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Applications of Higher-Order q-Derivative to Meromorphic q-Starlike Function Related to Janowski Function

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Abstract: By making use of a higher-order q-derivative operator, certain families of meromorphic q-starlike functions and meromorphic q-convex functions are introduced and studied. Several sufficient conditions and coefficient inequalities for functions in these subclasses are derived. The results presented in this article extend and generalize a number of previous results.

Keywords: q-derivative; analytic functions; differential subordination; meromorphic functions; meromorphic q-starlike functions; meromorphic q-convex functions

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1. Introduction

Recently, q-analysis has fascinated scholars due to various applications in many areas of physics and mathematics. The applications of q-analysis were first considered by Jackson [1,2]. In recent years, some scholars have written a number of papers [3–15] associated with q-starlike functions and the Janowski functions [16]. In particular, Srivastava [17,18] pointed out some applications and mathematical explanations of q-derivatives in GFT. In this paper, we consider several families of meromorphically multivalent q-starlike functions by making use of the Janowski function and the higher-order q-derivative. Certain sufficient conditions and coefficient inequalities for functions in these subclasses are derived. Moreover, several previous results are generalized.

Let $\Sigma(p)(p \in \mathbb{N})$ denote the family of p-valent analytic functions in $U^* = \{z : 0 < |z| < 1\}$ which have the following form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}. \tag{1}$$

Furthermore, we let $\Sigma(1) = \Sigma$.

If $f \in \Sigma(p)$ satisfies the condition

$$\mathrm{Re}\left(-\frac{zf'(z)}{f(z)}\right) > 0, \tag{2}$$

then f is called a meromorphic p-valent starlike function. We note the family by $M\Sigma^*(p)$ and write $M\Sigma^*(1) = M\Sigma^*$. The class $M\Sigma^*$ was studied by Pommerenke [19].

If $f \in \Sigma(p)$ satisfies the condition

$$\mathrm{Re}\left(-\frac{(zf'(z))'}{f'(z)}\right) > 0, \tag{3}$$

then f is called a meromorphic p-valent convex function. We note this family by $MC(p)$ and write $MC(1) = MC$.

If a function ψ is analytic in $U = U^* \cup \{0\}$ and satisfies

$$\psi(z) = 1 + \sum_{k=1}^{\infty} \psi_k z^k \tag{4}$$

and

$$\mathrm{Re}\big(\psi(z)\big) > 0,$$

then ψ is said to be in the family P.

Let φ be analytic in U and $\varphi(0) = 1$. If φ satisfies

$$\varphi(z) \prec \frac{Az+1}{Bz+1} \quad (-1 \leqq B < A \leqq 1),$$

then φ is said to be in $P[A, B]$.

In [16], Janowski studied the family $P[A, B]$ and obtained that φ is in $P[A, B]$ if

$$\varphi(z) = \frac{(1-A) + (1+A)\psi(z)}{(1-B) + (1+B)\psi(z)} \quad (\psi \in P; -1 \leqq B < A \leqq 1).$$

Let f and g be analytic in U. If there exists w analytic in U with $w(0) = 0$ and $|w(z)| < 1$, so that $f(z) = g(w(z))$, then we say that f is subordinate to g, written by $f \prec g$. Further, if the function g is analytic and univalent in U, then

$$f \prec g \quad (z \in U) \iff f(U) \subset g(U) \text{ and } f(0) = g(0).$$

Definition 1. *Let $f \in \Sigma$. Then f is called to be in $M\Sigma^*[A, B]$ if*

$$-\frac{zf'(z)}{f(z)} = \frac{(1-A) + (1+A)\psi(z)}{(1-B) + (1+B)\psi(z)} \quad (-1 \leqq B < A \leqq 1; \psi \in P). \tag{5}$$

In [20], Karunakaran studied the family $M\Sigma^*[A, B]$.

