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Approximation of Fixed Points for Enriched Suzuki Nonexpansive Operators with an Application in Hilbert Spaces

Kifayat Ullah [1], Junaid Ahmad [2], Muhammad Arshad [2] and Zhenhua Ma [3,*]

[1] Department of Mathematics, University of Lakki Marwat, Lakki Marwat 28420, Pakistan; kifayatmath@yahoo.com

[2] Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan; ahmadjunaid436@gmail.com (J.A.); marshadzia@iiu.edu.pk (M.A.)

[3] Department of Mathematics and Physics, Hebei University of Architecture, Zhangjiakou 075024, China

* Correspondence: mazhenghua_1981@163.com

Abstract: In this article, we introduce the class of enriched Suzuki nonexpansive (ESN) mappings. We show that this new class of mappings properly contains the class of Suzuki nonexpansive as well as the class of enriched nonexpansive mappings. We establish existence of fixed point and convergence of fixed point in a Hilbert space setting under the Krasnoselskii iteration process. One of the our main results is applied to solve a split feasibility problem (SFP) in this new setting of mappings. Our main results are a significant improvement of the corresponding results of the literature.

Keywords: enriched mapping; Krasnoselskii iteration; demicompact; suzuki map; hilbert space

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1. Introduction

For many different types of problems in applied physics and mathematical engineering, one faces too many difficulties to guarantee the ability to find solutions using the already known analytical methods (see, e.g., [1–5] and others). In such cases, fixed point theory suggests some alternative techniques for obtaining the sought after solutions. We first express the problem in the form of a fixed point equation in such a way that the fixed point set of the expressed equation and the solution set of the given problem become equal. Once the existence of the fixed point for the expressed equation is established, then it can be said that the existence of the solution for the given equation is established. After this, we use the Picard iteration [6] which is the simplest iterative method used for computing the value of the fixed point of the expressed equation and hence the solution of the given probelem. We use other iterations such as the Krasnoselskii iteration [7] if the Picard iteration fails to approach the fixed point as discussed in this paper.

Consider a Hilbert space X with $\langle .,. \rangle$. Let $||.||$ be the norm on X induced by the inner product $\langle .,. \rangle$. Suppose $\emptyset \neq M \subseteq X$ is closed and convex. An operator $T : M \to M$ is called contraction in the case when for every choice of $u, v \in M$, there is a real constant $0 \leq \theta < 1$, such that:

$$||Tu - Tv|| \leq \theta ||u - v||. \tag{1}$$

It ia well known that the Banach fixed point theorem (BFPT) [8] asserts that the contraction operator T as defined in (1) attains a unique fixed point, namely, z in M and its Picard iterates [6] $x_{k+1} = Tx_k$, $(k = 0, 1, 2, \ldots)$, essentially converges to z in the strong sense for every choice of $x_0 \in M$. In (1), if θ takes the value 1, then T is called nonexpansive.

The following theorem has been shown by Browder [9] in 1965.

Theorem 1. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is nonexpansive, then set $F(T)$ is nonempty closed and convex.*

A natural question arises: Does the Picard iteration converge to the fixed point of T in Theorem 2? Here, we answer this question in the negative using the following example.

Example 1. *Let $M = [0,1]$ and $Tu = 1 - u$. Then T is nonexpansive but not contraction having a unique fixed point $z = 0.5$. It is easy to show that the Krasnoselskii iteration [7] converges to the fixed point of T but Picard iteration fails to converge. To see this, let $x_0 = x \neq 0.5$, the Picard iteration produce the following non convergent sequence:*

$$x, 1-x, x, 1-x, \ldots$$

The class of nonexpansive maps has been extended in many different ways. In particular, Aoyama and Kohsaka [10], Bae [11], Bogin [12], Garcia-Falset et al. [13], Goebel and Kirk [14], Goebel et al. [15], Karapinar and Tas [16], Llorens-Fuster and Moreno-Galvez [17], Pant and Shukla [18], Patir et al. [19] suggested and studied different types of extensions of nonexpansive mappings. Among these generalizations, Suzuki [20] suggested an important extension of nonexpansive mappings.

Definition 1. *A mapping $T : M \to M$ is said to be Suzuki nonexpansive if*

$$\frac{1}{2}||u - Tu|| \leq ||u - v|| \Rightarrow ||Tu - Tv|| \leq ||u - v||, \tag{2}$$

for all two points u, v in M.

