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Supersymmetric Polynomials and a Ring of Multisets of a Banach Algebra

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Abstract: In this paper, we consider rings of multisets consisting of elements of a Banach algebra. We investigate the algebraic and topological structures of such rings and the properties of their homomorphisms. The rings of multisets arise as natural domains of supersymmetric functions. We introduce a complete metrizable topology on a given ring of multisets and extend some known results about structures of the rings to the general case. In addition, we consider supersymmetric polynomials and other supersymmetric functions related to these rings. This paper contains a number of examples and some discussions.

Keywords: set of multisets; topological rings; supersymmetric polynomials; symmetric bases

MSC: 46H15; 46G20; 46G25

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1. Introduction

In recent years, symmetric structures and mappings in infinite-dimensional spaces have been studied by numerous authors [1–11]. In many problems of algebra and analysis [1,6], as well as in applications in symmetric neural networks (see, e.g., [12–15]), it is crucial to know the invariants of a given (semi)-group $\mathcal{S}$ acting on a Banach space X. The invariants can be described as elements of algebras of $\mathcal{S}$-symmetric functions on X. The Classical Invariant Theory, which was developed in the middle of the last century, investigated polynomial invariants of a group acting on a finite-dimensional linear space. The famous Nagata counterexample to the general case of Hilbert's fourteenth problem shows that polynomial algebras on $\mathbb{C}^n$ may be not finitely generated.

Symmetric polynomials and analytic functions on infinite-dimensional Banach spaces were investigated first by [16–19]. In particular, in [16,17], algebraic bases were described in algebras of symmetric polynomials on various Banach spaces with symmetric structures. These investigations were continued in [19–26] and others. To describe the spectrum of a uniform algebra of $\mathcal{S}$-symmetric functions on X, it is important to have more information about the quotient set $X/\sim$, where "$\sim$" is the relation of equivalence "up to the action of $\mathcal{S}$" on X. Such a quotient set may be interesting in itself and has applications in informatics and neural networks. If X is a sequence space and $\mathcal{S}$ is the group of permutations of elements of the sequences, then $X/\sim$ can be considered as a set of nonzero multisets—completed in a metrizable topology—induced from X. The set $X/\sim$ has a semiring structure with respect to natural algebraic operations. The commutative semiring can be extended to a ring by using a standard procedure from K-theory (see, e.g., [27]). Such a ring $\mathcal{M}$ of multisets for the case $X = \ell_1$ was investigated in [7,28]. In particular, homomorphisms and ideals of $\mathcal{M}$ were considered, and it was shown that each supersymmetric polynomial on $\ell_1 \times \ell_1$ can be extended to the ring $\mathcal{M}$. In [29], the properties of the ring of multisets of integer numbers were studied, and some applications to cryptography were found.

In this paper, we consider possible generalizations of the results obtained in [7] for more general cases. Instead of the sequence space ℓ_1, we consider the space of sequences $(x_1, x_2, \ldots, x_n, \ldots)$, where x_n are elements of a Banach algebra $\mathcal{A}$ and each sequence of norms, $(\|x_1\|_{\mathcal{A}}, \|x_2\|_{\mathcal{A}}, \ldots, \|x_n\|_{\mathcal{A}}, \ldots)$, is a vector in a Banach space X with a norm $\|\cdot\|_X$ and a symmetric basis $\{e_n\}$. Let us recall (see [30] for details) that a sequence $\{e_n\}$ is a *topological* (or *Schauder*) basis in a Banach space X if every element $x \in X$ can be uniquely expressed by

$$x = \sum_{n=1}^{\infty} x_n e_n = \lim_{m\to\infty} \sum_{n=1}^{m} x_n e_n,$$

where the limit is taken in $(X, \|\cdot\|_X)$. From here, in particular, we have that $x_n \to 0$ as $n \to \infty$.

A topological basis is called *symmetric* if it is equivalent to the basis $\{e_{\sigma(n)}\}$ for every permutation σ on the set of natural numbers $\mathbb{N}$. This means that for every σ, a series $\sum_{n=1}^{\infty} x_n e_n$ converges if and only if $\sum_{n=1}^{\infty} x_n e_{\sigma(n)}$ converges. It is known [30] (p. 114) that every Banach space X with a symmetric basis has an equivalent so-called *symmetric* norm such that

$$\left\| \sum_{n=1}^{\infty} x_n \theta_n e_{\sigma(n)} \right\|_X = \left\| \sum_{n=1}^{\infty} x_n e_n \right\|_X$$

for every permutation σ and sequence of numbers $\{\theta_n\}$ such that $|\theta_n| = 1$. Throughout this paper, we assume that X is endowed with a symmetric norm. In this case, we know that for every $x \in X$, $|x_n| \le 2\|x\|$.

In Section 2, we construct a ring of multisets $\mathcal{M}_X(\mathcal{D})$ of sets from a multiplicative semigroup $\mathcal{D}$ of $\mathcal{A}$ and investigate the basic properties. In particular, we show that $\mathcal{M}_X(\mathcal{D})$ is complete in a metrizable topology induced from X. In Section 3, we investigate homomorphisms of $\mathcal{M}_X(\mathcal{D})$ and related supersymmetric polynomials. In addition, we consider some examples and make discussions. We refer the reader to [31] for more information about polynomials on Banach spaces and to [32] for details on the classical theory of symmetric functions.

2. Group Rings of Multisets

Let X be a Banach space with a normalized symmetric basis $\{e_n\}$ and a symmetric norm $\|\cdot\|_X$, let $\mathcal{A}$ be a Banach algebra with an identity $\mathfrak{e}$, and let $\mathcal{D}$ be a closed multiplicative subgroup in $\mathcal{A}$ containing $\mathfrak{e}$. We denote by $X(\mathcal{D})$ the set of sequences $u = (x_1, \ldots, x_n, \ldots)$, $x_i \in \mathcal{D}$, and

$$\|u\| = \left\| \sum_{i=1}^{\infty} e_n \|x_n\|_{\mathcal{A}} \right\|_X.$$

In addition, let us denote by $\Lambda_X(\mathcal{D}) = X(\mathcal{D}) \times X(\mathcal{D})$, and we represent each element $v \in \Lambda_X(\mathcal{D})$ as

$$v = (y|x) = (\ldots, y_n, \ldots, y_2, y_1 | x_1, x_2, \ldots, x_n, \ldots),$$

$x, y \in X(\mathcal{D})$. Clearly, $\Lambda_X(\mathcal{A})$ is a Banach space with respect to the norm

$$\|v\| = \|x\| + \|y\|,$$

and $\Lambda_X(\mathcal{D})$ is its closed subset.