Let $0 < q < 1$ and define $[\tau]_q$ as the following:

$$[\tau]_q = \begin{cases} \dfrac{q^\tau - 1}{q-1} & (\tau \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\tau = n \in \mathbb{N}). \end{cases}$$

Let $0 < q < 1$. The q-factorial $[m]_q!$ is defined by

$$[m]_q! = \begin{cases} 1 & (m = 0) \\ \Pi_{k=1}^{m} [k]_q & (m \in \mathbb{N}). \end{cases}$$

Let $\gamma \in \mathbb{N}$. We define q-Pochhammer symbol $[\gamma]_{q,n}$ by (see [21])

$$[\gamma]_{q,n} = \begin{cases} 1 & (n = 0) \\ \Pi_{k=\gamma}^{\gamma+n-1} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Specifically, we write $[0]_{q,n} = 0$.

Next, we define the q-derivative D_q $(0 < q < 1)$ for $f \in \Sigma(p)$ by

$$\begin{aligned} (D_q f)(z) &= \frac{f(zq) - f(z)}{z(q-1)} \\ &= -\sum_{k=1}^{p} \left(\frac{[k]_{q,1}}{q^k} \right) a_{-k} z^{-1-k} + \sum_{k=1}^{\infty} [k]_q a_k z^{-1+k}, \end{aligned} \tag{6}$$

where $a_{-p} = 1$.

From (6), we can see that

$$\lim_{q\to 1}(D_q f)(z) = \lim_{q\to 1} \frac{f(zq) - f(z)}{z(q-1)} = f'(z).$$

Further, one can find that

$$(D_q^{(2)} f)(z) = \sum_{k=1}^{p} \left(\frac{[k]_{q,2}}{q^{2k+1}} \right) a_{-k} z^{-k-2} + \sum_{k=2}^{\infty} [k-1]_{q,2} a_k z^{k-2}, \tag{7}$$

$$(D_q^{(3)} f)(z) = -\sum_{k=1}^{p} \left(\frac{[k]_{q,3}}{q^{3k+3}} \right) a_{-k} z^{-k-3} + \sum_{k=3}^{\infty} [k-2]_{q,3} a_k z^{k-3}, \tag{8}$$

$$\cdots\cdots\cdots$$

$$(D_q^{(p)} f)(z) = (-1)^p \sum_{k=1}^{p} \left(\frac{[k]_{q,p}}{q^{p\left[k+\frac{1}{2}(p-1)\right]}} \right) a_{-k} z^{-k-p} + \sum_{k=p}^{\infty} [k-p+1]_{q,p} a_k z^{k-p}, \tag{9}$$

where $a_{-p} = 1$ and $D_q^{(p)}$ is called pth order q-derivatives.

Definition 2. *Let $f \in \Sigma$. If f satisfies*

$$\left| \frac{qzD_q f(z)}{f(z)} + \frac{1}{1-q} \right| < \frac{1}{1-q}, \tag{10}$$

then f is called to be in the meromorphic q-starlike function family $M\Sigma_q^*$.

It is easily seen that, when $q \to 1^-$, the disk given by (10) becomes

$$\mathrm{Re}\left(-\frac{zf'(z)}{f(z)} \right) > 0.$$

Thus, the class $M\Sigma_q^*$ reduces to the meromorphic starlike function family $M\Sigma^*$ (see [19]). Furthermore, we can rewrite (10) as the following:

$$-\frac{qzD_q f(z)}{f(z)} \prec \widehat{h}(z) \quad \text{where} \quad \widehat{h}(z) = \frac{z+1}{1-qz}.$$