It easy that each nonexpansive selfmap T satisfies (2). Hence we deduced that the class of nonexpansive mappings is properly contained in the class of mappings due to Suzuki [20], however, the converse is precisely not valid in general (see an example below).

Example 2. *[21] Let $M = [0,1]$ and set Tu by*

$$Tu = \begin{cases} 1 - u & \text{for } u < \frac{1}{5}, \\ \frac{4+u}{5} & \text{otherwise.} \end{cases}$$

Since nonexpansive maps are continuous, so here we must have T is not nonexpansive. However T is Suzuki nonexpansive.

Suzuki [20] improved and extended Thoerem 2 to the setting of mappings satisfying (2) in Banach space setting. The Hilbert space version of the Suzuki result [20] can be stated as follows.

Theorem 2. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is Suzuki nonexpansive, then set $F(T)$ is nonempty closed and convex.*

Very recently, Berinde [22] suggested the concept of enriched nonexpansive mappings.

Definition 2. *A mapping $T : M \to M$ is said to be enriched nonexpansive if there is a $b \in [0, \infty)$ such that*

$$||b(u - v) + Tu - Tv|| \leq (b+1)||u - v||, \tag{3}$$

for all two points u, v in M.

Obviously, any nonexpansive mapping satisifies (3) with $b = 0$. Berinde [22] used the Krasnoselskii iteration process [7] for establishing convergence (weak and strong) and existence of fixed point for these mappings in a Hilbert space setting. He also noted that enriched nonexpansive mappings are essentially continuous.

The main result of the Berinde [22] is stated as follows. This theorem extended Theorem 2 from nonexpansive mappings to the enriched nonexpansive mappings.

Theorem 3. *[22] Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is demi-compact and enriched nonexpansive. Then $F(T)$ is nonempty closed and convex. Additionally, one can choose a $\lambda \in (0,1)$ such that for $x_0 \in M$, the sequence of Krasnoselskii iterates given by*

$$x_{k+1} = (1-\lambda)x_k + \lambda T x_k,\ (k = 0,1,2,\ldots), \tag{4}$$

converges to an element of $F(T)$ in the strong sense.

Remark 1. *Theorem 3 is the remarkable extension of the Theorem 2. Because each nonexpansive map is enriched nonexpansive with $b = 0$.*

Now, we consider the following interesting problem.

Problem 1. *Is there exist a class of mappings which includes all the Suzuki nonexpansive and enriched nonexpansive mappings?*

To answer the Problem 1 in the affirmative, we introduce the concept of ESN mappings and show that these mappings are essentially more general than the concept of Suzuki nonexpansive and enriched nonexpansive mappings. We improve and extend several theorems including Theorem 3.

In many cases, a given problem can not solved by any analytical method. In such situations, one is interested to obtain an approximate solution. Although, BFPT [8] gives the guarantee for the convergence of the Picard iterates [6] in the case of contractions but we have noted in the Example 1 that in the case of nonexpansive operators, Picard iterates may fails to converge. Thus, in this paper, we shall use Krasnoselskii iteration [7] instead of Picard iteration [6], to study the existence of fixed point, fixed point set, weak and strong convergence theorems in a Hilbert space setting.

2. Preliminaries

In this section, we present some already known definitions and results, which will be used in establishing the main outcome of the paper.

Definition 3. *[23] Consider a Hilbert space X, $\emptyset \neq M \subseteq X$ and T a selfmap of M. Then T is known as demi-compact if and only if for any $\{x_k\} \subseteq X$ if bounded and $\{Tx_k - x_k\}$, $(k = 0,1,2,\ldots)$, converges in the strong sense, one can find a strongly convergent subsequence $\{x_{k_i}\}$ of $\{x_k\}$.*

Definition 4. *[24] Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and T a selfmap of T. Then T is known as asymptotically regular if and only if for every choice of $u \in M$, one has $\lim_{k\to\infty} ||T^{k+1}u - T^k u|| = 0$, $(k = 0,1,2,\ldots)$.*

The following facts are in [20].

