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Homothetic Symmetries of Static Cylindrically Symmetric Spacetimes—A Rif Tree Approach

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Abstract: In this paper, we find all static, cylindrically symmetric spacetime metrics admitting homothetic symmetries. For this purpose, first we analyze the homothetic symmetry equations by an algorithm developed in Maple which gives all possible static, cylindrically symmetric metrics that may possess proper homothetic symmetry. After that, we have solved the homothetic symmetry equations for all these metrics to get the final form of homothetic symmetry vector fields. Comparing the obtained results with those of direct integration technique, it is observed that the Rif tree approach not only recovers the metrics already found by direct integration technique, but it also produces some new metrics.

Keywords: homothetic symmetries; Killing symmetries; static cylindrically symmetric spacetime

MSC: 83C15



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1. Introduction

The theory of general relativity was presented by Albert Einstein in 1915, and until now it has been considered as a widely accepted theory among all the existing theories of gravitation. This theory gives a generalization of the special theory of relativity, which is another scientific theory presented by Einstein in 1905 giving a relationship between space and time. Moreover, general relativity refines Newton's law of universal gravitation and treats gravity as a warping of spacetime rather than a force that attracts objects toward each other. The warping of spacetime, known as curvature, is because of the presence of matter in the spacetime. The curvature of a spacetime is given by a tensor quantity, called an Einstein tensor, and it is denoted by G_{ab} . In terms of metric tensor g_{ab} , Ricci tensor R_{ab} , and the Ricci scalar R , the Einstein tensor can be expressed as $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$. The distribution of matter and energy in a spacetime is given by energy-momentum tensor T_{ab} , and it is related to the Einstein tensor G_{ab} via a system of non-linear partial differential equations, known as Einstein's field equations (EFEs) [1],

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = kT_{ab}, \quad (1)$$

where $k = \frac{8\pi G}{c^4}$ is a constant with G as a Newtonian constant of gravitation and c as the speed of light. The apparent look of EFEs is simple, but finding their exact solutions is much more difficult because of their nonlinear nature. However, many exact solutions of these equations have been found in the literature, and have played a pivotal role in the discussion of physical problems. Some of the known solutions of EFEs include Schwarzschild, Kerr, Reissner and Nordström, de Sitter, Tolman, Friedmann–Robertson–Walker, and plane wave solutions. Out of these solutions, Schwarzschild and Kerr solutions are found to be very helpful in the study of black holes. Similarly, Friedmann solutions play a key role in the field of cosmology and the plane wave solutions in discovering the existence of gravitational

radiations [1]. All these exact solutions of EFEs are obtained under some assumptions, the most common being the symmetry restrictions on the metric of spacetime. Such restrictions not only help in finding the exact solution of EFEs, they are also used in the classification of the existing solutions. Killing symmetry, also called the Killing vector field (KVF), is considered to be the most basic spacetime symmetry and is defined in terms of a smooth vector field V such that the Lie derivative of the metric tensor along V turns out to be zero; that is, $\mathcal{L}_V g_{ab} = 0$ [2]. In order to explore KVFs for any spacetime, one requires to derive a set of Killing equations by using the relation $\mathcal{L}_V g_{ab} = 0$ and to solve these equations to get the explicit form of KVFs. These vector fields have been studied for some physically important spacetimes such as static, cylindrically symmetric [3], static, spherically symmetric [4] and plane symmetric spacetimes [5].

Corresponding to the KVFs admitted by a spacetime, there exist conservation laws [6]. In the literature, it is observed that most of the conservation laws in a spacetime are given by the KVFs they admit. However, there are certain conservation laws which are not given by KVFs. For example, in the Friedmann metric, there is no timelike KVF giving conservation of energy, but the same conservation law can be achieved in this metric with the help of conformal symmetries [7]. Like Killing symmetry, conformal symmetry is also defined in terms of a smooth vector field V satisfying the relation [2]

$$\mathcal{L}_V g_{ab} = 2\alpha g_{ab}, \quad (2)$$

where α denotes a real valued function on the spacetime. In explicit form, Equation (2) gets the form

$$g_{ab,c} V^c + g_{ac} V^c{}_{,b} + g_{bc} V^c{}_{,a} = 2\alpha g_{ab}. \quad (3)$$

Varying a, b , and c from 0 to 3 in the above equation and using the Einstein's summation convention over the repeated indices, we get a system of ten equations. The solution of these ten equations leads to the explicit form of the vector field V . In cases where α in Equation (2) is a constant, the conformal vector field becomes a homothetic vector field (HVF). Conformal symmetries, also called conformal vector fields (CVFs) have many applications. These symmetries are used in literature as a mathematical tool for the integration of EFEs [8]. Moreover, these symmetries have interesting applications in cosmology and astrophysics [9,10]. Due to these applications, conformal symmetry attracted researchers and different spacetimes were classified via this symmetry [11–14].