For a given $x \in \Lambda_X(\mathcal{D})$, we denote by $\operatorname{supp} x$ the subset of all natural numbers $n \in \mathbb{N}$ such that $x_n \neq 0$.

Let σ, μ be permutations on $\mathbb{N}$ and $(y|x) \in X(\mathcal{D})$. We define

$$(\sigma, \mu)(y|x) = (\ldots, y_{\sigma(n)}, \ldots, y_{\sigma(1)} | x_{\mu(1)}, \ldots, x_{\mu(n)}, \ldots).$$

Let $u = (y|x)$ and $w = (d|b)$ be in $\Lambda_X(\mathcal{D})$. Then,

$$u \bullet w = (y \bullet d|x \bullet b) = (\ldots, d_n, y_n, \ldots, d_1, y_1|x_1, b_1, \ldots, x_n, b_n, \ldots).$$

Note that if $x, b \in X(\mathcal{D})$, then $\|x \bullet b\| \leq \|x\| + \|b\|$. Hence, $u \bullet w \in \Lambda_X(\mathcal{D})$ for all $u, w \in \Lambda_X(\mathcal{D})$.

Let us consider an equivalence defined as $(y|x) \sim (y'|x')$ if and only if there are vectors $(a|a), (c|c) \in \Lambda_X(\mathcal{D})$, and bijections σ and μ such that σ maps $\operatorname{supp} x \bullet c$ onto $\operatorname{supp} x' \bullet a$ and μ maps $\operatorname{supp} y \bullet c$ onto $\operatorname{supp} y' \bullet a$; in addition,

$$(\sigma, \mu)\big((y'|x') \bullet (a|a)\big) = (y|x) \bullet (c|c). \tag{1}$$

Let us denote by $\mathcal{M}(\mathcal{D}) = \mathcal{M}_X(\mathcal{D})$ the quotient set $\Lambda_X(\mathcal{D})/\sim$ with respect to the equivalence "$\sim$". We denote by $[(y|x)] \in \mathcal{M}(\mathcal{D})$ the class of equivalence containing element $(y|x)$. Clearly, for every $a \in X(\mathcal{D})$, $(a|a) \sim (0|0)$, and so $[(y|x) \bullet (x|y)] = [(0|0)]$. In addition, we denote $\mathcal{M}^+(\mathcal{D}) = \{[(0|x)] \colon x \in \Lambda_X(\mathcal{D})\}$.

Let us explain the definition of the equivalence in a more detailed form. The requirement that σ and μ act bijectively between supports of corresponding vectors means that zero coordinates "do not matter", that is, for example,

$$(\ldots, y_n, \ldots, y_2, y_1|x_1, x_2, \ldots, x_n, \ldots) \sim (\ldots, y_n, 0, \ldots, 0, y_2, 0, y_1|x_1, 0, x_2, 0, \ldots, 0, x_n, \ldots).$$

In addition, for example,

$$(\ldots, y_n, \ldots, y_2, y_1|x_1, x_2, \ldots, x_n, \ldots) \sim (\ldots, y_n, \ldots, y_2, y_1, \lambda|\lambda, x_1, x_2, \ldots, x_n, \ldots)$$

for any $\lambda \in \mathbb{C}$. In addition, the classes of equivalence are invariant with respect to permutations of coordinates of x and of y separately. This approach allows us to consider $\mathcal{M}^+(\mathcal{D})$ as a set of multisets of $\mathcal{D}$. More exactly, the subset $\mathcal{M}^+_{00}(\mathcal{D})$ consisting of all elements in $\mathcal{M}^+(\mathcal{D})$ with finite supports can be naturally identified with the set of all finite multisets of *nonzero* elements in $\mathcal{D}$, and the operation "$\bullet$" is actually the union of multisets.

We say that $(y'|x')$ is an *irreducible* representative of $[u] \in \mathcal{M}(\mathcal{D})$ if $[(y'|x')] = [u]$, and $(y'|x') \sim (y|x)$ implies that

$$(y|x) = (\sigma, \mu)\big((y'|x') \bullet (a|a)\big)$$

for some permutations σ, μ on $\mathbb{N}$ and $(a|a) \in \Lambda_X(\mathcal{D})$. In other words, for every nonzero coordinate x'_i of x', we have $x'_i \neq y'_j$ for all coordinates y'_j of y'.

Proposition 1. *For every $[u] \in \mathcal{M}(\mathcal{D})$, there exists an irreducible representative.*

Proof. Let $(y|x)$ be a representative of $[u]$. Since elements $\sum_n e_n \|x_n\|_{\mathcal{A}}$ and $\sum_n e_n \|y_n\|_{\mathcal{A}}$ belong to the Banach space X with the Schauder basis e_n, it follows that $\|x_n\|_{\mathcal{A}} \to 0$, and $\|y_n\|_{\mathcal{A}} \to 0$ as $n \to \infty$. Without loss of generality, we may assume that the coordinates of x are ordered so that $\|x_1\|_{\mathcal{A}} \geq \|x_2\|_{\mathcal{A}} \geq \cdots \geq \|x_n\|_{\mathcal{A}} \geq \cdots$. If there is j such that $x_1 = y_j$, then let us remove the coordinate x_1 in x and y_j in y, and we denote by $x^{(1)}$ and $y^{(1)}$ the resulting vectors. If such a number j does not exist, we denote $x^{(1)} = x$ and $y^{(1)} = y$. Suppose that $x^{(n)}$ and $y^{(n)}$ are already constructed. If there is j such that $x_{n+1} = y_j$, then we remove the coordinate x_{n+1} in $x^{(n)}$ and y_j in $y^{(n)}$ and denote by $x^{(n+1)}$ and $y^{(n+1)}$ the resulting vectors. Otherwise, we set $x^{(n+1)} = x^{(n)}$ and $y^{(n+1)} = y^{(n)}$. Thus, we obtain the sequence $(y^{(n)}|x^{(n)})$ in $\Lambda_X(\mathcal{D})$, which is obviously fundamental. By the completeness of $\Lambda_X(\mathcal{D})$, there exists a limit

$$(y'|x') = \lim_{n\to\infty} (y^{(n)}|x^{(n)}).$$

Let a be a vector in $X(\mathcal{D})$ such that its coordinates a_n are exactly removed coordinates from x. Then, $(y|x) = (y' \bullet a|x' \bullet a)$, and so $(y'|x')$ is a representative of $[u]$. By the construction, $(y'|x')$ is irreducible. □

Now, we can introduce a commutative operation "+" on $\mathcal{M}(\mathcal{D})$.