Lemma 1. *Consider a Hilbert space X and $\emptyset \neq M \subseteq X$, and $T : M \to M$ a Suzuki nonexpansive operator. Then, for every two elements $u, v \in M$, it follows that:*

$$||u - Tv|| \leq 3||u - Tu|| + ||u - v||.$$

3. Enriched Suzuki Nonexpansive Mappings

Now, we introduce the notion of ESN mappings as follows. A mapping $T : M \to M$ is said to be ESN if there is a $b \in [0,\infty)$ such that:

$$\frac{1}{2}||u - Tu|| \leq (b+1)||u - v|| \Rightarrow ||b(u - v) + Tu - Tv|| \leq (b+1)||u - v||, \tag{5}$$

for each $u, v \in M$.

Remark 2. *We may note that every Suzuki nonexpansive mapping satisfies (5). We also note that every enriched nonexpansive mapping satisifes (5). The converse is not valid in general as shown by examples in the last section.*

First, we establish an important result.

Lemma 2. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and U a selfmap of U. If U is Suzuki nonexpansive with $F(U) \neq \emptyset$, then for every choice of $\lambda \in (0,1)$, the selfmap $U_\lambda = \lambda I + (1-\lambda)U$ is essentially asymptotically regular with $F(U_\lambda) = F(U)$.*

Proof. Select any element $x \in M$ and define $x_k = U_\lambda^k x$, $(k = 0, 1, 2, \ldots)$. Then for $z \in F(T)$, and hence it is also the fixed point for U_λ. Now,

$$x_{k+1} - z = \lambda x_k + (1-\lambda)Ux_k - z = \lambda(x_k - z) + (1-\lambda)(Ux_k - z).$$

Additionally, for any choice of a constant a, we have:

$$a(x_k - Ux_k) = a(x_k - z) - a(Ux_k - z).$$

Now, $\frac{1}{2}||z - Uz|| = 0 \leq ||x_k - z||$, and so $||Ux_k - Uz|| \leq ||x_k - z||$ because U is Suzuki nonexpansive. Hence,

$$\begin{aligned} ||x_{k+1} - z||^2 &= \lambda^2 ||x_k - z||^2 + (1-\lambda)^2 ||Ux_k - z||^2 + 2\lambda(1-\lambda) \\ &\quad < Ux_k - z, x_k - z > \\ &\leq \lambda^2 ||x_k - z||^2 + (1-\lambda)^2 ||x_k - z||^2 + 2\lambda(1-\lambda) \\ &\quad < Ux_k - z, x_k - z > \\ &= (\lambda^2 + (1-\lambda)^2)||Ux_k - z||^2 + 2\lambda(1-\lambda) < Ux_k - z, x_k - z > . \end{aligned}$$

Moreover,

$$\begin{aligned} a^2 ||x_k - Ux_k||^2 &= a^2 ||x_k - z||^2 + a^2 ||Ux_k - z||^2 - 2a^2 < Ux_k, x_k - z > \\ &\leq a^2 ||x_k - z||^2 + a^2 ||x_k - z||^2 - 2a^2 < Ux_k, x_k - z > \\ &= 2a^2 ||x_k - z||^2 - 2a^2 < Ux_k, x_k - z > . \end{aligned}$$

By adding the above, we obtain:

$$\begin{aligned} ||x_{k+1} - z||^2 + a^2 ||x_k - Ux_k||^2 &\leq (2a^2 + \lambda^2 + (1-\lambda)^2)||x_k - z||^2 + 2(\lambda(1-\lambda) \\ &\quad - a^2) < Ux_k - z, x_k - z > . \end{aligned}$$

If one supposes that: $a^2 = \lambda(1-\lambda)$, then it is obvious that $a^2 > 0$. Set $\sum_{k=0}^{k=K}$, we have:

$$\begin{aligned} \lambda(1-\lambda) \sum_{k=0}^{k=K} ||x_k - Ux_k||^2 &\leq \sum_{k=0}^{k=K} (||x_k - z||^2 - ||x_{k+1} - z||^2) \\ &= ||x_0 - z||^2 - ||x_{K+1} - z||^2) \\ &\leq ||x_0 - z||^2. \end{aligned}$$