The method of direct integration was used for finding all the above-defined symmetries in the aforementioned references, which is a cumbersome and time-consuming approach that also runs the risk of losing some important metrics. In order to overcome these problems and to ensure that no important spacetime metric is lost during the classification, the recent literature of general relativity uses some computer algorithms for determining all possible metrics possessing the desired symmetry. These metrics are then used to solve the symmetry equations under consideration. This approach of investigating spacetime symmetries is known as the Rif tree approach, and it has been recently used to classify some spacetime metrics according to different symmetries [15–17], where it has been observed that this new approach not only recovers all the metrics obtained by direct integration technique but also gives some new metrics. In this paper, we use this approach to study homothetic symmetries of the most general static, cylindrically symmetric metric.

2. Homothetic Symmetry Equations

The metric of static, cylindrically symmetric spacetimes is given by [1]

$$ds^2 = -A^2(r) dt^2 + dr^2 + B^2(r) d\theta^2 + C^2(r) dz^2, \quad (4)$$

where $A \neq 0$, $B \neq 0$ and $C \neq 0$. This metric admits at least three KVFs, given by $V_{(1)} = \partial_t$, $V_{(2)} = \partial_\theta$ and $V_{(3)} = \partial_z$. Out of the three metric coefficients A , B , and C , if any two are same, the above metric reduces to the metric of static plane symmetric spacetimes, admitting

the same three KVF's along with an additional rotational symmetry. The homothetic symmetries of static plane symmetric spacetimes were explored by using the Rif tree approach in Ref. [18]. Thus, throughout this paper we only consider static, cylindrically symmetric metrics with $A \neq B \neq C$. By using $g_{00} = -A^2$, $g_{11} = 1$, $g_{22} = B^2$ and $g_{33} = C^2$ from the above metric in Equation (2) with $\alpha = \text{const.}$, we obtain the following set of homothetic symmetry equations:

$$A' V^1 + A V_{,0}^0 = \alpha A, \quad (5)$$

$$A^2 V_{,1}^0 - V_{,0}^1 = 0, \quad (6)$$

$$B^2 V_{,0}^2 - A^2 V_{,2}^0 = 0, \quad (7)$$

$$C^2 V_{,0}^3 - A^2 V_{,3}^0 = 0, \quad (8)$$

$$V_{,1}^1 = \alpha, \quad (9)$$

$$V_{,2}^1 + B^2 V_{,1}^2 = 0, \quad (10)$$

$$V_{,3}^1 + C^2 V_{,1}^3 = 0, \quad (11)$$

$$B' V^1 + B V_{,2}^2 = \alpha B, \quad (12)$$

$$B^2 V_{,3}^2 + C^2 V_{,2}^3 = 0, \quad (13)$$

$$C' V^1 + C V_{,3}^3 = \alpha C. \quad (14)$$

The solution of the above system leads to the explicit form of HVF's for $\alpha \neq 0$, whereas for $\alpha = 0$ it gives KVF's. These equations were solved in Ref. [18] by considering $\alpha \neq 0$ and using the direct integration approach. The authors obtained only two static, cylindrically symmetric metrics (other than plane symmetric) possessing proper HVF's.

In the direct integration technique, one needs to decouple the set of symmetry equations and then integrate it. Moreover, some conditions on the metric functions are also needed to solve this system. The main drawback of this approach of solving symmetry equations is that it does not provide a complete classification of the spacetime under consideration because there is no criteria for choosing the values of metric functions. On the other hand, a newly developed Rif tree approach provides a systematic way to solve the set of symmetry equations and provides many more metrics than those given by the direct integration technique. In this way, one gets a complete classification of the spacetime under consideration.