Definition 1. *For a given* $\mathbf{u} = [u] = [(y|x)]$ *and* $\mathbf{w} = [w] = [(d|b)]$ *in* $\mathcal{M}(\mathcal{D})$*, we define*

$$\mathbf{u} + \mathbf{w} := [u \bullet w] = [(y \bullet d|x \bullet b)].$$

In addition, we set $-\mathbf{u} = -[(y|x)] := [(x|y)]$.

Proposition 2. *The operation "+" is well defined on* $\mathcal{M}(\mathcal{D})$*, and* $\big(\mathcal{M}(\mathcal{D}), +\big)$ *is a commutative group with zero (the neutral element),* $0 = [(0|0)] = [(\ldots, 0|0, \ldots)]$.

Proof. From definition of the operation, it follows that $\mathbf{u} + 0 = \mathbf{u}$ and $\mathbf{u} - \mathbf{u} = 0$. If $\mathbf{u} = [(y'|x')]$ and $\mathbf{w} = [(d'|b')]$ are the irreducible representatives $\mathbf{u}$ and $\mathbf{w}$, then, according to (1) and Proposition 1, $(y|x) = (y' \bullet a|x' \bullet a)$ and $(d|b) = (d' \bullet c|b' \bullet c)$ for some a and c. Hence,

$$[(y|x)] + [(d|b)] = [(y'|x') \bullet (a|a)] + [(d'|b') \bullet (c|c)]$$
$$= [(y'|x')] + [(d'|b')] + [(a|a)] + [(c|c)] = [(y'|x')] + [(d'|b')].$$

So, the result does not depend of representatives. □

Let $x, y \in X(\mathcal{D})$. By $x \diamond y$, we denote the resulting sequence of ordering the set $\{x_i y_j : i, j \in \mathbb{N}\}$ with one single index in some fixed order.

Proposition 3. *Let* $x, y \in X(\mathcal{D})$*. Then,* $x \diamond y \in X(\mathcal{D})$ *and* $\|x \diamond y\| \le 2\|x\|\|y\|$*. Moreover, if* $\mathcal{D}$ *is such that* $\|ab\| = \|a\|\|b\|$ *for every* $a, b \in \mathcal{D}$*, and* $X = c_0$ *or* ℓ_p *for some* $1 \le p < \infty$*, then* $\|x \diamond y\| = \|x\|\|y\|$.

Proof. Let $k(i,j)$ be a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. According to the straightforward calculations,

$$\|x \diamond y\| = \Big\| \sum_{i,j=1}^{\infty} \|x_i y_j\|_{\mathcal{A}} e_{k(i,j)} \Big\|_X \le \sup_i \|x_i\|_{\mathcal{A}} \Big\| \sum_{i,j=1}^{\infty} \|y_j\|_{\mathcal{A}} e_j \Big\|_X \le 2\|x\|\|y\|.$$

Let $\mathcal{D}$ be such that $\|ab\| = \|a\|\|b\|$ for every $a, b \in \mathcal{D}$. If $X = \ell_p(\mathcal{D})$, then

$$\|x \diamond y\|^p = \sum_{i,j=1}^{\infty} \|x_i y_j\|_{\mathcal{A}}^p = \sum_{i,j=1}^{\infty} \|x_i\|_{\mathcal{A}}^p \|y_j\|_{\mathcal{A}}^p = \|x\|^p \|y\|^p.$$

If $X = c_0$, then

$$\|x \diamond y\| = \sup_{i,j} \|x_i y_j\|_{\mathcal{A}} = \sup_{i,j} \|x_i\|_{\mathcal{A}} \|y_j\|_{\mathcal{A}} = \|x\|\|y\|.$$

□

Next, let us define a multiplication on $\mathcal{M}(\mathcal{D})$.

Definition 2. *If* $\mathbf{u} = [(0|x)]$ *and* $\mathbf{v} = [(0|y)]$*, we define* $\mathbf{uv} = [(0|x \diamond y)]$*. Finally, if* $\mathbf{u} = [(y|x)]$ *and* $\mathbf{v} = [(d|b)]$ *are in* $\mathcal{M}(\mathcal{D})$*, then we define*

$$\mathbf{uv} = [((y \diamond b) \bullet (x \diamond d)|(y \diamond d) \bullet (x \diamond b))].$$

Using routine calculations, it is easy to check (cf. [7,29]) that the multiplication is well defined and associative and that the distributive low with the addition holds on $\mathcal{M}(\mathcal{D})$. If $\mathcal{A}$ is a commutative Banach algebra, then the introduced multiplication is commutative. So, we have the following proposition.

Proposition 4. *$\big(\mathcal{M}(\mathcal{D}),+,\cdot\big)$ is a ring with zero, $0=[(0|0)]$, and unity, $\mathbb{I}=[(0|\mathfrak{e},0,\ldots)]$. If $\mathcal{A}$ is commutative, then $\big(\mathcal{M}(\mathcal{D}),+,\cdot\big)$ is commutative.*

Note that $\mathcal{M}(\mathcal{D})$ is not an algebra, even if $\mathcal{D}=\mathbb{C}$, because it is not a linear space (see, e.g., [7]). However, it is possible to introduce a norm on a given ring that has natural properties and induces a metrizable topology. Let us recall the following definition (cf. [33]).

Definition 3. *If R is any ring, then a real-valued function $\|z\|$ defined on R is called a norm for R if it satisfies the following conditions for all $z,r\in R$:*

1. $\|z\|\geq 0$ *and* $\|z\|=0$ *if and only if* $z=0$,
2. $\|z+r\|\leq\|z\|+\|r\|$,
3. $\|-z\|=\|z\|$,
4. $\|zr\|\leq C\|z\|\|r\|$ *for some constant* $C>0$.

Definition 4. *Let us define a norm on $\mathcal{M}(\mathcal{D})$ in the following way:*

$$\|\mathbf{u}\|=\|[(y|x)]\|:=\|(y'|x')\|=\|x'\|+\|y'\|,$$

where $(y'|x')$ is an irreducible representative of $\mathbf{u}$.

