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Wiman's Type Inequality in Multiple-Circular Domain

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Abstract: In the paper we prove for the first time an analogue of the Wiman inequality in the class of analytic functions $f \in \mathcal{A}^p_0(\mathbb{G})$ in an arbitrary complete Reinhard domain $\mathbb{G} \subset \mathbb{C}^p$, $p \in \mathbb{N}$ represented by the power series of the form $f(z) = f(z_1, \cdots, z_p) = \sum_{\|n\|=0}^{+\infty} a_n z^n$ with the domain of convergence $\mathbb{G}$. We have proven the following statement: If $f \in \mathcal{A}^p(\mathbb{G})$ and $h \in \mathcal{H}^p$, then for a given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p) \in \mathbb{R}^p_+$ and arbitrary $\delta > 0$ there exists a set $E \subset |G|$ such that $\int_{E\cap\Delta_\varepsilon} \frac{h(r)dr_1\cdots dr_p}{r_1\cdots r_p} < +\infty$ and for all $r \in \Delta_\varepsilon \setminus E$ we have $M_f(r) \le \mu_f(r)(h(r))^{\frac{p+1}{2}} \ln^{\frac{p}{2}+\delta} h(r) \ln^{\frac{p}{2}+\delta}\{\mu_f(r)h(r)\} \prod_{j=1}^p \left(\ln \frac{er_j}{\varepsilon_j}\right)^{\frac{p-1}{2}+\delta}$. Note, that this assertion at $p = 1$, $\mathbb{G} = \mathbb{C}$, $h(r) \equiv \text{const}$ implies the classical Wiman–Valiron theorem for entire functions and at $p = 1$, the $\mathbb{G} = \mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$, $h(r) \equiv 1/(1-r)$ theorem about the Kővari-type inequality for analytic functions in the unit disc $\mathbb{D}$; $p > 1$ implies some Wiman's type inequalities for analytic functions of several variables in $\mathbb{C}^n \times \mathbb{D}^k$, $n, k \in \mathbb{Z}_+, n + k \in \mathbb{N}$.

Keywords: maximum modulus; maximal term; double power series; Wiman's type inequality

MSC: 32A10; 32A17; 32A37; 30H99; 30A05

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1. Introduction: Notations and Preliminaries

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ be sets of complex numbers, real numbers, integers, and positive integers, respectively, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We denote by $\mathcal{A}^p_0(\mathbb{G})$, $p \in \mathbb{N}$, the class of an analytic functions f in a complete Reinhardt domain $\mathbb{G} \subset \mathbb{C}^p$, represented by the power series of the form

$$f(z) = f(z_1, \ldots, z_p) = \sum_{\|n\|=0}^{+\infty} a_n z^n, \tag{1}$$

with the domain of convergence $\mathbb{G}$, where $z^n = z_1^{n_1} \ldots z_p^{n_p}$, $z = (z_1, \ldots, z_p) \in \mathbb{G}$, $n = (n_1, \ldots, n_p) \in \mathbb{Z}^p_+$, $\|n\| = \sum_{j=1}^p n_j$; $\mathcal{E}^p := \mathcal{A}^p_0(\mathbb{C}^p)$ is the class of entire functions of several variables (i.e., analytic functions in $\mathbb{C}^p$); by $\mathcal{E}_R := \mathcal{A}^1_0(\mathbb{D}_R)$ $(0 < R \le +\infty)$ we denote the class of analytic functions of one complex variable in a disk $\mathbb{D}_R = \{z \in \mathbb{C} \colon |z| < R\}$. In particular, $\mathcal{E} := \mathcal{E}_{+\infty} = \mathcal{E}^1$ is the class of entire functions of one complex variable.

For a function $f \in \mathcal{A}^p_0(\mathbb{G})$ of form (1) with domain of convergence $\mathbb{G}$ and $r = (r_1, \cdots, r_p) \in |G| := \{r = (r_1, \ldots, r_p) \colon r_j = |z_j|, z = (z_1, \ldots, z_p) \in \mathbb{G}\}$ we denote

$$\Delta_{r_0} = \{t \in |G| \colon t_j \ge r^0_j,\ j \in \{1, \cdots, p\}\}, \quad \mu_f(r) = \max\{|a_n| r_1^{n_1} \cdots r_p^{n_p} \colon n \in \mathbb{Z}^p_+\},$$

$$M_f(r) = \max\{|f(z)| \colon |z_1| = r_1, \cdots, |z_p| = r_p\}, \quad \mathfrak{M}_f(r) = \sum_{\|n\|=0}^{+\infty} |a_n| r^n.$$

On the one hand, it is well-known that every analytic function f in the complete Reinhardt domain $\mathbb{G}$ with a center at $z = 0$ can be represented in $\mathbb{G}$ by the series of form (1). On the other hand, the domain of convergence of each series of form (1) is the logarithmically-convex complete Reinhardt domain with the center $z = 0$.