It follows that:

$$\sum_{k=0}^{\infty} ||x_k - Ux_k||^2 < \infty.$$

Now $x_{k+1} - x_k = (1-\lambda)(Ux_k - x_k)$, and we have:

$$\sum_{k=0}^{\infty} ||x_k - x_{k+1}||^2 \leq \frac{(1-\lambda)||x_0 - z||^2}{\lambda}.$$

Accordingly, $\sum_{k=0}^{\infty} ||x_k - x_{k+1}||^2 < \infty$ and so $\lim_{k\to\infty} ||U_\lambda^{k+1}x - U_\lambda^k x|| = 0$. Hence U_λ is asymptotically regular. $\square$

Theorem 4. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is demi-compact and ESN. Then $F(T)$ is nonempty closed and convex. Additionally, one can choose a $\lambda \in (0,1)$ such that for $x_0 \in M$, the sequence of Krasnoselskii iterates given by:*

$$x_{k+1} = (1-\lambda)x_k + \lambda T x_k, \ (k = 0,1,2,\ldots), \tag{6}$$

converges to an element of $F(T)$ in the strong sense.

Proof. First we want to show that the averaged operator $T_\mu x = (1-\mu)x + \mu T x$ is Suzuki nonexpansive mapping. Since T is ESN, so one has a constant $b \in [0,\infty)$, such that

$$\frac{1}{2}||u - Tu|| \leq (b+1)||u - v|| \Rightarrow ||b(u-v) + Tu - Tv|| \leq (b+1)||u - v||, \text{for all } u, v \in M.$$

Now we may put $b = \frac{1-\mu}{\mu} = \frac{1}{\mu} - 1$. Then $b + 1 = \frac{1}{\mu}$. It is easy to see that $\mu \in (0,1]$. The above condition becomes:

$$\frac{1}{2}||u - Tu|| \leq \frac{1}{\mu}||u - v|| \Rightarrow ||(\frac{1-\mu}{\mu})(u-v) + Tu - Tv|| \leq \frac{1}{\mu}||u - v||.$$

It follows that:

$$\frac{1}{2}||\mu u - \mu Tu|| \leq ||u - v|| \Rightarrow ||(1-\mu)(u-v) + \mu Tu - \mu Tv|| \leq ||u - v||.$$

Or

$$\frac{1}{2}||u - [(1-\mu)u + \mu Tu]|| \leq ||u - v|| \Rightarrow ||[(1-\mu)u + \mu Tu] - [(1-\mu)v + \mu Tv]|| \leq ||u - v||.$$

Since $(1-\mu)u + \mu Tu = T_\mu u$. We have:

$$\frac{1}{2}||u - T_\mu u|| \leq ||u - v|| \Rightarrow ||T_\mu u - T_\mu v|| \leq ||u - v||, \text{for all } u, v \in M.$$

Thus, we have observed that the averaged operator T_μ form a Suzuki nonexpansive operator. Hence according to the Theorem 2, we have $F(T_\mu)$ is nonempty closed and convex. However, by Lemma 2, $F(T) = F(T_\mu)$, we have proved the first part of the theorem.

Now we want to establish the final part of the theorem. Since $\{x_k\}$ is generated by:

$$x_{k+1} = (1-\lambda)x_k + \lambda T x_k$$

Since M is convex, we may conclude that $\{x_k\}$ contained in the set M and also bounded. Set

$$U_\lambda = (1-\lambda)I + \lambda T_\mu,$$

here I stands for the identity selfmap. Now we have established already that the mapping T_μ is Suzuki nonexpansive. By Lemma 2, we have U_λ is asymptotically regular, that is,

$$\lim_{k\to\infty} ||x_k - U_\lambda x_k|| = 0.$$

Now,

$$U_\lambda u - u = \lambda(T_\mu u - u) = \lambda\mu(Tu - u). \tag{7}$$

Thus,

$$\lim_{k\to\infty} ||x_k - T_\mu x_k|| = 0.$$

Since T is demi-compact, according to (7), we have T_μ is demi-compact too. Thus, we may choose a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $\lim_{i\to\infty} x_{k_i} = z$ for some $z \in M$. Since T_μ is Suzuki nonexpansive mapping, we have from Lemma 1 that

$$||x_{k_i} - T_\mu z|| \le 3||x_{k_i} - T_\mu x_{k_i}|| + ||x_{k_i} - z||.$$

Applying limit, we obtain $\lim_{i\to\infty} x_{k_i} = T_\mu z$. Consequently, $z = T_\mu z$. This shows that $z \in F(T_\mu) = F(T)$.