Proposition 5. *The norm in Definition 4 is well defined on $\mathcal{M}(\mathcal{D})$ and satisfies the conditions of Definition 3. In addition,*

$$\|\mathbf{u}\|=\min_{(y|x)\in\mathbf{u}}(\|x\|+\|y\|).$$

Proof. Note that an irreducible representative of $\mathbf{u}$ is not unique in general. However, if $(y'|x')$ and $(y''|x'')$ are irreducible representatives of $\mathbf{u}$, then they consist of the same coordinates (up to a permutation (σ,μ) of nonzero coordinates), and so, $\|(y'|x')\|=\|(y''|x'')\|$. Thus, the norm is well-defined.

Clearly, if $\mathbf{u}=0$, then $[(0|0)]$ is its irreducible representative, and so, $\|\mathbf{u}\|=0$. Otherwise, $\|\mathbf{u}\|\geq 0$. The second property of the norm evidently follows from the corresponding triangle property of the norm on a linear space. In addition, $\|-\mathbf{u}\|=\|(x'|y')\|=\|(y'|x')\|=\|\mathbf{u}\|$.

For any representative $(y|x)$ of $\mathbf{u}$, we have that $\|(y|x)\|\geq\|(y'|x')\|$, where $\|(y'|x')\|$ is an irreducible representative of $\mathbf{u}$. So,

$$\|\mathbf{u}\|=\min_{(y|x)\in\mathbf{u}}(\|x\|+\|y\|).$$

Let $\mathbf{u}=[(y|x)] and \mathbf{w}=[(d|b)]\in\mathcal{M}(\mathcal{D})$, and let $(y'|x')$ and $(b'|d')$ be corresponding irreducible representatives. Then, by Proposition 3,

$$\begin{aligned}\|\mathbf{uw}\|&=\|[(y'|x')(b'|d')]\|=\|[((y'\diamond b')\bullet(x'\diamond d')|(y'\diamond d')\bullet(x'\diamond b'))]\|\\&\leq\|((y'\diamond b')\bullet(x'\diamond d'))\|+\|((y'\diamond d')\bullet(x'\diamond b'))\|\\&\leq 2\|y'\|\|b'\|+2\|x'\|\|d'\|+2\|y'\|\|b'\|+2\|x'\|\|b'\|=2\|\mathbf{u}\|\|\mathbf{w}\|.\end{aligned}$$

Thus, $\|\cdot\|$ satisfies Condition 4 in Definition 3 for $C=2$. In addition, by Proposition 3, we can put $C=1$ if $X=c_0$ or ℓ_p, $1\leq p<\infty$. □

We define a metric ρ on $\mathcal{M}(\mathcal{D})$, associated with the norm in the natural way. Let $\mathbf{u}, \mathbf{w}$ be in $\mathcal{M}(\mathcal{D})$. We set

$$\rho(\mathbf{u}, \mathbf{w}) = \|\mathbf{u} - \mathbf{w}\|.$$

It is well known and easy to check that ρ is a metric.

Example 1. *Let $\mathbf{u}^{(n)} = [(0|h_n, 0, \ldots)]$, $h_n \in \mathcal{D}$ be a sequence in $\mathcal{M}(\mathcal{D})$ such that $h_n \to h$ as $n \to \infty$. If $h \neq 0$, then $\mathbf{u}^{(n)} \to [(0|h, 0, \ldots)]$ if and only if $h_n = h$ for all values of n that are big enough. Indeed, if $h_n \neq h$, then*

$$\|[(0|h_n, 0, \ldots)] - [(0|h, 0, \ldots)]\| = \|[(\ldots, 0, h_n|h, 0, \ldots)]\| = \|h_n\|_{\mathcal{A}} + \|h\|_{\mathcal{A}} \geq \|h\|_{\mathcal{A}}.$$

On the other hand, if $h = 0$, then $\|\mathbf{u}^{(n)} - 0\| = \|h_n\|_{\mathcal{A}} \to 0$ as $n \to \infty$.

Proposition 6. *The quotient map $(y|x) \mapsto [(y|x)]$ is discontinuous as a map from the Banach space $\Lambda_X(\mathcal{D})$ to the metric space $\big(\mathcal{M}(\mathcal{D}), \rho\big)$ at each point of $\Lambda_X(\mathcal{D})$, except for zero.*

Proof. Example 1 can be easily modified to show the discontinuity of the quotient map at any nonzero point. Indeed, let $v = (y|x) \neq 0$; then, without loss of generality, we can assume that $x_1 \neq 0$. Consider $u^{(n)} = (y|(1 - 1/n)x_1, x_2, \ldots, x_m, \ldots) \in \Lambda_X(\mathcal{D})$. Then, $u^{(n)} \to v$ in $\Lambda_X(\mathcal{D})$ as $n \to \infty$, but

$$\|[u^{(n)}] - [v]\| = \|[\ldots, 0, x_1|(1 - 1/n)x_1, 0, \ldots\| = 2\|x_1\|_{\mathcal{A}} - \frac{\|x_1\|_{\mathcal{A}}}{n} > \|x_1\|_{\mathcal{A}} > 0,$$

and so the quotient map is discontinuous at v. On the other hand, if a sequence $u^{(n)}$ tends to zero, then $\|[u^{(n)}]\| \to 0$ as $n \to \infty$, and thus, the quotient map is continuous at zero. $\square$

Theorem 1. *The metric space $\big(\mathcal{M}(\mathcal{D}), \rho\big)$ is complete.*

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be in $\mathcal{M}(\mathcal{D})$ and let $(y|x)$ be an irreducible representative of $\mathbf{u}$. We claim that there exists an irreducible representative $(d'|b') \in \mathbf{v}$ such that in $\Lambda_X(\mathcal{D})$, $\|(y|x) - (d'|b')\| < \varepsilon$. Indeed, let $(d|b)$ be any irreducible representative of $\mathbf{v}$. The inequality

$$\|\mathbf{u} - \mathbf{v}\| = \|[(y \bullet b|x \bullet d)]\| < \varepsilon$$

implies that there is an irreducible representative $(c|a)$ of $(y \bullet b|x \bullet d)$ such that $\|c\| + \|a\| < \varepsilon$. Note that $(y \bullet b|x \bullet d)$ is not necessary irreducible. However, since both $(y|x)$ and $(d|b)$ are irreducible, it may happen that some coordinates of y are the same as some coordinates of d and that some coordinates of x are the same as some coordinates of b. Let us construct $(d'|b')$ such that d' is obtained by permutating the coordinates of d, and b' is obtained by permutating the coordinates of b, so the coordinates of d that are equal to some coordinates of y have the same positions in d' as the corresponding coordinates in y, and the coordinates of b that are equal to some coordinates of x have the same positions in b' as the corresponding coordinates in x. Then, $(d'|b') \sim (b|d)$ and

$$\|(y|x) - (d'|b')\| = \|[(y \bullet b'|x \bullet d')]\| = \|c\| + \|a\| < \varepsilon.$$