The Rif tree approach is based on a Maple algorithm, known as Rif algorithm, whose idea was first given by Reid et al. [19]. This algorithm was developed for the purpose of reduction of nonlinear systems of differential equations to reduced involutive form by using some algebraic and differentiation operations. As a result, the simplified involutive form of the system satisfies the constant-rank condition, and it contains all the integrability conditions. To start the procedure, one needs to consider a system of differential equations along with a matrix representing the ranking of the derivatives used in the equations. In the second step, the basic operations of algebra and derivatives are utilized for transforming the system to a specific form that includes all integrability conditions. The Rif algorithm is a powerful tool that can be used for reducing the complexity of the system. It also gives information about the number of solutions of a system before solving it. Though this calculation is simple, it is quite lengthy. However, one may use the "rifsimp" command with the "Exterior" package in Maple for all these calculations. The Maple command "caseplot" is used to view the output of the Rif algorithm graphically. The resulting plot is always in the form of a tree, known as Rif tree or classification tree. While solving the symmetry equations by this method, each branch of the Rif tree gives a unique metric. After that, one needs to solve the symmetry equations for all these metrics. Here, we use the Rif tree approach to solve the set of symmetry equations and prove that this approach produces many new metrics which were not listed in Ref. [18].

In order to find all possible static cylindrically symmetric metrics admitting HVFs, we analyze Equations (5)–(14) through the Rif algorithm. The algorithm imposes some conditions on A , B , and C , which we then use to solve Equations (5)–(14) to get the explicit form of HVFs. These conditions are displayed in the form of branches of the Rif tree, given in Figure 1. The nodes of the Rif tree, denoted by $p_1, p_1, \dots, p_{12}$ are called pivots and are given in (15). Moreover, the symbols " $=$ " and " $<>$ " in the tree signify whether the corresponding p_i is zero or non-zero, respectively. In this way, every branch restricts the metric functions A , B , and C to satisfy some conditions which are then used to solve Equations (5)–(14).

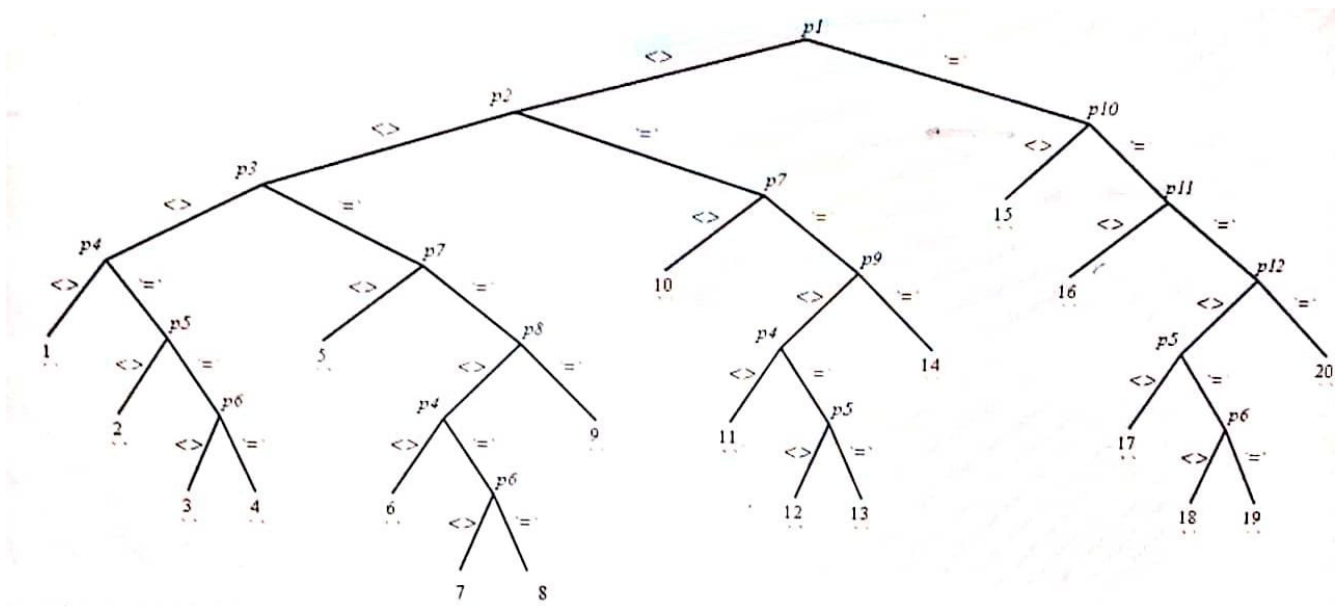


Figure 1. Rif Tree.

$$\begin{aligned} p1 &= A', & p2 &= C', & p3 &= B', \\ p4 &= A''A - A'^2, & p5 &= B'^2 - BB'', & p6 &= C'^2 - CC'', \\ p7 &= AA''' - A''A', & p8 &= A''C - A'C', & p9 &= B'A' - BA'', \\ p10 &= CC''' - C'C'', & p11 &= B'B'' - BB''', & p12 &= C'B' - BC''. \end{aligned} \quad (15)$$

For the conditions of some branches, the solution of Equations (5)–(14) gives the minimum three KVF's with no proper homothety. Such branches are labeled by 7, 12, 17, and 18 in the Rif tree. In the remaining branches, we have either obtained one proper HVF along with three minimum KVF's or four or seven KVF's with no proper HVF. We summarize our results in the forthcoming sections.