Since U_λ is a Suzuki nonexpansive mapping (because T_μ is Suzuki nonexpansive) and so $\frac{1}{2}||z - U_\lambda z|| = 0 \le ||x_k - z||$, we have $||U_\lambda x_k - U_\lambda z|| \le ||x_k - z||$. Hence,

$$\begin{aligned} ||x_{k+1} - z|| &= ||(1-\lambda)x_k + \lambda U_\lambda x_k - z|| \\ &\le (1-\lambda)||x_k - z|| + \lambda||U_\lambda x_k - z|| \\ &\le (1-\lambda)||x_k - z|| + \lambda||x_k - z|| \\ &= ||x_k - z||. \end{aligned}$$

The strong convergence of the whole sequence to this z now clearly follows from the facts that $||x_{k+1} - z|| \le ||x_k - z||$. Thus, for starting x_0 in M, the Krasnoselskii scheme:

$$\begin{aligned} x_{k+1} &= U_\lambda x_k \\ &= (1-\lambda)x_k + \lambda T_\mu x_k \\ &= (1-\lambda)x_k + \lambda[(1-\mu)x_k + \mu T x_k] \\ &= (1-\lambda\mu)x_k + \lambda\mu T x_k \end{aligned}$$

converges strongly to the fixed point z of T, for denoting $\lambda = \lambda\mu$ to get the exact Formula (6). □

Corollary 1. *Consider a Hilbert space X and assume that $\emptyset \ne M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is demi-compact and Suzuki nonexpansive. Then $F(T)$ is nonempty closed and convex. Then $F(T)$ is nonempty closed and convex. Additionally, one can choose a $\lambda \in (0,1)$ such that for $x_0 \in M$, the sequence of Krasnoselskii iterates given by*

$$x_{k+1} = (1-\lambda)x_k + \lambda T x_k, \ (k = 0, 1, 2, \dots),$$

converges to an element of $F(T)$ in the strong sense.

Proof. Since a Suzuki nonexpansive mappings is essentially ESN with the constant $b = 0$. Thus, Corollary 1 now follows directly form the Theorem 4 by choosing $b = 0$, that is, for $\mu = 1$. □

Corollary 2. *Consider a Hilbert space X and assume that $\emptyset \ne M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is demi-compact and enriched nonexpansive. Then $F(T)$ is nonempty closed and convex. Then $F(T)$ is nonempty closed and convex. Additionally, one can choose a $\lambda \in (0,1)$ such that for $x_0 \in M$, the sequence of Krasnoselskii iterates given by*

$$x_{k+1} = (1-\lambda)x_k + \lambda T x_k, \ (k = 0, 1, 2, \dots),$$

converges to an element of $F(T)$ in the strong sense.

Proof. Since the condition that a mapping should be ESN is weaker than the condition that a mapping should be enriched nonexpansive. Thus, Corollary 2 is a consequence of the Theorem 4. □

Remark 3. *It is to be noted that the Theorem 4 extends and improves ([22], Theorem 2.2) form the case of enriched nonexpansive to the case of ESN maps and ([23], Lemma 3) (see also ([24], Theorem 6) form the case of nonexpansive maps to the case of ESN maps.*

4. Weak Convergence

This section is devoted to the some weak convergence theorems.

Theorem 5. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is ESN with $\{z\} = F(T)$. Then, for each starting $x_0 \in M$ and $\lambda \in (0,1)$ the Krasnoselskij iteration $\{x_k\}$ provided by*

$$x_{k+1} = (1-\lambda)x_k + \lambda T x_k,\ (k = 0,1,2,\ldots), \tag{8}$$

converges to an element of $F(T)$ in the weak sense.

Proof. According to the arguments provided in the Theorem 4, $T_\mu = (1-\lambda)x_k + \lambda T x_k$ is Suzuki nonexpansive. By Lemma 2, $F(T_\mu) = F(T) = \{z\}$.

Now to obtain the required result, we show that if $\{x_{k_j}\}$ is generated by:

$$x_{k_j+1} = (1-\lambda)x_{k_j} + \lambda T x_{k_j},$$

is weakly convergent to a certain q, we must have in this case that q is fixed point for the operator T_μ (also of $U_\lambda = (1-\lambda)I + \lambda T_\mu$ and similarly for T) and so $q = z$.