Let $\mathbf{u}^{(m)}$, $m \in \mathbb{N}$ be a Cauchy sequence in $\big(\mathcal{M}(\mathcal{D}), \rho\big)$. Taking a subsequence, if necessary, we can assume that if $n \geq N$ and $m \geq N$, then $\rho(\mathbf{u}^{(m)}, \mathbf{u}^{(n)}) < \frac{1}{2^{N+1}}$. Let us choose irreducible representatives $(y^{(m)}|x^{(m)})$ of $u^{(m)}$ with

$$\|(y^{(m+1)}|x^{(m+1)}) - (y^{(m)}|x^{(m)})\| = \rho(\mathbf{u}^{(m+1)}, \mathbf{u}^{(m)}) < \frac{1}{2^{m+1}}.$$

Thus, if $n \geq N$ and $m \geq N$, then

$$\|(y^{(m)}|x^{(m)}) - (y^{(n)}|x^{(n)})\| < \frac{1}{2^N}.$$

Hence, $(y^{(m)}|x^{(m)})$, $m \in \mathbb{N}$ is a Cauchy sequence in $X(\mathcal{D})$, so it has a limit $z^{(0)} = (y^{(0)}|x^{(0)})$. Let $z_i^{(m)}$ be the ith coordinate of $z^{(m)} = (y^{(m)}|x^{(m)})$, $i \in \mathbb{Z} \setminus \{0\}$, that is, $z_i^{(m)} = x_i^{(m)}$ if $i > 0$ and $z_i^{(m)} = y_{-i}^{(m)}$ if $i < 0$. Clearly, $z_i^{(m)} \to z_i^{(0)}$ as $m \to \infty$. We claim that if $z_i^{(0)} = c \neq 0$, then there is a number N such that for every $m > N$, $z_i^{(m)} = c$. Indeed, if it is not so, then for every $n, m \in \mathbb{N}$, that is big enough, $\rho(\mathbf{u}^{(m)}, \mathbf{u}^{(n)}) > c$, and we have a contradiction.

For a given $\varepsilon > 0$, we denote by z^ε a vector in $X(\mathcal{D})$ such that z^ε has a finite support, $z_i^\varepsilon = z_i^{(0)}$ or $z_i^\varepsilon = 0$, and

$$\rho\Big(\big[z^\varepsilon\big], \big[z^{(0)}\big]\Big) < \frac{\varepsilon}{3}.$$

Note that for this case, $\rho\Big(\big[z^\varepsilon\big], \big[z^{(0)}\big]\Big) = \|z^\varepsilon - z^{(0)}\|$. Let N be a number such that for every $n > N$, $z_i^\varepsilon = z_i^{(n)}$ for all $i \in \operatorname{supp} z^\varepsilon$ and $\|z^{(n)} - z^{(0)}\| < \frac{\varepsilon}{3}$. So,

$$\rho\Big(\big[z^{(n)}\big], \big[z^\varepsilon\big]\Big) = \|z^\varepsilon - z^{(n)}\| \leq \|z^\varepsilon - z^{(0)}\| + \|z^{(n)} - z^{(0)}\| < \frac{2}{3}\varepsilon.$$

Thus,

$$\rho\Big(\big[z^{(n)}\big], \big[z^{(0)}\big]\Big) \leq \rho\Big(\big[z^{(n)}, z^\varepsilon\big]\Big) + \rho\Big(\big[z^\varepsilon, z^{(0)}\big]\Big) < \varepsilon.$$

Therefore, $\mathbf{u} = \big[z^{(0)}\big]$ is the limit of $\mathbf{u}^{(m)}$, and thus, $\big(\mathcal{M}(\mathcal{D}), \rho\big)$ is complete. □

From the triangle and multiplicative triangle inequalities of the norm, we have that the algebraic operations are jointly continuous in $\big(\mathcal{M}(\mathcal{D}), \rho\big)$. Indeed, let $\rho(\mathbf{u}, \mathbf{u}') < \varepsilon_1$ and $\rho(\mathbf{v}, \mathbf{v}') < \varepsilon_2$; then,

$$\rho(\mathbf{u} + \mathbf{v}, \mathbf{u}' + \mathbf{v}') < \|(\mathbf{u} + \mathbf{v}) - (\mathbf{u}' + \mathbf{v}')\| < \varepsilon_1 + \varepsilon_2$$

and

$$\rho(\mathbf{uv}, \mathbf{u}'\mathbf{v}') < 2\varepsilon_2\|\mathbf{u}\| + 2\varepsilon_1\|\mathbf{v}\| + 4\varepsilon_1\varepsilon_2.$$

The continuity of the addition implies that if Φ is an additive map from $\mathcal{M}(\mathcal{D})$ to an additive topological group and Φ is continuous at zero, then it is continuous at any point.

3. Homomorphisms and Supersymmetric Polynomials

Let $\mathcal{U}$ be a closed multiplicative semigroup of another Banach algebra $\mathcal{B}$ and let Y be a Banach space with a symmetric basis.

Theorem 2. *Let γ be a multiplicative map from $\mathcal{D}$ to $\mathcal{U}$. If there is a constant C_γ, such that $\|\gamma(z)\|_{\mathcal{B}} \leq C_\gamma \|z\|_{\mathcal{A}}$, $z \in \mathcal{D}$, then there exists a continuous ring homomorphism*

$$\Phi_\gamma \colon \mathcal{M}_X(\mathcal{D}) \to \mathcal{M}_Y(\mathcal{U})$$

defined by

$$\Phi_\gamma(\mathbf{u}) = \Phi_\gamma([(y|x)]) = [(\ldots, \gamma(y_n), \ldots, \gamma(y_2), \gamma(y_1)|\gamma(x_1), \gamma(x_2), \ldots, \gamma(x_n), \ldots)].$$

Proof. It is clear that $\Phi_\gamma([(y|x)])$ is additive and does not depend on the representative. In addition,

$$\|\Phi_\gamma([(y|x)])\| = \|[(\ldots, \gamma(y_n), \ldots, \gamma(y_2), \gamma(y_1)|\gamma(x_1), \gamma(x_2), \ldots, \gamma(x_n)\ldots)]\| \leq C_\gamma\|\mathbf{u}\|.$$

Let $\mathbf{u} \in \mathcal{M}_X(\mathcal{D})$ and let $(y|x)$ be its irreducible representative. Then,

$$\|\Phi_\gamma(\mathbf{u})\| = \|\gamma(x)\| + \|\gamma(y)\| \leq C_\gamma(\|x\| + \|y\|) = C_\gamma\|\mathbf{u}\|.$$

Hence, Φ_γ is continuous at zero, and according to the additivity, it is continuous at each point of $\mathcal{M}_X(\mathcal{D})$.

