_6 satisfying the last conditions is

$$q_3=\frac{r(\varsigma-1)}{\sigma_1-\sigma_2},\ \ q_4=\frac{2n(\varsigma-1)}{r(\varsigma-1)-2(\sigma_1-\sigma_2)},\ \ q_5=\frac{r(\varsigma-1)}{\sigma_1-\sigma_2},\ \ q_6=\frac{2r}{r-2(\sigma_1-\sigma_2)}.$$

These choices imply the condition

$$\begin{array}{rl} 1+\frac{2(\sigma_1-\sigma_2)}{r}\le\varsigma & \text{if}\ \ \sigma_1\in\left[\frac{r}{2},\frac{r}{2}+\sigma_2\right),\\ 1+\frac{2(\sigma_1-\sigma_2)}{r}\le\varsigma\le 1+\frac{2(\sigma_1-\sigma_2)}{r-2\sigma_1} & \text{if}\ \ \sigma_1\in\left(0,\frac{r}{2}\right).\end{array}$$

Consequently, we obtain for $0\le\tau\le\lambda$ the estimate

$$\begin{aligned}&\left\||\zeta(\tau,\cdot)|^{\varsigma}-|\xi(\tau,z)|^{\varsigma}\right\|_{\mathcal{H}^{\sigma_2}(\mathbb{R}^r)}\\ &\lesssim\int_0^1(1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{4}-\frac{\sigma_2}{2}+(1-\gamma)\varsigma}\|\zeta-\xi\|_{\chi(\lambda)}\|\zeta-\iota(\zeta-\xi)\|^{\varsigma-1}_{\chi(\lambda)}dr\\ &\lesssim\int_0^1(1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{4}-\frac{\sigma_2}{2}(1-\gamma)\varsigma}\|\zeta-\xi\|_{\chi(\lambda)}\left(\|u\|^{\varsigma-1}_{\chi(\lambda)}+\|v\|^{\varsigma-1}_{\chi(\lambda)}\right)dr\\ &\lesssim(1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{4}-\frac{\sigma_2}{2}(1-\gamma)\varsigma}\|\zeta-\xi\|_{\chi(\lambda)}\left(\|u\|^{\varsigma-1}_{\chi(\lambda)}+\|v\|^{\varsigma-1}_{\chi(\lambda)}\right).\end{aligned}$$

All together similar to the control of $\|\mathbf{N}\zeta-\mathbf{N}\zeta\|_{\chi(\lambda)}$ in Theorem 1 leads to

$$\begin{aligned}&\left\||\mathcal{D}|^{\sigma}(\mathbf{N}\zeta-\mathbf{N}\xi)(\lambda,\cdot)\right\|_{\mathcal{L}^2}\\ &\lesssim(1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)-\frac{\sigma}{2}}\|\zeta-\xi\|_{\chi(\lambda)}\left(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)}+\|\xi\|^{\varsigma-1}_{\chi(\lambda)}\right),\end{aligned}\tag{60}$$

$$\begin{aligned}&\big\|(\mathbf{N}\zeta-\mathbf{N}\zeta)(\lambda,\cdot)\big\|_{\mathcal{L}^2(\mathbb{R}^r)}\\&\lesssim(1+\lambda)^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)}\|\zeta-\xi\|_{\chi(\lambda)}\big(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)}+\|\xi\|^{\varsigma-1}_{\chi(\lambda)}\big).\end{aligned}\tag{61}$$

The proof is completed.