Further, the meromorphic q-convex function family MC_q could be derived by

$$f(z) \in MC_q \Leftrightarrow -qzD_q f(z) \in M\Sigma_q^*.$$

Definition 3. *If a function $f \in \Sigma(p)$ satisfies*

$$-\frac{q^{2p-1} z (D_q^{(p)} f)(z)}{[2p-1]_q (D_q^{(p-1)} f)(z)} \prec \frac{(A+1)\widehat{h}(z) + (1-A)}{(B+1)\widehat{h}(z) + (1-B)} \quad \left(\widehat{h}(z) = \frac{z+1}{1-qz} - 1; \leqq B < A \leqq 1 \right),$$

or, equivalently,

$$-\frac{q^{2p-1} z (D_q^{(p)} f)(z)}{[2p-1]_q (D_q^{(p-1)} f)(z)} \prec s(z), \tag{11}$$

where

$$s(z) = \frac{z(1+A) - zq(1-A) + 2}{z(1+B) - zq(1-B) + 2} \quad (0 < q < 1;\ -1 \leqq B < A \leqq 1), \tag{12}$$

then f is in the family $M\Sigma_q^*[p, A, B]$.

Remark 1. *We write the following special cases:*

(i) $M\Sigma_q^*[1, A, B] = M\Sigma_q^*[A, B]$, *when* $p = 1$.

(ii) $\lim_{q\to 1^-} M\Sigma_q^*[1, A, B] = M\Sigma^*[A, B]$, *when* $p = 1$.

(iii) $M\Sigma_q^*[p, A, B] = M\Sigma_q^*[\alpha]$, *when* $p = 1$, $A = 1 - 2\alpha$ $(0 \leqq \alpha < 1)$ *and* $B = -1$. *In [19], Pommerenke considered the family* $M\Sigma_q^*[\alpha]$.

Now we define the meromorphic q-convex function family $MC_q[p, A, B]$ by

$$f \in MC_q[p, A, B] \iff \frac{(-1)^p q^{\frac{1}{2}p(3p-1)}}{[p]_{q,p}} z^p D_q^{(p)} f \in M\Sigma_q^*[p, A, B].$$

In particular, we write $MC_q[p, A, B] = MC_q[A, B]$ when $p = 1$.

Lemma 1 ([22])**.** *Let* $\psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \cdots$ *belong to the family* P. *Then*

$$|\psi_2 - \nu\psi_1^2| \leqq \begin{cases} 4\nu - 2 & (\nu > 1) \\ 2 & (0 \leqq \nu \leqq 1) \\ -4\nu + 2 & (\nu < 0). \end{cases} \tag{13}$$

Lemma 2 ([23])**.** *Let* $h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k$ *be analytic in* U. *Furthermore, let* $H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k$ *be univalent convex in* U. *If* $h(z) \prec H(z)$, *then*

$$|h_k| \leqq |C_1| \quad (k \geqq 1).$$

2. Main Results

Theorem 1. *If*

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} a_{-p+k} z^{-p+k} \in M\Sigma_q^*[p, A, B] \quad (p \geqq 2),$$

then

$$|a_{2-p} - \mu a_{1-p}^2| \leqq \begin{cases} \left(\frac{A-B}{2}\right)\left(\frac{[2p-3]_{q,3}}{q^{2p-2}[p-2]_{q,2}}\right)\Lambda(q) & (\mu > \sigma_1) \\ \left(\frac{A-B}{2}\right)\left(\frac{[2p-3]_{q,3}}{q^{2p-2}[p-2]_{q,2}}\right) & (\sigma_2 \leqq \mu \leqq \sigma_1) \\ \left(\frac{B-A}{2}\right)\left(\frac{[2p-3]_{q,3}}{q^{2p-2}[p-2]_{q,2}}\right)\Lambda(q) & (\mu < \sigma_2), \end{cases}$$

where

$$\Lambda(q) = \frac{\left\{ \begin{array}{l} \{(1 + Bq[2p-2]_q - A[2p-1]_q)(q+1) - 2\}[p-1]_q[2p-3]_q \\ +\mu(q+1)^2(A-B)[2p-2]_{q,2}[p-2]_q \end{array} \right\}}{2[p-1]_q[2p-3]_q},$$

$$\sigma_1 = \frac{[p-1]_q[2p-3]_q\{4 + (q+1)(A[2p-1]_q - qB[2p-2]_q - 1)\}}{(q+1)^2(A-B)[p-2]_q[2p-2]_{q,2}}$$

and

$$\sigma_2 = \frac{[p-1]_q[2p-3]_q\{(q+1)(A[2p-1]_q - 1 - qB[2p-2]_q)\}}{4(q+1)^2(A-B)[p-2]_q[2p-2]_{q,2}}.$$