2.1. Four HVFs

For branches 1, 5, 6, 10, 15, and 16, the solution of Equations (5)–(14) yields four-dimensional homothetic algebra, with one proper homothety and three minimum KVF. Moreover, some of these branches produce more than one metric, each possessing the same number of HVFs. All these metrics along with the explicit form of homothetic vector field V and the proper homothety are listed in Table 1.

Comparing our results with those of Ref. [18], one can easily observe that here we have obtained 12 static, cylindrically symmetric metrics possessing four HVFs, whereas in Ref. [18] the authors obtained only two metrics whose homothetic algebra is four-dimensional. The remaining metrics given in Ref. [18] are static plane symmetric metrics. This proves the significance of the Rif tree approach, in that it gives a complete classification of spacetime under consideration with respect to homothetic symmetries.

All the obtained metrics of this section have a non-zero Weyl tensor of Petrov type I. To check whether the obtained spacetime metrics are regular or they contain some singularity, we find the corresponding Kretschmann scalar for all these metrics. The Kretschmann scalar, denoted by K , is a quadratic scalar invariant defined by $K = R_{abcd}R^{abcd}$, where R_{abcd} is the Riemann curvature tensor and Einstein's summation convention is applied on the repeated indices a, b, c , and d . The Kretschmann scalar for the metric 4a is obtained as

$$K = \frac{4}{(\alpha r + b_1)^4} \left[b_2^2(\alpha - b_2)^2 + b_3^2(\alpha - b_3)^2 + b_4^2(\alpha - b_4)^2 + (\alpha - b_2)^2(\alpha - b_3)^2 + (\alpha - b_2)^2(\alpha - b_4)^2 + (\alpha - b_3)^2(\alpha - b_4)^2 \right]. \quad (16)$$

As the metric functions A, B , and C are non-zero, thus $\alpha r + b_1 \neq 0$. Therefore, the Kretschmann scalar is finite and hence the spacetime has no singularity. In addition, because $\alpha \neq b_2 \neq b_3 \neq b_4$, the values of K is always non-zero.

For the metric 4b, the Kretschmann scalar becomes

$$K = \frac{4b_1^4}{(b_1r + b_2)^4} \left[b_3^2 + b_4^2 + b_3^2b_4^2 + b_3^2(b_3 - 1)^2 + b_4^2(b_4 - 1)^2 \right], \quad (17)$$

which is again finite, as $b_1r + b_2 \neq 0$; otherwise, the metric functions vanish. Hence the metric 4b is regular. The structure of the Kretschmann scalar for the metrics 4c and 4d is the same as that of the metric 4b with only the difference of constants involved in the values of A, B , and C . Hence, the metrics 4c and 4d have no singularity.

Similarly, the value of K for the metric 4e is found to be

$$K = \frac{4}{(\alpha r + b_1)^4} \left[b_2^2(\alpha - b_2)^2 + b_5^2(\alpha - b_5)^2 + (\alpha - b_2)^2(\alpha - b_5)^2 \right]. \quad (18)$$

Like the previous cases, here K is finite as $\alpha r + b_1 \neq 0$. Thus the metric is regular. Similarly, the Kretschmann scalar for the metrics 4h and 4j, being similar in structure to that of the metric 4e, is finite, and hence these two metrics are also regular.

The Kretschmann scalar for the metrics 4f, 4g, 4i, 4k, and 4l have a similar structure with only the difference of parameters involved in the values of A, B , and C . Out of these metrics, the value of K for metric 4f is given by

$$K = \frac{4b_3^2(b_1 - \alpha)^2(b_1^2 + \alpha^2)}{\alpha^4(b_3r + b_4)^4}, \quad (19)$$

which is finite because $b_3r + b_4 \neq 0$ and $\alpha \neq 0$. Hence, the metrics 4f, 4g, 4i, 4k, and 4l have no singularity.