3.4. *Proof of Theorem 4*

As we did in the proof to Theorem 3 we need just to modify the estimate of the norm $\||\mathcal{D}|^{\sigma_2}\zeta^{n\mathcal{L}}_{\lambda}(\lambda,\cdot)\|_{\mathcal{L}^2}$ by using the Lemma A1 introduced by D'Abbicco in [27]. Then we get for $\sigma^*<\frac{r}{2}$ the following estimate:

$$\begin{aligned}\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{H}^{\sigma_2}(\mathbb{R}^r)}&\lesssim\|\zeta(\tau,\cdot)\|_{\mathcal{H}^{\sigma_2}(\mathbb{R}^r)}\|\zeta(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{L}^\infty(\mathbb{R}^r)}\\&\lesssim\|\zeta(\tau,\cdot)\|_{\mathcal{H}^{\sigma_2}(\mathbb{R}^r)}\big(\|\zeta(\tau,\cdot)\|_{\mathcal{H}^{\sigma^*}(\mathbb{R}^r)}+\|\zeta(\tau,\cdot)\|_{\mathcal{H}^{\sigma_2}(\mathbb{R}^r)}\big)^{\varsigma-1}\\&\lesssim\|\zeta(\tau,\cdot)\|^{\varsigma}_{\mathcal{H}^{\sigma_2}(\mathbb{R}^r)}+\|\zeta(\tau,\cdot)\|_{\mathcal{H}^{\sigma_2}(\mathbb{R}^r)}\|\zeta(\tau,\cdot)\|^{\varsigma-1}_{\mathcal{H}^{\sigma^*}(\mathbb{R}^r)}.\end{aligned}$$

Gagliardo-Nirenberg inequality and the definition of the solution space $\chi(\lambda)$ lead us to conclusion

$$\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{H}^{\sigma_2}}\lesssim(1+B(\tau,0))^{-\frac{r}{2}\left(\frac{1}{k}-\frac{1}{2}\right)p-\frac{\sigma_2}{2}-\frac{\sigma^*}{2}(\varsigma-1)+(1-\gamma)\varsigma}\|\zeta\|^{\varsigma}_{\chi(\lambda)}.\tag{62}$$

If we choose $\sigma^*=\frac{r}{2}-\varepsilon$, then we get

$$-\frac{r}{2}\Big(\frac{1}{k}-\frac{1}{2}\Big)\varsigma-\frac{\sigma_2}{2}-\frac{\sigma^*}{2}(\varsigma-1)\le-\frac{r}{2k}\varsigma+\frac{r}{2k}.$$

Thus, we obtain

$$\||\zeta(\tau,\cdot)|^{\varsigma}\|_{\mathcal{L}^k\cap\mathcal{H}^{\sigma_2}}\lesssim(1+\tau)^{-\frac{r}{2k}\varsigma+\frac{r}{2k}+(1-\gamma)\varsigma}\|\zeta\|^{\varsigma}_{\chi(\lambda)}.$$

Using the most recent estimate, the same steps as in the proof of Theorem 3 can be used to conclude the proof.

4. Concluding Remarks

There are a number of Semiliear Cauchy problems in the literature which have the same decay estimates for the homogeneous problem as that of the Cauchy problem (12) with friction, viscoelastic damping. The results that we have acquired also hold for the semilinear Cauchy problems that already exist in mathematical literature. In this study, we have discussed the global existence in time of small data solutions to the Cauchy problem (12) with friction, viscoelastic damping and a fractional nonlinearity, where the data are supposed to belong to different classes of regularity and $\mathcal{I}^{\gamma}$ denote the Caputo fractional integral of order γ defined by $\mathcal{I}^{\gamma}(\zeta)\approx\int_0^{\lambda}(\lambda-s)^{-\gamma}\zeta(s)ds$ for some $\gamma\in(0,1)$. We have also shown the influence of the fractional nonlinearity to the admissible range of exponent ς comparing with power nonlinearity and also the generating of loss of decay. Indeed the Cauchy problem studied in this paper is more general than the Cauchy problems (13)–(15) since it contains the global existence in time of small data solutions when the data are supposed to belong to different classes of regularity.

Author Contributions: Conceptualization, A.M.D.; Formal analysis, A.M.D.; Writing—original draft, A.M.D.; Writing—review & editing, M.A.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Project No. GRANT492], (The Number of the old project that is transferred to the new project is No. AN000441).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no competing interest.