Proof. From the assumption of the theorem, we obtain

$$-\frac{q^{2p-1} z(D_q^{(p)} g)(z)}{[2p-1]_q (D_q^{(p-1)} g)(z)} \prec \phi(z),$$

where

$$\phi(z) = \frac{2 - q(1-A)z + (1+A)z}{2 - q(1-B)z + (1+B)z}.$$

This gives that

$$-\frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} = \phi(w(z)),$$

where $w(z)$ is a Schwarz function. Now a function $h(z)$ is defined as follows:

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + \sum_{n=1}^{\infty} h_n z^n \in P.$$

□

Furthermore, one can see that

$$\begin{aligned}\phi(w(z)) &= \frac{2(h(z)+1-q(h(z)-1)) + (1+A)(q+1)(h(z)-1)}{2(h(z)+1-q(h(z)-1)) + (1+B)(q+1)(h(z)-1)} \\ &= 1 + \frac{1}{4}(1+q)(A-B)h_1 z + \frac{1}{16}(1+q)(A-B)\Big\{4h_2 - (B(1+q)-q+3)h_1^2\Big\}z^2 + \cdots.\end{aligned}$$

Similarly, we find that

$$\begin{aligned}-\frac{q^{2p-1}z(D_q^{(p)}g)(z)}{[2p-1]_q(D_q^{(p-1)}g)(z)} &= 1 - \frac{[p-1]_q}{[2p-2]_{q,2}} q^{p-1} a_{1-p} z \\ &\quad + \frac{[p-1]_q}{[2(p-1)]_{q,2}} q^{2(p-1)} \left(\frac{[p-1]_q}{[2(p-1)]_q} a_{1-p}^2 - \frac{[p-2]_q}{[2p-3]_q}(1+q)a_{2-p} \right) z^2 + \cdots\end{aligned}$$

for $p \geqq 2$. Therefore, for $p \geqq 2$, we obtain

$$a_{1-p} = -\frac{(q+1)(A-B)[2(p-1)]_{q,2}}{4q^{p-1}[p-1]_q} h_1 \tag{14}$$

and

$$a_{2-p} = \frac{(A-B)[2p-3]_{q,3}}{q^{2p-2}[p-2]_{q,2}} \left(\frac{1}{16} k_1(q) h_1^2 - \frac{1}{4} h_2 \right), \tag{15}$$

where

$$k_1(q) = (1+q)\{A[2p-1]_q - B([2p-1]_q - 1) - 1\} + 4. \tag{16}$$

Hence we obtain for $p \geqq 2$ that

$$|a_{2-p} - \mu a_{1-p}^2| = \left(\frac{A-B}{4} \right) \left(\frac{[2p-3]_{q,3}}{q^{2p-2}[p-2]_{q,2}} \right) |h_2 - k_2 h_1^2|, \tag{17}$$

where

$$k_2 = \frac{[p-1]_q[2p-3]_q k_1(q) - \mu[2(p-1)]_{q,2}[p-2]_q(A-B)(q+1)^2}{4[2p-3]_q[p-1]_q}$$

with $k_1(q)$ given by (16).

Now we can see that the conditions $\mu > \sigma_1$, $\sigma_2 \leqq \mu \leqq \sigma_1$ and $\mu < \sigma_2$ in Theorem 1 imply that $k_2 < 0$, $0 \leqq k_2 \leqq 1$ and $k_2 > 1$, respectively. By applying Lemma 1 in (17), the desired result is obtained. This proves Theorem 1.