Table 1. Metrics admitting four HVFs.

No.	Metric	$V = V^a \partial_{x^a}$	Proper HVF
4a. (Branch 1)	$A = (\alpha r + b_1)^{1-\frac{b_2}{\alpha}}, B = (\alpha r + b_1)^{1-\frac{b_3}{\alpha}}$ $C = (\alpha r + b_1)^{1-\frac{b_4}{\alpha}}, b_2 \neq \alpha \neq b_3 \neq b_4.$	$\{b_2t + b_5\}\partial_t + \{\alpha r + b_1\}\partial_r +$ $\{b_3\theta + b_6\}\partial_\theta + \{b_4z + b_7\}\partial_z.$	$r\partial_r.$
4b. (Branch 1)	$A = b_1r + b_2, B = A^{b_3}$ $C = A^{b_4}, b_3 \neq b_4 \neq 0, 1.$	$\{b_5\}\partial_t + \{\alpha \frac{A}{b_1}\}\partial_r +$ $\{\theta(1 - b_3)\alpha + b_6\}\partial_\theta$ $+ \{\alpha z(1 - b_4) + b_7\}\partial_z.$	$\frac{A}{b_1}\partial_r + \theta(1 - b_3)\partial_\theta$ $+ z(1 - b_4)\partial_z.$
4c. (Branch 1)	$A = B^{b_1}, B = b_2r + b_3$ $C = A^{b_4}, b_1 \neq b_4 \neq 0, 1.$	$\{\alpha t(1 - b_1) + b_5\}\partial_t + \{\alpha \frac{B}{b_2}\}\partial_r +$ $\{b_6\}\partial_\theta + \{\alpha z(1 - b_4) + b_7\}\partial_z.$	$t(1 - b_1)\partial_t + \frac{B}{b_2}\partial_r$ $+ z(1 - b_4)\partial_z.$
4d. (Branch 1)	$A = C^{b_1}, B = C^{b_2}$ $C = b_3r + b_4, b_1 \neq b_2 \neq 0, 1.$	$\{\alpha t(1 - b_1) + b_5\}\partial_t + \{\alpha \frac{C}{b_3}\}\partial_r +$ $\{\theta(1 - b_2)\alpha + b_6\}\partial_\theta + \{b_7\}\partial_z.$	$t(1 - b_1)\partial_t + \frac{C}{b_3}\partial_r$ $+ \theta(1 - b_2)\partial_\theta.$

Table 1. Cont.

No.	Metric	$V = V^a \partial_{x^a}$	Proper HVF
4e. (Branch 5)	$A = (\alpha r + b_1)^{1-\frac{b_2}{\alpha}}, B = c_1 \neq 0$ $C = (\alpha r + b_1)^{1-\frac{b_5}{\alpha}}, b_2 \neq \alpha \neq b_5$	$\{b_2 t + b_3\} \partial_t + \{\alpha r + b_1\} \partial_r +$ $\{\alpha \theta + b_4\} \partial_\theta + \{b_5 z + b_6\} \partial_z.$	$r \partial_r + \theta \partial_\theta.$
4f. (Branch 5)	$A = C^{1-\frac{b_1}{\alpha}}, B = b_2 \neq 0$ $C = (b_3 r + b_4), b_1 \neq \alpha.$	$\{b_5\} \partial_t + \{\alpha \frac{C}{b_3}\} \partial_r +$ $\{\alpha \theta + b_6\} \partial_\theta + \{b_7\} \partial_z.$	$\frac{C}{b_3} \partial_r + \theta \partial_\theta.$
4g. (Branch 6)	$A = (b_1 r + b_2), B = b_3 \neq 0$ $C = A^{1-\frac{b_4}{\alpha}}, b_4 \neq \alpha.$	$\{b_5\} \partial_t + \{\alpha \frac{A}{b_1}\} \partial_r +$ $\{\alpha \theta + b_6\} \partial_\theta + \{b_7\} \partial_z.$	$\frac{A}{b_1} \partial_r + \theta \partial_\theta.$
4h. (Branch 10)	$A = (\alpha r + b_5)^{1-\frac{b_1}{\alpha}}, B = (\alpha r + b_5)^{1-\frac{b_3}{\alpha}}$ $C = c_1 \neq 0, b_1 \neq \alpha \neq b_3$	$\{b_1 t + b_2\} \partial_t + \{\alpha r + b_5\} \partial_r +$ $\{b_3 \theta + b_4\} \partial_\theta + \{\alpha z + b_6\} \partial_z.$	$r \partial_r + z \partial_z.$
4i. (Branch 10)	$A = B^{1-\frac{b_1}{\alpha}}, B = (b_2 r + b_3)$ $C = b_4 \neq 0, b_1 \neq \alpha$	$\{b_5\} \partial_t + \frac{B}{b_2} \partial_r$ $\{b_6\} \partial_\theta + \{\alpha z + b_7\} \partial_z.$	$\frac{B}{b_2} \partial_r + z \partial_z.$
4j. (Branch 15)	$A = c_1 \neq 0, B = (\alpha r + b_4)^{1-\frac{b_2}{\alpha}}$ $C = (\alpha r + b_4)^{1-\frac{b_5}{\alpha}}, b_2 \neq \alpha \neq b_5.$	$\{\alpha t + b_1\} \partial_t + \{\alpha r + b_4\} \partial_r +$ $\{b_2 \theta + b_3\} \partial_\theta + \{b_5 z + b_6\} \partial_z.$	$t \partial_t + r \partial_r.$
4k. (Branch 15)	$A = b_1 \neq 0, B = (b_2 r + b_3)$ $C = B^{1-\frac{b_4}{\alpha}}, b_4 \neq \alpha$	$\{\alpha t + b_5\} \partial_t + \{\alpha \frac{B}{b_2}\} \partial_r +$ $\{b_6\} \partial_\theta + \{b_7\} \partial_z.$	$t \partial_t + \frac{B}{b_2} \partial_r.$
4l. (Branch 16)	$A = b_1 \neq 0, B = C^{1-\frac{b_2}{\alpha}}$ $C = (b_3 r + b_4), b_5 \neq \alpha.$	$\{\alpha t + b_5\} \partial_t + \{\alpha \frac{C}{b_3}\} \partial_r +$ $\{b_6\} \partial_\theta + \{b_7\} \partial_z.$	$t \partial_t + \frac{C}{b_3} \partial_r.$