Appendix A

Here we state some inequalities which come into play in our proofs.

Proposition A1. *Let $1 < \varsigma, \varsigma_0, \varsigma_1 < \infty$, $\varrho > 0$ and $s \in [0, \varrho)$. Then the following fractional Gagliardo-Nirenberg inequality holds for all $\zeta \in \mathcal{L}^{\varsigma_0} \cap \mathcal{H}^{\varrho}_{\varsigma_1}$:*

$$\|\zeta\|_{\mathcal{H}^{\varrho}_{\varsigma}} \lesssim \|\zeta\|^{1-\theta}_{\mathcal{L}^{\varsigma_0}} \|\zeta\|^{\theta}_{\mathcal{H}^{\varrho}_{\varsigma_1}}, \tag{A1}$$

where

$$\theta = \theta_{s,\varrho} := \frac{\frac{1}{\varsigma_0} - \frac{1}{\varsigma} + \frac{\varrho}{r}}{\frac{1}{\varsigma_0} - \frac{1}{\varsigma_1} + \frac{\varrho}{r}} \quad \text{and} \quad \frac{\varrho}{\varrho} \leq \theta \leq 1.$$

For the proof see [28–34].

Proposition A2. *Let us assume $\sigma > 0$ and $1 \leq m \leq \infty, 1 < \varsigma_1, \varsigma_2, q_1, q_2 \leq \infty$ satisfying the relation*

$$\frac{1}{m} = \frac{1}{\varsigma_1} + \frac{1}{\varsigma_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then the following fractional Leibniz rule holds:

$$\||\mathcal{D}|^{\sigma}(fg)\|_{\mathcal{L}^m} \lesssim \||\mathcal{D}|^{\sigma} f\|_{\mathcal{L}^{\varsigma_1}} \|g\|_{\mathcal{L}^{\varsigma_2}} + \|f\|_{\mathcal{L}^{q_1}} \||\mathcal{D}|^{\sigma} g\|_{\mathcal{L}^{q_2}},$$

for all $f \in \mathcal{H}^{\sigma}_{\varsigma_1} \cap \mathcal{L}^{q_1}$ and $g \in \mathcal{H}^{\sigma}_{q_2} \cap \mathcal{L}^{\varsigma_2}$.

For more details concerning fractional Leibniz rule see [29].

Proposition A3. *Let us choose $\sigma > 0, \varsigma > \lceil \sigma \rceil$ and $1 < m, m_1, m_2 < \infty$ satisfying*

$$\frac{1}{m} = \frac{\varsigma - 1}{m_1} + \frac{1}{m_2}.$$

Let us denote by $F(u)$ one of the functions $|\zeta|^{\varsigma}$, $\pm|\zeta|^{\varsigma-1}\zeta$. Then the following fractional chain rule holds:

$$\||\mathcal{D}|^{\sigma} F(u)\|_{\mathcal{L}^m} \lesssim \|\zeta\|^{\varsigma-1}_{\mathcal{L}^{m_1}} \||\mathcal{D}|^{\sigma}\zeta\|_{\mathcal{L}^{m_2}}, \tag{A2}$$

For the proof see [25].

Lemma A1. *Let $0 < 2\sigma^* < r < 2\sigma$. Then for any function $f \in \mathcal{H}^{\sigma^*} \cap \mathcal{H}^{\sigma}$ one has the estimate*

$$\|f\|_{\mathcal{L}^{\infty}} \leq \|f\|_{\mathcal{H}^{\sigma^*}} + \|f\|_{\mathcal{H}^{\sigma}}.$$

For the proof see [6].