Applying the same method as in the proof of Theorem 1, we obtain the following theorem for the case $p = 1$.

Theorem 2. *If*

$$g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1} \in M\Sigma_q^*[A, B],$$

then

$$|a_1 - \mu a_0^2| \leqq \begin{cases} \left(\frac{A-B}{4}\right)[(1-A)q + \mu(q+1)^2(A-B) - A - 1] & \left(\mu > \frac{(A-1)q+A+3}{(A-B)(q+1)^2}\right) \\ \frac{A-B}{2} & \left(\frac{A-1}{(A-B)(q+1)} \leqq \mu \leqq \frac{(A-1)q+A+3}{(A-B)(q+1)^2}\right) \\ \left(\frac{B-A}{4}\right)[(1-A)q + \mu(q+1)^2(A-B) - A - 1] & \left(\mu < \frac{A-1}{(A-B)(q+1)}\right). \end{cases}$$

Letting $q \to 1^-$, $A = 1$ and $B = -1$ in Theorem 2, we obtain a result of the known family $M\Sigma^*$.

Corollary 1. *If*

$$g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1} \in M\Sigma^*,$$

then

$$\left|a_1 - \mu a_0^2\right| \leqq \begin{cases} 4\mu - 1 & (\mu > \frac{1}{2}) \\ 1 & (0 \leqq \mu \leqq \frac{1}{2}) \\ 1 - 4\mu & (\mu < 0). \end{cases}$$

Theorem 3. *Let $p \geqq 2$. If*

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} a_{-p+k} z^{-p+k} \in M\Sigma_q^*[p, A, B],$$

then

$$\left|a_{-p+k}\right| \leqq \prod_{j=1}^{k} \frac{2[j-1]_q + [2p-1]_q(q+1)(A-B)}{2[j]_q q^{p-1}[p-j]_q} \tag{18}$$

for $1 \leqq k \leqq p-1$.

Proof. If g belongs to $M\Sigma_q^*[p, A, B]$, then

$$\psi(z) := -\frac{q^{2p-1} z (D_q^{(p)} g)(z)}{[2p-1]_q (D_q^{(p-1)} g)(z)} \prec \phi(z), \tag{19}$$

where

$$\phi(z) = \frac{z(1+A) - zq(1-A) + 2}{z(1+B) - zq(1-B) + 2} = 1 - \frac{1}{2}(1+q)(B-A)z + \frac{1}{4}(1+q)(B-A)\{1 + B(1+q) - q\}z^2 + \cdots.$$

□

Let

$$\psi(z) = 1 + \sum_{k=1}^{\infty} \psi_k z^k.$$

Applying Lemma 2, we obtain

$$|\psi_k| \leqq \frac{1}{2}(q+1)(A-B) \quad (k \geqq 1). \tag{20}$$

Furthermore, from (19), we have

$$-q^{2p-1} z (D_q^{(p)} g)(z) = \{[2p-1]_q (D_q^{(p-1)} g)(z)\}\psi(z),$$

which implies that

$$
\begin{aligned}
&-q^{2p-1}\left(\sum_{k=1}^{p}\frac{(-1)^p[k]_{q,p}}{q^{p\left[k+\frac{1}{2}(p-1)\right]}}a_{-k}z^{-p-k+1}+\sum_{k=p}^{\infty}[-p+k+1]_{q,p}a_kz^{-p+k+1}\right)\\
&=[2p-1]_q\left(1+\sum_{k=1}^{\infty}\psi_kz^k\right)\left(\sum_{k=1}^{p}\frac{(-1)^{p-1}[k]_{q,p-1}}{q^{\left[k+\frac{1}{2}(p-2)\right](p-1)}}a_{-k}z^{-p-k+1}\right.\\
&\qquad\left.+\sum_{k=p-1}^{\infty}[2+k-p]_{q,p-1}a_kz^{-p+k+1}\right),
\end{aligned}
$$

where $a_{-p}=1$.