We assume that $\{x_{k_j}\}$ is not weakly convergent to z. Now as in Theorem 4, the operator U_λ is Suzuki nonexpansive and so one has it is asymptotically regular, as follows:

$$\lim_{j\to\infty} ||x_{k_j} - U_\lambda x_{k_j}|| = 0.$$

Now according to Lemma 1, we have:

$$||x_{k_j} - U_\lambda q|| \leq 3||x_{k_j} - U_\lambda x_{k_j}|| + ||x_{k_j} - q||.$$

It follows that:

$$\limsup_{j\to\infty}(||x_{k_j} - U_\lambda q|| + ||x_{k_j} - q||) \leq 0. \tag{9}$$

Now,

$$\begin{aligned} ||x_{k_j} - U_\lambda q||^2 &= ||(x_{k_j} - q) + (q - U_\lambda q||^2 \\ &\leq ||x_{k_j} - q||^2 + ||q - U_\lambda q||^2 + 2 < x_{k_j} - q, q - U_\lambda q >, \end{aligned}$$

Now $\{x_{k_j}\}$ is weakly convergent q, one has from the above:

$$\lim_{j\to\infty}(||x_{k_j} - U_\lambda q||^2 - ||x_{k_j} - q||^2) = ||q - x_{k_j}||^2. \tag{10}$$

Moreover,

$$||x_{k_j} - U_\lambda q||^2 - ||x_{k_j} - q||^2 = (||x_{k_j} - U_\lambda q|| - ||x_{k_j} - q||)(||x_{k_j} - U_\lambda q|| + ||x_{k_j} - q||). \tag{11}$$

Since M is bounded, the sequence $\{||x_{k_j} - U_\lambda q|| + ||x_{k_j} - q||\}$ is bounded, too, and therefore by combining (9), (10) and (11), we get:

$$||q - U_\lambda q|| = 0.$$

This shows that $q \in F(U_\lambda) = F(T) = \{z\}$. □

Corollary 3. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is Suzuki nonexpansive with $\{z\} = F(T)$. Then, for each starting $x_0 \in M$ and $\lambda \in (0,1)$ the Krasnoselskij iteration $\{x_k\}$ provided by*

$$x_{k+1} = (1-\lambda)x_k + \lambda T x_k, \ (k = 0, 1, 2, \ldots),$$

converges to an element of $F(T)$ in the weak sense.

Proof. Since a Suzuki nonexpansive mappings is essentially ESN with the constant $b = 0$. Thus, Corollary 3 now follows directly form the Theorem 5 by choosing $b = 0$, that is, for $\mu = 1$. □

Corollary 4. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is enriched nonexpansive with $\{z\} = F(T)$. Then, for each starting $x_0 \in M$ and $\lambda \in (0,1)$ the Krasnoselskii iteration $\{x_k\}$ provided by*

$$x_{k+1} = (1-\lambda)x_k + \lambda T x_k, \ (k = 0, 1, 2, \ldots),$$

converges to an element of $F(T)$ in the weak sense.

Proof. Since the condition that a mapping should be ESN is weaker than the condition that a mapping should be enriched nonexpansive. Thus, Corollary 4 is a consequence of the Theorem 5. □

Remark 4. *It is to be noted that the Theorem 5 extends and improves ([22], Theorem 3.3) form the case of enriched nonexpansivene maps to the case of ESN maps and ([24], Theorem 7) (see also ([25], Theorem 3.3) form the case of nonexpansive to the case of ESN maps.*

Now we dropt the strong assumption $F(T) = \{z\}$ and show another weak convergence theorem as follows.