Proposition A4. *The operator $\mathbf{N}$ maps $\chi(\lambda)$ into itself and has one and only one fixed point $\zeta \in \chi(\lambda)$ if the following inequalities hold:*

$$\|\mathbf{N}\zeta\|_{\chi(\lambda)} \leq C_0(\lambda)\|(\zeta_0, \zeta_1)\|_{\mathbb{A}_{m,s}} + C_1(\lambda)\|\zeta\|^{\varsigma}_{\chi(\lambda)}, \tag{A3}$$

$$\|\mathbf{N}\zeta - \mathbf{N}\zeta\|_{\chi(\lambda)} \leq C_2(\lambda)\|\zeta - \zeta\|_{\chi(\lambda)}\big(\|\zeta\|^{\varsigma-1}_{\chi(\lambda)} + \|\zeta\|^{\varsigma-1}_{\chi(\lambda)}\big), \tag{A4}$$

where $C_1(\lambda), C_2(\lambda) \longrightarrow 0$ for $\lambda \longrightarrow +0$ and $C_0(\lambda), C_1(\lambda), C_2(\lambda) \leq C$ for all $\lambda \in [0, \infty)$.

For the proof see [35].

References

1. Caputo, M.; Fabrizio, M.A. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
2. Abdeljawad, T.; Baleanu, D. On fractional derivatives with exponential kernel and their discrete versions. *Rep. Math. Phys.* **2017**, *80*, 11–27. [CrossRef]
3. Atangana, A.; Baleanu, D. New fractional derivatives with non-local and non-singular kernel, theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763–769. [CrossRef]
4. Abdeljawad, T.; Baleanu, D. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *J. Nonlinear Sci. Appl.* **2017**, *10*, 1098–1107. [CrossRef]
5. Podlubny, I. Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. In *Mathematics in Science and Engineering*; Elsevier: New York, NY, USA, 1999; Volume 198.
6. D'Abbicco, M. The threshold of effective damping for semilinear wave equations. *Math. Meth. Appl. Sci.* **2015**, *38*, 1032–1045. [CrossRef]
7. Cazenave, T.; Dickstein, F.; Weissler, F.B. An equation whose Fujita critical exponent is not given by scaling. *Nonlinear Anal.* **2008**, *68*, 862–874. [CrossRef]
8. Lai, N.A.; Zhou, Y. An elementary proof of Strauss conjecture. *J. Funct. Anal.* **2014**, *267*, 1364–1381.[CrossRef]
9. Wakasa, K.; Yordanov, B. Blow-up of solutions to critical semilinear wave equations with variable coefficients. *J. Differ. Equ.* **2019**, *266*, 5360–5376. [CrossRef]
10. Fino, A.Z.; Kirane, M. Qualitative properties of solutions to a nonlocal evolution system. *Math. Methods Appl. Sci.* **2011**, *34*, 1125–1143. [CrossRef]
11. Xu, Y.Q.; Tan, Z. Blow-up of solutions for a time-space fractional evolution system. *Acta Math. Sin.-Engl. Ser.* **2013**, *29*, 1067–1074. [CrossRef]
12. Mezadek, M.K.; Mezadek, M.K.; Reissig, M. Semilinear wave models with friction and viscoelastic damping. *Math. Meth. Appl. Sci.* **2020**, *47*, 3117–3147.
13. Chen, W.; D'Abbicco, M.; Girardi, G. Global small data solutions for semilinear waves with two dissipative terms. *Ann. Mat.* **2022**, *201*, 529–560. [CrossRef]
14. Nakao, M.; Ono, K. Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations. *Math. Z.* **1993**, *214*, 325–342. [CrossRef]
15. Todorova, M.; Yordanov, B. Critical exponent for a nonlinear wave equation with damping. *J. Diff. Eq.* **2001**, *174*, 464–489.[CrossRef]
16. Ikehata, R.; Miyaoka, Y.; Nakatake, T. Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^r$ with lower power nonlinearities. *J. Math. Soc. Jpn.* **2004**, *56*, 365–373. [CrossRef]