It is easily seen from the above formula that

$$
|a_{-p+k}|\leqq\frac{(A-B)(q+1)[2p-1]_q q^{\frac{1}{2}(p-1)(3p-2k-2)}}{2[k]_q[p-k]_{q,p-1}}\sum_{l=1}^{k}\frac{[p+l-k]_{q,p-1}}{q^{\frac{1}{2}(3p-2k+2l-2)(p-1)}}\left|a_{k-p-l}\right| \tag{21}
$$

for $1\leqq k\leqq p-1$.

Now

$$
\begin{aligned}
|a_{1-p}|&\leqq\frac{[2p-2]_{q,2}(1+q)(A-B)}{2q^{p-1}[p-1]_q},\\
|a_{2-p}|&\leqq\frac{q^{\frac{1}{2}(p-1)(3p-6)}[2p-1]_q(1+q)(A-B)}{2[p-2]_{q,p-1}[2]_q}\left\{\frac{[p]_{q,p-1}}{q^{\frac{1}{2}(3p-2)(p-1)}}+\frac{[p-1]_{q,p-1}}{q^{\frac{1}{2}(3p-4)(p-1)}}|a_{1-p}|\right\}\\
&=\frac{(1+q)(A-B)[2p-2]_{q,2}}{2q^{p-1}[p-1]_q}\cdot\frac{\{2+[2p-1]_q(1+q)(A-B)\}[2p-3]_q}{2q^{p-1}[2]_q[p-2]_q},\\
&\cdots\cdots\cdots\\
|a_{k-p}|&\leqq\prod_{j=1}^{k}\frac{2[j-1]_q+(1+q)(A-B)[2p-1]_q}{2q^{p-1}[j]_q[p-j]_q}
\end{aligned}
$$

for $1\leqq k\leqq p-1$. This proves Theorem 3.

Applying the same methods as in the proof of Theorem 3, we obtain the following Theorems 4 and 5.

Theorem 4. *Let $p\geqq 2$. If $g(z)=z^{-p}+\sum_{k=1}^{\infty}a_{-p+k}z^{-p+k}$ belongs to $MC_q[p,A,B]$, then*

$$
|a_{-p+k}|\leqq\frac{[p]_{q,p}}{q^{pk}[p-k]_{q,p}}\prod_{n=1}^{k}\frac{[2p-1]_q(1+q)(A-B)+2[n-1]_q}{2[n]_q q^{p-1}[p-n]_q}
$$

for $1\leqq k\leqq p-1$.

Theorem 5. *Let $g(z)=z^{-1}+\sum_{k=1}^{\infty}a_{k-1}z^{k-1}$ belong to $M\Sigma_q^*[A,B]$. Then*

$$
|a_{k-1}|\leqq\prod_{n=1}^{k}\frac{(1+q)(A-B)+2[n-1]_q}{2[n]_q}
$$

for $k\geqq 2$.

Letting $q\to 1-$, $A=1-2\alpha$ $(0\leqq\alpha<1)$ and $B=-1$ in Theorem 5, we have a result of the known family $M\Sigma^*(\alpha)$.

Corollary 2. *Let $g(z)=z^{-1}+\sum_{k=1}^{\infty}a_{k-1}z^{k-1}\in M\Sigma^*(\alpha)$. Then*

$$|a_{k-1}| \leqq \prod_{j=1}^{k} \frac{j-2\alpha+1}{j}$$

for $k \geqq 2$.