2.2. Four KVF

This section contains the results of those branches of the Rif tree for which the solution of Equations (5)–(14) leads to one additional KVF, along with the three minimum ones, and no proper homothety. Such branches are labeled by 4, 8, 13, and 19 in the Rif tree. The exact form of the metrics of these branches, the components of the Killing vector field V and additional KVFs are given in Table 2.

Table 2. Metrics admitting four KVFs.

No.	Metric	Vector Field Components	Additional KVFs
4(i). (Branch 4)	$A = e^{c_1 r + c_2},$ $B = e^{c_3 r + c_4},$ $C = e^{c_5 r + c_6},$ where $c_1 \neq c_3 \neq c_5$	$V^0 = -c_1 b_1 t + b_2$ $V^1 = b_1$ $V^2 = -c_3 b_1 \theta + b_3$ $V^3 = -c_5 b_1 z + b_4$	$V_{(4)} = -c_1 t \partial_t + \partial_r - c_3 \theta \partial_\theta$ $- c_5 z \partial_z$
4(ii). (Branch 8)	$A = c_4 e^{c_2 r},$ $B = c_1,$ $C = c_5 e^{c_3 r},$ where $c_1 \neq c_2 \neq c_3$	$V^0 = -c_2 b_1 t + b_2$ $V^1 = b_1$ $V^2 = b_3$ $V^3 = -c_3 b_1 z + b_4$	$V_{(4)} = -c_2 t \partial_t + \partial_r - c_3 z \partial_z$
4(iii). (Branch 13)	$A = c_3 e^{c_2},$ $B = c_5 e^{c_4 r},$ $C = c_1,$ where $c_1 \neq c_2 \neq c_4$	$V^0 = -c_2 b_1 t + b_2$ $V^2 = -c_4 b_1 \theta + b_3$ $V^3 = b_4$	$V_{(4)} = -c_2 t \partial_t + \partial_r - c_4 \theta \partial_\theta$
4(iv). (Branch 19)	$A = c_1,$ $B = c_2 e^{c_3 r},$ $C = c_4 e^{c_5 r},$ where $c_1 \neq c_3 \neq c_5$	$V^0 = b_1$ $V^1 = b_2$ $V^2 = -c_3 b_2 \theta + b_3$ $V^3 = -c_5 b_2 z + b_4$	$V_{(4)} = \partial_r - c_3 \theta \partial_\theta - c_5 z \partial_z$

A complete classification of the static cylindrically symmetric spacetimes via their KVFs was presented in Ref. [3], where the authors solved the set of Killing equations by direct integration approach. However, like the case of homotheties, the metrics obtained here by Rif tree approach were not listed in Ref. [3].