17. Ikehata, R.; Ohta, M. Critical exponents for semilinear dissipative wave equations in $\mathbb{R}^N$. *J. Math. Anal. Appl.* **2002**, *269*, 87–97. [CrossRef]
18. Ikehata, R.; Tanizawa, K. Global existence of solutions for semilinear damped wave equations in $\mathbb{R}^r$ with noncompactly supported initial data. *Nonlinear Anal.* **2005**, *61*, 1189–1208. [CrossRef]
19. Shibata, Y. On the rate of decay of solutions to linear viscoelastic equation. *Math. Meth. Appl. Sci.* **2000**, *23*, 203–226. [CrossRef]
20. Ikehata, R. Asymptotic profiles for wave equations with strong damping. *J. Diff. Equ.* **2014**, *257*, 2159–2177. [CrossRef]
21. D'Abbicco, M.; Reissig, M. Semilinear structural damped waves. *Math. Methods Appl. Sci.* **2014**, *37*, 1570–1592. [CrossRef]
22. Ikehata, R.; Sawada, A. Asymptotic profile of solutions for wave equations with frictional and viscoelastic damping terms. *Asymptot Anal.* **2016**, *98*, 59–77. [CrossRef]
23. D'Abbicco, M. $L^1 - L^1$ estimates for a doubly dissipative semilinear wave equation. *Nonlinear Differ. Equ. Appl. (NoDEA)* **2017**, *24*, 1–23.
24. Ikehata, R.; Takeda, H. Exponent for nonlinear wave equations with frictional and viscoelastic damping terms. *Nonlinear Anal.* **2017**, *148*, 228–253. [CrossRef]
25. Palmieri, A.; Reissig, M. Semi-linear wave models with power non-linearity and scale invariant time-dependent mass and dissipation II. *Math. Nachr.* **2018**, *291*, 1859–1892. [CrossRef]
26. D'Abbicco, M.; Girardi, G. A structurally damped σ-evolution equation with nonlinear memory. *Math. Meth. Appl. Sci.* **2020** . [CrossRef]
27. D'Abbicco, M.; Ebert, M.R.; Lucente, S. Self-similar asymptotic profile of the solution to a nonlinear evolution equation with critical dissipation. *Math. Meth. Appl. Sci.* **2017**, *40*, 6480. [CrossRef]
28. Christ, F.; Weinstein, M. Dispersion of Small-Amplitude Solutions of the Generalized Korteweg-de Vries Equation. *J. Funct. Anal.* **1991**, *100*, 87–109. [CrossRef]
29. Grafakos, L. *Classical and Modern Fourier Analysis*; Prentice Hall: Upper Saddle River, NJ, USA, 2004.
30. Grafakos, L.; Oh, S. The Kato Ponce inequality. *Commun. Partial. Differ. Equ.* **2014**, *39*, 1128–1157. [CrossRef]
31. Gulisashvili, A.; Kon, M. Exact smoothing properties of Schrndinger semigroups. *Am. J. Math.* **1996**, *118*, 1215–1248.

32. Hajaiej, H.; Molinet, L.; Ozawa, T.; Wang, B. Necessary and sufficient conditions for the fractional Gagliardo-Nirenberg inequalities and applications to Navier-Stokes and generalized boson equations. In *Harmonic Analysis and Nonlinear Partial Differential Equations B26*; 2011 ; pp. 159–175. Available online: https://arxiv.org/pdf/1004.4287v2.pdf (accessed on 30 August 2022).
33. Kato, T.; Ponce, G. Well-posedness and scattering results for the generalized Kortewegde-Vries equation via the contraction principle. *Commun. Pure Appl. Math.* **1993**, *46*, 527–620.
34. Kenig, C.E.; Ponce, G.; Vega, L. Commutator estimates and the Euler and Navier Stokes equations. *Commun. Pure Appl. Math.* **1988**, *41*, 891–907.
35. Djaouti, A.M. Semilinear Systems of Weakly Coupled Damped Waves. Ph.D. Thesis, TU Bergakademie Freiberg, Freiberg, Germany, 2018.