The following equivalence could help us to study the family $M\Sigma_q^*[p,A,B]$:

$$g \in M\Sigma_q^*[p,A,B] \iff \left| \frac{(1-B)\left\{ \frac{q^{2p-1} z (D_q^{(p)} g)(z)}{[2p-1]_q (D_q^{(p-1)} g)(z)} \right\} + (1-A)}{(1+B)\left(-\frac{q^{2p-1} z (D_q^{(p)} g)(z)}{[2p-1]_q (D_q^{(p-1)} g)(z)} \right) - (1+A)} - \frac{1}{1-q} \right| < \frac{1}{1-q}.$$

Theorem 6. *If a function* $g(z) = z^{-p} + \sum_{k=1}^{\infty} a_{-p+k} z^{-p+k} \in \Sigma(p)$ *satisfies*

$$\begin{aligned} &\sum_{k=1}^{p} \left(\frac{[k]_{q,p-1}}{q^{\left[k+\frac{1}{2}(p-2)\right](p-1)}} \right) \left\{ \left| [2p-1]_q (1+A) - [p+k-1]_q q^{p-k} (1+B) \right| + 2[p-k]_q \right\} |a_{-k}| \\ &\quad + \sum_{k=p}^{\infty} [k-p]_{q,p-1} \left\{ \left| [2p-1]_q (1+A) - [p+k-1]_q q^{2p-1} (1+B) \right| + 2[k+p]_q \right\} |a_k| \\ &\quad + [2p-1]_q [p-1]_q! (2+A) |a_{p-1}| < \frac{(A-B)[p]_{q,p}}{q^{\frac{1}{2}(3p-2)(p-1)}}, \end{aligned} \tag{22}$$

then $g \in M\Sigma_q^*[p,A,B]$.

Proof. By a simple calculation, we obtain

$$\begin{aligned} &\left| \frac{(1-B)\left\{ \frac{q^{2p-1} z (D_q^{(p)} g)(z)}{[2p-1]_q (D_q^{(p-1)} g)(z)} \right\} + (1-A)}{(1+B)\left\{ -\frac{q^{2p-1} z (D_q^{(p)} g)(z)}{[2p-1]_q (D_q^{(p-1)} g)(z)} \right\} - (1+A)} - \frac{1}{1-q} \right| \\ &\leqq \left| \frac{\{-q^{2p-1} z (D_q^{(p)} g)(z)\}(1-B) - [2p-1]_q (1-A)(D_q^{(p-1)} g)(z)}{\{-q^{2p-1} z (D_q^{(p)} g)(z)\}(1+B) - [2p-1]_q (1+A)(D_q^{(p-1)} g)(z)} + 1 \right| + \frac{q}{1-q} \\ &= 2 \left| \frac{\left\{ \begin{array}{l} \sum_{k=1}^{p} \frac{(-1)^{p-1} [k]_{q,p-1} [2p-1]_q}{q^{(p-1)\left[k+\frac{1}{2}(p-2)\right]}} a_{-k} z^{p-k} + \sum_{k=p-1}^{\infty} [k+2-p]_{q,p} [2p-1]_q a_k z^{k+p} \\ + \sum_{k=1}^{p} \frac{(-1)^p [k]_{q,p} q^{2p-1}}{q^{p\left[k+\frac{1}{2}(p-1)\right]}} a_{-k} z^{p-k} + \sum_{k=p}^{\infty} q^{2p-1} [k+1-p]_{q,p} a_k z^{n+p} \end{array} \right\}}{\left\{ \begin{array}{l} (A-B) \frac{(-1)^{p-1} [p]_{q,p}}{q^{\frac{1}{2}(p-1)(3p-2)}} + (A+1)[p-1]_q! [2p-1]_q a_{p-1} z^{2p-1} \\ + \sum_{k=1}^{p-1} \frac{(-1)^{p-1} [k]_{q,p-1}}{q^{(p-1)\left[k+\frac{1}{2}(p-2)\right]}} \{ [2p-1]_q (A+1) - (1+B) q^{p-k} [p-1+k]_q \} a_{-k} z^{p-k} \\ + \sum_{k=p}^{\infty} \{ [2p-1]_q (A+1) - (B+1) q^{2p-1} [p-1+k]_q \} [k-p]_{q,p+1} a_k z^{k+p} \end{array} \right\}} \right| \\ &\quad + \frac{q}{1-q} \\ &\leqq 2 \frac{\sum_{k=1}^{p-1} \frac{[k]_{q,p-1} [p-k]_q}{q^{(p-1)\left[k+\frac{1}{2}(p-2)\right]}} |a_{-k}| + [p-1]_q! [2p-1]_q |a_{p-1}| + \sum_{k=p}^{\infty} [k-p]_{q,p+1} [k+p]_q |a_k|}{\left\{ \begin{array}{l} \left(\frac{[p]_{q,p}}{q^{\frac{1}{2}(p-1)(3p-2)}} (A-B) - (1+A)[p-1]_q! [2p-1]_q |a_{p-1}| \right) \\ - \sum_{k=1}^{p-1} \frac{[k]_{q,p-1}}{q^{\left[k+\frac{1}{2}(p-2)\right](p-1)}} \left| \left\{ [2p-1]_q (A+1) - (B+1)[p+k-1+]_q q^{p-k} \right\} \right| |a_{-k}| \\ - \sum_{k=p}^{\infty} [k-p]_{q,p+1} \left| \{ [2p-1]_q (A+1) - (B+1) q^{2p-1} [p+k-1]_q \} \right| |a_k| \end{array} \right\}} + \frac{q}{1-q}. \end{aligned} \tag{23}$$