Theorem 6. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is ESN. Then, for each starting $x_0 \in M$ and $\lambda \in (0,1)$ the Krasnoselskii iteration $\{x_k\}$ provided by (8) converges to an element of $F(T)$ in the weak sense.*

Proof. According to the arguments we have noted in the proof of Theorem 4, one has $F(T) = F(T_\mu) \neq \emptyset$, where $T_\mu x = (1-\mu)x + \mu T x$ as usual. According to Theorem 4, $F(T_\mu)$ is nonempty and convex. Since T_μ is Suzuki nonexpansive, thus for every choice of $z \in F(T_\mu)$, $\frac{1}{2}||z - T_\mu z|| = 0 \leq ||x_k - z||$, we have $||T_\mu x_k - T_\mu z|| \leq ||x_k - z||$. Accordingly

$$\begin{aligned} ||x_{k+1} - p|| &= ||(1-\lambda)x_k + \lambda T_\mu x_k - z|| \\ &\leq (1-\lambda)||x_k - z|| + \lambda||T_\mu x_k - z|| \\ &\leq (1-\lambda)||x_k - z|| + \lambda||x_k - z|| \\ &= ||x_k - z||. \end{aligned}$$

Consequently, we showd $||x_{k+1} - z|| \leq ||x_k - z||$. It follows that the map

$$h(z) = \lim_{k\to\infty} ||x_k - z||, \text{for each } z \in F(T_\mu),$$

is lower semi-continuous convex and well defined on the set $F(T_\mu)$. The remaining proof now closely follows the proof of ([24], Theorem 8). □

Corollary 5. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is Suzuki nonexpansive. Then, for each starting $x_0 \in M$ and $\lambda \in (0,1)$ the Krasnoselskii iteration $\{x_k\}$ provided by (8) converges to an element of $F(T)$ in the weak sense.*

Proof. Since a Suzuki nonexpansive mappings is essentially ESN with the constant $b = 0$. Thus, Corollary 5 now follows directly form the Theorem 6 by choosing $b = 0$, that is, for $\mu = 1$. □

Corollary 6. *Consider a Hilbert space X and assume that $\emptyset \neq M \subseteq X$ is convex closed and bounded and $T : M \to M$. If T is enriched nonexpansive. Then, for each starting $x_0 \in M$ and $\lambda \in (0,1)$ the Krasnoselskii iteration $\{x_k\}$ provided by (8) converges to an element of $F(T)$ in the weak sense.*

Proof. Since the condition that a mapping should be ESN is weaker than the condition that a mapping should be enriched nonexpansive. Thus, Corollary 6 is a consequence of the Theorem 6. □

Remark 5. *Noticed that Theorem 6 extends and improves ([22], Theorem 3.4) form the case of enriched nonexpansive to the case of ESN maps.*

5. Examples

Now we show by examples that the class of ESN mappings properly includes the class of Suzuki nonexpansive and the class of enriched nonexpansive maps.

Example 3. *Set a selfmap T of a bounded closed convex subset $M = [0.5, 2]$ by $Tu = u^{-1}$ for each $u \in M$. We show that T is ESN and not Suzuki nonexpansive.*

Proof. Then for $u = 1$ and $v = 0.5$, we have

$$\frac{1}{2}||u - Tu|| = \frac{1}{2}||1 - 1^{-1}|| \leq ||u - v||,$$

and

$$||Tu - Tv|| = ||u^{-1} - v^{-1}|| = ||0.5^{-1} - 1^{-1}|| = 1 > ||u - v||.$$

On the other hand, T is ESN. Choose $b = 1.5$, then for any $u, v \in M$ with $\frac{1}{2}||u - Tu|| \leq (b+1)||u - v||$ or $\frac{1}{2}||v - Tv|| \leq (b+1)||u - v||$, we have:

$$||b(u - v) + Tu - Tv|| \leq (b+1)||x - y||.$$

It follows that:

$$||u - v|| \times ||b - (uv)^{-1}|| \leq (b+1)||u - v||.$$

Consequently:

$$||b - (uv)^{-1}|| \leq (b+1).$$

This equation now holds for $b = 1.5$. Therefore T is 1.5–ESN. By our main results, for any $x_0 \in M$, Krasnosleskii iterates converge to the fixed point 1 of T. Note that if $u_0 = u \neq 1$, we obtain the following non convergent sequence:

$$u, u^{-1}, u, u^{-1}, \ldots$$

This shows that Picard iterates fails to converge in the case of ESN mappings in general. □

We finish the paper with the following example.

Example 4. *Let $M = [0, 3]$ and set T by*

$$Tu = \begin{cases} 0 & \text{if } \ u \neq 3 \\ 1 & \text{if } \ u = 3. \end{cases}$$

Since enriched nonexpansive maps are continuous (see [22]) , so we must have T is not enriched nonexpansive. However T is ESN mappings. This example also shows that that ESN operators are not necessary to be continuous on their domains.