The Weyl tensor for all the metrics of this section vanishes; therefore, all these metrics are conformally flat and are of Petrov type O. The Kretschmann scalar for the metric 4(i) is obtained as

$$K = 4 \left[c_1^4 + c_3^4 + c_5^4 + c_1^2 c_3^2 + c_3^2 c_5^2 + c_1^2 c_5^2 \right], \quad (20)$$

which is always finite. Thus, the metric 4(i) is regular. The structure of Kretschmann scalar for the remaining three metrics is similar, with only the difference of parameters involved in the metric functions. For metric 4(ii), it has the value

$$K = 4 \left[c_2^4 + c_3^4 + c_2^2 c_3^2 \right], \quad (21)$$

which is clearly finite. Hence, all three metrics given by 4(ii), 4(iii) and 4(iv) are regular.

2.3. Seven KVF's

The branches 9, 14, and 20 of the Rif tree produce metrics admitting seven KVF's, out of which three are the minimum ones and the extra four KVF's along with the components of vector field V , are given in Tables 3–5. Like the previous section, these three metrics were not found in Ref. [3].

For all the three metrics of this section, the Weyl tensor vanishes; thus, all these metrics are conformally flat and of Petrov type O. The Kretschmann scalar for all these three metrics is given by $K = 12c_1^4$, showing that these metrics have no singularity.

Table 3. Metrics admitting seven KVF's.

No.	Metric	Vector Field Components	Additional KVF's
7a. (Branch 9)	$A(r) = c_2 e^{c_1 r} + c_3 e^{-c_1 r},$ $C(r) = c_4 A'(r),$ $B(r) = c_5,$	$V^0 = \frac{B}{mA} (-b_1 \sin mt + b_2 \cos mt) \cos m\theta,$ $+ \frac{B}{mA} (-b_3 \sin mt + b_4 \cos mt) \sin m\theta + b_5,$ $V^1 = (b_1 \sin mt + b_2 \cos mt) \cos m\theta$ $+ (b_3 \sin mt + b_4 \cos mt) \sin m\theta,$ $V^2 = -\frac{A}{mB} (b_1 \sin mt + b_2 \cos mt) \cos m\theta$ $+ \frac{A}{mB} (b_3 \sin mt + b_4 \cos mt) \sin m\theta + b_6,$ $V^3 = b_7.$ where $m = 2c_1 \sqrt{c_2 c_3},$	$V_{(4)} = -\frac{B}{mA} \sin mt \cos m\theta \partial_t$ $+ \frac{B}{mA} \cos mt \cos m\theta \partial_r,$ $- \frac{A}{mB} \cos mt \sin m\theta \partial_\theta,$ $V_{(5)} = \frac{B}{mA} \cos mt \cos m\theta \partial_t,$ $+ \frac{B}{mA} \sin mt \cos m\theta \partial_r,$ $- \frac{A}{mB} \sin mt \sin m\theta \partial_\theta,$ $V_{(6)} = -\frac{B}{mA} \sin mt \sin m\theta \partial_t$ $+ \frac{B}{mA} \cos mt \cos m\theta \partial_r,$ $+ \frac{A}{mB} \cos mt \cos m\theta \partial_\theta$ $V_{(7)} = \frac{B}{mA} \cos mt \sin m\theta \partial_t$ $+ \frac{B}{mA} \sin mt \sin m\theta \partial_r,$ $+ \frac{A}{mB} \sin mt \cos m\theta \partial_\theta$

Table 4. Metrics admitting seven KVF's.