By the multiplicativity of γ,

$$\Phi_\gamma([(0|x)][(0|x')]) = [(0|\gamma(x_1)\gamma(x_1'), \ldots, \gamma(x_n)\gamma(x_j') \ldots)] = \Phi_\gamma([(0|x)])\Phi_\gamma([(0|x')]).$$

Thus,

$$\begin{aligned} &\Phi_\gamma([(y|x)][(y'|x')]) \\ = {}&\Phi_\gamma([(y|0)][(y'|0)]) + \Phi_\gamma([(0|x)][(0|x')]) - \Phi_\gamma([(0|x)][(0|y')]) - \Phi_\gamma([(0|0)][(0|x')]) \\ = {}&\Phi_\gamma([(y|x)])\Phi_\gamma([(y'|x')]). \end{aligned}$$

□

Note that in Theorem 2, we do not need the continuity of γ.

Example 2. *Let $\mathcal{D} = B$ be an open unit ball centered at the origin of a Banach algebra $\mathcal{A}$ and $\mathcal{U} = B_\varepsilon \cup \{\mathfrak{e}\}$, where $\mathfrak{e}$ is the unity of $\mathcal{A}$, and B_ε is an open ball of radius $0 < \varepsilon < 1$, which is centered at the origin of $\mathcal{A}$. In addition, let $X = Y$. We define $\gamma\colon \mathcal{D} \to \mathcal{U}$ by*

$$\gamma(z) = \begin{cases} z & \textit{if } z \in \mathcal{U}, \\ 0 & \textit{otherwise.} \end{cases}$$

Then, Φ_γ satisfies the conditions of Theorem 2 and, thus, is continuous.

Corollary 1. *Any continuous homomorphism φ from a Banach algebra $\mathcal{A}$ to a Banach algebra $\mathcal{B}$ can be extended to a continuous homomorphism from $\mathcal{M}_X(\mathcal{A})$ to $\mathcal{M}_Y(\mathcal{B})$ for any infinite-dimensional Banach space Y with a symmetric basis.*

Proof. Since φ is a continuous linear and multiplicative operator from $\mathcal{A}$ to $\mathcal{B}$, it follows that

$$\|\varphi\|_{\mathcal{B}} \leq \|\varphi\|\|z\|_{\mathcal{A}}, \quad z \in \mathcal{A}.$$

Hence, Φ_γ satisfies the conditions of Theorem 2 for $\gamma = \varphi$; thus, Φ_φ is a continuous homomorphism from $\mathcal{M}_X(\mathcal{A})$ to $\mathcal{M}_Y(\mathcal{B})$. The map $z \mapsto [(0|z, 0, \ldots)]$ is an embedding of $\mathcal{A}$ to $\mathcal{M}_X(\mathcal{A})$ and

$$\Phi_\varphi[(0|z, 0, \ldots)] = [(0|\varphi(z), 0, \ldots)].$$

Thus, we can consider Φ_φ as an extension of φ. Note that $z \mapsto [(0|z, 0, \ldots)]$ is not a homomorphism of rings because it is not additive. □

The following example shows that for some cases, the condition $\|\gamma(z)\|_{\mathcal{B}} \leq C_\gamma\|z\|_{\mathcal{A}}$ is not necessary for the continuity of Φ_γ.

Example 3. *Let $X = \ell_p$ for $1 \leq p < \infty$, let $Y = \ell_1$, and let n be a natural number, $n \geq p$. We set $\gamma(z) = z^n$, $z \in \mathcal{A}$. Then, for every Banach algebra $\mathcal{A}$, the mapping Φ_γ from $\mathcal{M}_{\ell_p}(\mathcal{A})$ to $\mathcal{M}_{\ell_1}(\mathcal{A})$ is a continuous homomorphism. Indeed, since $n \geq p$, $\Phi_\gamma(\mathbf{u}) \in \mathcal{M}_{\ell_1}(\mathcal{A})$ for every $\mathbf{u} \in \mathcal{M}_{\ell_p}(\mathcal{A})$ and*

$$\|\Phi_\gamma(\mathbf{u})\| \leq \|\mathbf{u}\|^n.$$

Thus, Φ_γ is continuous at zero and, thus, continuous.

Example 4. *Let $\gamma(z) = \|z\|_{\mathcal{A}}$. Then, Φ_γ maps $\mathcal{M}_X(\mathcal{D})$ to $\mathcal{M}_X(\mathbb{C})$, and it is continuous and additive. If the norm $\mathcal{A}$ is multiplicative, then Φ_γ is multiplicative.*

Note that if Φ is a homomorphism from $\mathcal{M}_X(\mathcal{D})$ to $\mathcal{M}_Y(\mathcal{U})$ and for every $z \in \mathcal{D}$,

$$\Phi([0|z,0,\ldots]) = ([0|w,0,\ldots])$$

for some $w \in \mathcal{U}$, then the map $\gamma\colon z \mapsto w$ is multiplicative. However, we do not know if every homomorphism from $\mathcal{M}_X(\mathcal{D})$ to $\mathcal{M}_Y(\mathcal{U})$ is of the form in Theorem 2.

Let us consider vector-valued mappings on $\mathcal{M}(\mathcal{D})$. Let E be a linear normed space. We say that a mapping $f\colon \Lambda_X(\mathcal{D}) \to E$ is *supersymmetric* if $f(y|x) = f(y'|x')$ whenever $(y|x) \sim (y'|x')$. In fact, every supersymmetric function can be defined on $\mathcal{M}(\mathcal{D})$ by $\widetilde{f}([(y|x)]) = f(y|x)$. It is easy to check that if f is of the form

$$f(y|x) = \sum_{i=1}^{\infty} \gamma(x_i) - \sum_{j=1}^{\infty} \gamma(y_j), \tag{2}$$

where γ is a map from $\mathcal{M}(\mathcal{D})$ to E, then $\widetilde{f}$ is supersymmetric and additive. If γ is multiplicative, then $\widetilde{f}$ is so.