□

Now, it follows from (22) that the last expression in (23) is less than $\frac{1}{1-q}$. This proves Theorem 6.

Theorem 7. *If a function* $g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1} \in \Sigma$ *satisfies*

$$\sum_{k=1}^{\infty} \left\{ 2[k]_q + \left| q[k-1]_q(1+B) - (1-A) \right| \right\} |a_{k-1}| < A - B, \tag{24}$$

then $g \in M\Sigma_q^*[A,B]$.

Proof. By simple calculation, we have

$$\begin{aligned}
&\left| \frac{(1-B)\left(\frac{qz(D_q g)(z)}{g(z)}\right) + (1-A)}{(1+B)\left(-\frac{qz(D_q g)(z)}{g(z)}\right) - (1+A)} - \frac{1}{1-q} \right| \\
&\leqq \frac{q}{1-q} + 2\left| \frac{g(z) + qz(D_q g)(z)}{q(1+B)z(D_q g)(z) + (1+A)} \right| \\
&= \frac{q}{1-q} + 2\left| \frac{\sum_{k=1}^{\infty} [k]_q |a_{k-1}|}{(A-B) - \sum_{k=1}^{\infty} [q(1+B)[k-1]_q + (1+A)] |a_{k-1}|} \right|.
\end{aligned} \tag{25}$$

From (24) we can see that (25) is less than $\frac{1}{1-q}$. This proves Theorem 7. □

Letting $A = 1 - 2\alpha$ $(0 \leqq \alpha < 1)$, $B = -1$ and $q \to 1^-$ in Theorem 7, we obtain a result of the known family $M\Sigma^*(\alpha)$.

Corollary 3. *If a function* $g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1} \in \Sigma$ *satisfies*

$$\sum_{k=1}^{\infty} (1 + k - \alpha)|a_{k-1}| < 1 - \alpha \quad (0 \leqq \alpha < 1),$$

then $g \in M\Sigma^*(\alpha)$.

Applying the same method as in the proof of Theorem 7, we obtain the following theorem.

Theorem 8. *If a function* $g(z) = z^{-1} + \sum_{k=1}^{\infty} a_{k-1} z^{k-1} \in \Sigma$ *satisfies*

$$\sum_{k=1}^{\infty} \left\{ 2[k]_q + \left| q[k-1]_q(1+B) - (1-A) \right| \right\} [k-1]_q |a_{k-1}| < A - B,$$

then $g \in MC_q[A,B]$.

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