6. Application to Split Feasibility Problems

We know that the SFP [3] (for shot, SFP) is stated in the following way:

$$\text{Search } z^* \in C : \mathcal{G}z^* \in Q, \tag{12}$$

the alphabats C and Q, respectively, stand for the closed convex subsets of any given real Hilbert spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ while the map $\mathcal{G} : \mathcal{X}_1 \to \mathcal{X}_2$ is any linear and bounded function. It is known from [5], that almost many of the problems of applied sciences can be solve by using the concept and techniques of SFPs.

In this research, we shall essentially assume that the SFP (12) admits a solution and thus the solution set shall be denoted by $\mathcal{S}$. By [5], it is known that any $z^* \in C$ is a solution for (12) if and only if it is a solution for the following equation

$$u = P_C(I_{id} - \xi\mathcal{G}^*(I_{id} - P_Q)\mathcal{G})u,$$

where the notions P_C and P_Q are used for the nearest point projections onto the sets C and Q, respectively. While $\xi > 0$ and the notion $\mathcal{G}^*$ is used for the adjoint operator of the corresponding operator $\mathcal{G}$. In [4], Byrne was the first who noted that if η denotes the spectral radius of $\mathcal{G}^*\mathcal{G}$ and $0 < \xi < \frac{2}{\eta}$, then the operator

$$T = P_C(I_{id} - \xi\mathcal{G}^*(I_{id} - P_Q)\mathcal{G}),$$

is essentially nonexpansive and the following CQ iterative scheme

$$x_{k+1} = P_C(I_{id} - \xi\mathcal{G}^*(I_{id} - P_Q)\mathcal{G})x_k, \ (k = 0, 1, 2, \ldots),$$

always converges weakly to a point of $\mathcal{S}$.

Once a weak convergence is established it is desirable to check the result for the case of strong convergence. To achieve the objective, one needs some more conditions (see, e.g., [5] and others) to study a recent survey on the Halpern type algorithms.

Here, we use a new approach to solve SFPs using the concept of ESN operators because these operators are generally discontinuous on the subsets they are defind (as we have shown by a numerical example in the paper), instead of nonexpansive operators, which are essentially continuous (uniformly) on the subsets they are defind. We show that the suggested scheme converges to the solution of the SFP.

Theorem 7. *Suppose SFP (12) is such that $\mathcal{S} \neq \emptyset$, $0 < \xi < \frac{2}{\eta}$ and $P_C(I_{id} - \xi\mathcal{G}^*(I_{id} - P_Q)\mathcal{G})$ is ESN operator. Consequently, for ome $\lambda \in (0, 1)$, the sequence $\{u_k\}$ produced by*

$$\begin{cases} x_0 \in C, \\ x_{k+1} = (1 - \lambda)x_k + \lambda P_C(I_{id} - \xi\mathcal{G}^*(I_{id} - P_Q)\mathcal{G})x_k, \ (k = 0, 1, 2, \ldots), \end{cases}$$

always converges in the strong sense to some solution, namely, z^* of the SFP given by (12).

Proof. We can set $T = P_C(I_{id} - \xi\mathcal{G}^*(I_{id} - P_Q)\mathcal{G})$, that is, the operator T is ESN. Hence applying Theorem 4, we get $\{x_k\}$ converges in the strong sense in the set $F(T)$. Since $F(T) = \mathcal{S}$, it follows that $\{u_k\}$ converges strongly to some solution, namely, z^* of the SFP given by (12). □

7. Conclusions and Future Plan

In this article, the following new outcome is obtained.

(i) We provided the concept of ESN mappings and proved that these mappings are more general than the concept of nonexpansive, enriched nonexpansive and Suzuki nonexpansive operators.
(ii) We provided the existence and approximation results for these mappings and improved some known results of the literature due to Berinde [22], Browder and Petryshyn [24] and Petryshyn [23] in this new setting of mappings.
(iii) One of the main results of the article is used to approximate the solutions for SFPs on the framework of Hilbert spaces.
(iv) In the next research papers, we will try to use the main results of this paper to solve some more problems of the literature.

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