Example 5. *Let $(y|x)$ be an irreducible representative of $u \in \Lambda_X(\mathcal{D})$. We set*

$$f(u) = \|x\| - \|y\|.$$

Then, f is a supersymmetric complex-valued function.

If $\mathcal{D} = \mathcal{A}$ is a Banach algebra, then $\Lambda_X(\mathcal{A})$ is a Banach space, and we can consider *supersymmetric polynomials* on $\Lambda_X(\mathcal{A})$, that is, polynomial mappings to a normed space E that are supersymmetric. Let us recall that a mapping P_n from a normed space Z to E is an n-homogeneous polynomial if there exists a multi-linear mapping $\overline{P}_n$ on the nth Cartesian degree Z^n of Z such that $P_n(x) = \overline{P}_n(x,\ldots,x)$. A finite sum of homogeneous polynomials is a polynomial. Continuous polynomials on Banach spaces were studied by many authors (see, e.g., [31]). The following example gives us supersymmetric polynomials on $\Lambda_{\ell_p}(\mathcal{A})$ for $1 \le p \le \infty$.

Example 6. *Let $X = \ell_p$ for some $1 \le p < \infty$, and $E = \mathcal{A}$. For any integer $n \ge p$, we define*

$$T_m(y|x) = \sum_{i=1}^{\infty} x_i^m - \sum_{i=1}^{\infty} y_i^m.$$

Clearly, polynomials T_m are supersymmetric. Since the mapping $x_i \mapsto x_i^m$ is multiplicative and $\|T_m(y|x)\| \le (\|x\| + \|y\|)^m$, mappings $\widetilde{T}_m$ are continuous ring homomorphisms from $\mathcal{M}(\mathcal{A})$ to $\mathcal{A}$.

A polynomial P on $\Lambda_{\ell_p}(\mathbb{C})$ is *separately symmetric* if P is invariant with respect to all permutations (σ,μ) acting by

$$\sigma\colon (x_1,\ldots,x_n,\ldots) \mapsto (x_{\sigma(1)},\ldots,x_{\sigma(n)},\ldots)$$

and

$$\mu\colon (y_1,\ldots,y_n,\ldots) \mapsto (y_{\mu(1)},\ldots,y_{\mu(n)},\ldots).$$

Clearly, if P is supersymmetric, then it is separately symmetric, but the inverse statement is not true.

Example 7. *Let*

$$P(y|x) = \sum_{i<j} x_i x_j - \sum_{i<j} y_i y_j.$$

Evidently, P is separately symmetric. Moreover, $P(x|y) = -P(y|x)$. However, P is not supersymmetric. Indeed, $P(\ldots, 0, -1|1, 0, \ldots) = 0$ while $P(\ldots, 0, 1, -1|1, 1, 0, \ldots) = 2$. However, $(\ldots, 0, -1|1, 0, \ldots) \sim (\ldots, 0, 1, -1|1, 1, 0, \ldots)$. Thus, P has different values on equivalent vectors, and thus, it cannot be supersymmetric.

The minimal algebra generated by polynomials T_m, $m \in \mathbb{N}$ was studied in [7,29] for the case of $X = \ell_1$ and $\mathcal{D} = \mathcal{A} = \mathbb{C}$. The next theorem shows that every supersymmetric polynomial can be represented as a finite algebraic combination of polynomials T_m.

Theorem 3. *Let P be a supersymmetric polynomial on $\Lambda_{\ell_1}(\mathbb{C})$. Then, P is an algebraic combination (that is, a linear combination of finite products) of polynomials T_m, $m \in \mathbb{N}$.*

Proof. Let P be a supersymmetric polynomial on $\Lambda_{\ell_1}(\mathbb{C})$; then, $P(y|x)$ is separately symmetric. According to [34], P is an algebraic combination of polynomials F_m^+ and F_m^-, $m \in \mathbb{N}$, where

$$F_m^+(y|x) = \sum_{k=1}^{\infty} x_k^m \quad \text{and} \quad F_m^-(y|x) = \sum_{k=1}^{\infty} y_k^m.$$

Thus, we have

$$P(y|x) = \sum_{\substack{k_1+2k_2+\cdots+ik_i+ \\ n_1+2n_2+\cdots jn_j=0}}^{m} c_{k_1\ldots k_i n_1 \ldots n_j} F_1^+(x)^{k_1} \cdots F_i^+(x)^{k_i} F_1^-(y)^{n_1} \cdots F_j^-(y)^{n_j}$$

for some constants $c_{k_1\ldots k_i n_1 \ldots n_j}$.

Clearly, $T_k = F_k^+ - F_k^-$. Denote $Q_k = F_k^+ + F_k^-$. Then, there is a polynomial $q\colon \mathbb{C}^n \to \mathbb{C}$ such that

$$P(y|x) = q(T_1(y|x), \ldots, T_m(y|x), Q_1(y|x), \ldots, Q_m(y|x)).$$

According to our assumption, $P(y \bullet a|x \bullet a) = P(y|x)$, $a \in \ell_1$. We can see that

$$T_k(y \bullet a|x \bullet a) = T_k(y|x) \quad \text{and} \quad Q_k(y \bullet a|x \bullet a) = Q_k(y|x) + 2F_k(a)$$

for every $k \in \mathbb{N}$. It is known that for every $(\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$, there exists a vector $a \in \ell_1$ such that $F_n(a) = \lambda_n$, $1 \le n \le m$ (see, e.g., [19]). Thus, for every $(\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$,

$$q(T_1(y|x), \ldots, T_m(y|x), Q_1(y|x), \ldots, Q_m(y|x))$$

$$= q(T_1(y|x), \ldots, T_m(y|x), Q_1(y|x) + \lambda_1, \ldots, Q_m(y|x) + \lambda_m).$$

However, this means that q does not depend on $Q_1, \ldots, Q_m$. Hence, P is an algebraic combination of polynomials T_m, $m \in \mathbb{N}$. $\square$

In particular, in [29], it was proved that $[(y|x)] = [(y'|x')]$ in $\mathcal{M}_{\ell_1}(\mathbb{C})$ if and only if $T_m(y|x) = T_m(y'|x')$ for all $m \in \mathbb{N}$. The next example shows that in a more general case, supersymmetric polynomials do not separate points of $\mathcal{M}(\mathcal{D})$.