No.	Metric	Vector Field Components	Additional KVF's
7b. (Branch 14)	$A(r) = c_2 e^{c_1 r} + c_3 e^{-c_1 r},$ $B(r) = c_4 A'(r),$ $C(r) = c_5,$	$V^0 = \frac{A'}{nA} (b_1 \sin my + b_2 \cos my) \cos nt$ $- \frac{A'}{nA} (b_3 \sin my + b_4 \cos my) \sin nt + b_5,$ $V^1 = (b_1 \sin my + b_2 \cos my) \sin nt$ $+ (b_3 \sin my + b_4 \cos my) \cos nt,$ $V^2 = -\frac{A}{mB} (b_1 \cos my - b_2 \sin my) \sin nt$ $+ \frac{A}{mB} (b_3 \cos my - b_4 \sin my) \cos nt + b_6,$ $V^3 = b_7.$ where $n = 2c_1 \sqrt{c_2 c_3}, m = 2c_4 \sqrt{c_2 c_3}$	$V_{(4)} = \frac{A'}{nA} \sin my \cos nt \partial_t$ $+ \sin my \sin nt \partial_r,$ $+ \frac{A}{mB} \cos my \sin nt \partial_\theta,$ $V_{(5)} = \frac{A'}{nA} \cos my \cos nt \partial_t,$ $+ \sin nt \cos my \partial_r,$ $- \frac{A}{mB} \sin my \sin nt \partial_\theta,$ $V_{(6)} = -\frac{A'}{nA} \sin my \sin nt \partial_t$ $+ \sin my \cos nt \partial_r,$ $+ \frac{A}{mB} \cos my \cos nt \partial_\theta$ $V_{(7)} = -\frac{A'}{nA} \cos my \sin nt \partial_t$ $+ \cos my \cos nt \partial_r,$ $- \frac{A}{mB} \sin my \cos nt \partial_\theta$

Table 5. Metrics admitting seven KVF.

No.	Metric	Vector Field Components	Additional KVFs
7c. (Branch 20)	$A(r) = c_5,$ $B(r) = c_4 C'(r),$ $C(r) = c_2 e^{c_1 r} + c_3 e^{-c_1 r},$	$V^0 = b_1,$ $V^1 = (b_2 \sin mz + b_3 \cos mz) e^{ny}$ $+ (b_4 \sin mz + b_5 \cos mz) e^{-ny},$ $V^2 = -\frac{B'}{nB} (b_2 \sin mz + b_3 \cos mz) e^{ny},$ $+ \frac{B'}{nB} (b_4 \sin mz + b_5 \cos mz) e^{-ny} + b_6$ $V^3 = \frac{B}{mC} (b_2 \cos mz - b_3 \sin mz) e^{ny}$ $+ \frac{B}{mC} (b_4 \cos mz - b_5 \sin mz) e^{-ny} + b_7.$ where $n = 2c_1 \sqrt{c_2 c_3}, m = 2c_4 \sqrt{c_2 c_3}$	$V_{(4)} = \sin mze^{ny} \partial_r$ $- \frac{B'}{nB} \sin mze^{ny} \partial_\theta$ $+ \frac{B}{mC} \cos mze^{ny} \partial_z,$ $V_{(5)} = \cos mze^{ny} \partial_r$ $- \frac{B'}{nB} \cos mze^{ny} \partial_\theta$ $+ \frac{B}{mC} \cos mze^{ny} \partial_z,$ $V_{(6)} = \sin mze^{-ny} \partial_r$ $+ \frac{B'}{nB} \sin mze^{-ny} \partial_\theta$ $+ \frac{B}{mC} \cos mze^{-ny} \partial_z,$ $V_{(7)} = \cos mze^{-ny} \partial_r$ $+ \frac{B'}{nB} \cos mze^{-ny} \partial_\theta$ $- \frac{B}{mC} \cos mze^{-ny} \partial_z$

3. Conclusions

In this paper, we have explored HVFs of static, cylindrically symmetric spacetimes. Instead of the frequently used method of directly integrating the homothetic symmetry equations, we have analyzed these equations by using the Rif algorithm that gives many static, cylindrically symmetric metrics possessing different dimensional homothetic and Killing algebras. Out of these, twelve metrics admit one proper HVF along with three minimum KVFs of the spacetimes under consideration. The remaining metrics possess three, four, or seven KVFs with no proper homothety. By finding the Lie algebra of the obtained vector fields by using the relation $[V_1, V_2] = V_1 V_2 - V_2 V_1$, one can easily check that the structure of Lie algebra for each metric is different from that of all other metrics. Thus, no two of these metrics are connected by coordinate transformations. We have also calculated the Weyl tensor for all the obtained metrics, and it is observed that for the metrics admitting proper HVFs, the Weyl tensor is Petrov type I. For the metrics of Sections 2.3 and 2.4, the Weyl tensor vanishes, and therefore these are Petrov type O metrics. By finding the Kretschmann scalar, it is conjectured that all the obtained metrics during our classification are regular.

Comparing our result with those obtained in an earlier study by direct integration technique, it is observed that the Rif tree approach recovers all the metrics given by direct integration method, and it also produces many extra metrics. Thus, the Rif tree approach is a better choice to be used for the classification of spacetimes via their Killing and homothetic symmetries.

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