Example 8. *Let $X = \ell_1$ and $\mathcal{A} = \mathbb{C}^2$ be the algebra with respect to the coordinate-wise multiplication. Then, the vector*

$$(y|x) = \left(\ldots, 0, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix} \middle| \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, 0, \ldots\right)$$

is not equivalent to $(0|0)$, but

$$T_m(y|x) = \begin{pmatrix}3^m + 1^m - 1^m - 3^m \\ 4^m + 2^m - 2^m - 4^m\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Let φ be a complex homomorphism of $\mathcal{A}$ and let Φ be a ring homomorphism from $\mathcal{M}(\mathcal{D})$ to $\mathcal{A}$; then, $\varphi \circ \Phi$ is a ring complex homomorphism of $\mathcal{M}(\mathcal{D})$. From the following example, we can see that there are complex homomorphisms of $\mathcal{M}(\mathcal{D})$ constructed in a different way.

Example 9. *Consider the case $\mathcal{M}_{\ell_1}(\mathbb{C}^2)$, as in Example 8. For arbitrary $k, n \in \mathbb{N}$, we set*

$$P_{kn}(y|x) = \sum_{i=1}^{\infty} x_i^k x_i'^n - \sum_{i=1}^{\infty} y_i^k y_i'^n,$$

where

$$(y|x) = \left(\cdots \ , \ \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} \ , \ \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \ \middle| \ \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \ , \ \begin{pmatrix} x_2 \\ x_2' \end{pmatrix} \ , \ \cdots \right).$$

Note that $\|P_{kn}(y|x)\| \le (\|x\| + \|y\|)^{k+n}$. Polynomials P_{kn} are of the form (2) for $\gamma(x) = x_i^k x_i'^n$, and the map γ is multiplicative. So, $\widetilde{P}_{kn}$ are continuous complex homomorphisms.

Polynomials P_{kn} in Example 9, which are restricted to elements $(0|x)$, are called *block-symmetric* polynomials on $\ell_1(\mathbb{C}^2)$ (see, e.g., [4,23,26]) or *MacMahon* polynomials in the literature [35].

Example 10. *Let $X = \ell_1$, and let $\mathcal{A} = M_m$ be the algebra of all square matrices $m \times m$ for some fixed $m \in \mathbb{N}$. Then, $\mathcal{M}_{\ell_1}(M_m)$ is a noncommutative ring of matrix multisets. Let D be the following map from $\mathcal{M}_{\ell_1}(M_m)$ to $\mathcal{M}_{\ell_1}(\mathbb{C})$:*

$$D([(y|x)]) = [(\ldots, \det(y_n), \ldots, \det(y_2), \det(y_1) | \det(x_1), \det(x_2), \ldots, \det(x_n), \ldots)].$$

Since the determinant $\det(x_i)$ ia a multiplicative mapping, D is a homomorphism. The continuity of D follows from the fact that $\|D(y|x)\| \le (\|x\| + \|y\|)^m$.

4. Discussions and Conclusions

We considered the ring of multisets $\mathcal{M}_X(\mathcal{D})$ consisting of elements in a given multiplicative semigroup $\mathcal{D}$ of a Banach algebra $\mathcal{A}$ and endowed with some natural "supersymmetric" operations of addition and multiplication. We constructed a complete metrizable topology of $\mathcal{M}_X(\mathcal{D})$ generated by a ring norm. In addition, we investigated homomorphisms of $\mathcal{M}_X(\mathcal{D})$ and their relations with supersymmetric polynomials. Note that $\mathcal{M}_X(\mathcal{D})$ is not a linear space over $\mathbb{C}$ or $\mathbb{R}$ because there is no natural multiplication by scalars (see, e.g., [7]).

Rings of multisets may have wide applications in neural networks and machine learning. Computer algorithms are often invariant with respect to permutations of input data instances. This observation suggests the use of permutation-invariant sets instead of vectors of a fixed dimension for the organization of input data (see, e.g., [12]). For this purpose, multisets (sets with possible repetitions of elements) are actually more suitable. However, classical multisets have a poor algebraic structure. For example, a very important operation of the union of two multisets has no inverse. On the other hand, we can consider a set of multisets as a natural domain of symmetric functions (with respect to permutations of variables) that are defined on a linear space. Since the union of multisets does not preserve cardinality, it is convenient to use infinite-dimensional linear spaces of sequences, such as Banach spaces with symmetric bases. All symmetric functions on X can be extended to the set of multisets, and if $X = \ell_1$, then symmetric polynomials separate different points of the multisets. To get an operation that is inverse to the union, we have to use Grothendieck's well-known idea, which is widely used in K-theory. It leads to the construction of classes of equivalences of pairs $(y|x)$, where y plays the role of a "negative part" (while components of both vectors x and y are complex numbers or, in the general case, elements of an abstract

Banach algebra $\mathcal{A}$). If we consider x as vector coding information, then y consists of "negative" information in the sense that if both x and y contain the same piece of information (the same coordinate), then this piece of information will be annulated. Therefore, the union can be extended to a commutative group operation on the classes of equivalence, and together with a natural symmetric multiplication, they form a ring structure on the set of classes. Such a ring of multisets of complex numbers was considered in [7] for the case of $X = \ell_1$. In this paper, we investigated the situation when the "coordinates" of x and y were in a Banach algebra $\mathcal{A}$ and sequences of their norms belonged to a Banach space X with a symmetric basis. It is interesting that the basic results in [7] can be extended to the general case. In particular, the ring $\mathcal{M}_X(\mathcal{D})$ that was obtained is a complete metric space in a metrizable topology, and it is naturally induced by norms of $\mathcal{A}$ and X. The main difference is that supersymmetric polynomials separate points of $\mathcal{M}_{\ell_1}(\mathbb{C})$, while in the general case, they do not.

One can compare the rings of multisets and fuzzy sets. In a fuzzy set, each element may have a partial membership (between 0 and 1) [36]. In a ring of multisets, elements may have multiple memberships, and even negatively multiple memberships. Note that the ring $\mathcal{M}_X(\mathcal{D})$ is never algebra, even if $\mathcal{D} = \mathbb{C}$ (see [7]). However, it is known [33] that under some natural conditions, any metric ring R can be embedded into a normed algebra over the field of fractions over R. It would be interesting to construct such an algebra for the ring $\mathcal{M}_X(\mathcal{D})$ and compare it with fuzzy sets and other algebraic structures.

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