We say that a domain $\mathbb{G} \subset \mathbb{C}^p$ is the complete Reinhardt domain if:

(a) $z = (z_1, \ldots, z_p) \in \mathbb{G} \Longrightarrow (\forall R = (R_1, \ldots, R_p) \in [0,1]^p)\colon Rz = (R_1 z_1, \ldots, R_p z_p) \in \mathbb{G}$ (*a complete domain*);

(b) $(z_1, \ldots, z_p) \in \mathbb{G} \Longrightarrow (\forall (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p) : (z_1 e^{i\theta_1}, \ldots, z_p e^{i\theta_p}) \in \mathbb{G}$ (*a multiple-circular domain*).

The Reinhardt domain $\mathbb{G}$ is called logarithmically-convex if the image of the set $G^* = \{z \in \mathbb{G}\colon z_1 \cdot \ldots \cdot z_p \neq 0\}$ under the mapping $Ln\colon z \to Ln(z) = (\ln|z_1|, \ldots, \ln|z_p|)$ is a convex set in the space $\mathbb{R}^p$. In one complex variable ($p = 1$), a logarithmically-convex Reinhardt domain is a disc. The following complete Reinhardt domains ($p \geq 2$) are considered most frequently:

$$C_p(R) := \{z \in \mathbb{C}^p\colon |z_1| < R_1, \ldots, |z_p| < R_p\},\ R = (R_1, \ldots, R_p) \in (0, +\infty)^p,\ \text{(polydisk)},$$
$$\mathbb{B}_p(r) := \{z \in \mathbb{C}^p\colon |z| := \sqrt{|z_1|^2 + \ldots + |z_p|^2} < r\} \quad \text{(ball)},$$
$$\Pi_p(r) := \{z \in \mathbb{C}^p\colon |z_1| + \ldots + |z_p| < r\}, \quad r > 0.$$

Note, that $C_p(R) \subset \mathbb{G}$ for every $w = (w_1, \ldots, w_p) \in \mathbb{G}$ and $R = (|w_1|, \ldots, |w_p|)$. The domains $C_p(re_1)$, $e_1 = (1, \ldots, 1) \in \mathbb{R}^p$, $\mathbb{B}_p(r)$, $\Pi_p(r)$ $(r > 0)$ are the logarithmically-convex complete Reinhardt domains. However, for example, the complete Reinhardt domain

$$G_{1,2} = \{z = (z_1, z_2)\colon |z_1| < 1, |z_2| < 2\} \cup \{z = (z_1, z_2)\colon |z_1| < 2, |z_2| < 1\}$$

is not a logarithmically-convex domain.

2. Wiman's Type Inequality for Analytic Functions of One Variable

In article [1] the following statement is proved.

Theorem 1 ([1])**.** *Let a nondecreasing function* $h\colon [0, R) \to [10, \infty)$ *such that* $\int_{r_0}^R h(r) d\ln r = +\infty$ *for some* $r_0 \in (0, R)$*. If* $f \in \mathcal{E}_R$, $R \in (0, +\infty]$ *is an analytic function represented by a power series of the form* $f(z) = \sum_{n=0}^{+\infty} a_n z^n$*, then* $(\forall\ \delta > 0)$ $(\exists E(\delta, f, h) = E \subset (0, R))$ $(\exists r_0 \in (0, R))$ $(\forall\ r \in (r_0, R) \setminus E)$

$$M_f(r) \leq h(r)\mu_f(r)\{\ln h(r) \ln(h(r)\mu_f(r))\}^{1/2+\delta} \ \textit{and} \ \int_{E \cap (r_0, R)} \frac{h(r)}{r} dr < +\infty,$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$ *is the maximum modulus and* $\mu_f(r) = \max\{|a_n| r^n : n \geq 0\}$ *is the maximal term of power series.*

For nonconstant entire functions $f \in \mathcal{E}$ we can choose $h(r) = 10$ and $\delta = \varepsilon/2$ for an arbitrarily given $\varepsilon > 0$. Then, from Theorem 1 we obtain the assertion of the classical Wiman–Valiron theorem on Wiman's inequality (for example see [2], [3] (p. 9), [4,5], [6] (p. 28), [7–10]), i.e., that for all $r \in (r_0, +\infty) \setminus E$, $\int_E d\ln r < +\infty$, we have

$$M_f(r) \leq 10\mu_f(r)\{\ln 10 \ln(10\mu_f(r))\}^{1/2+\delta} \leq \mu_f(r) \ln^{1/2+\varepsilon} \mu_f(r). \tag{2}$$

For analytic functions $f \in \mathcal{E}_1$ in the unit disk $\mathbb{D}_1$ we can choose $h(r) = \frac{r}{1-r}$. Then,

$$M_f(r) \leq \frac{r\mu_f(r)}{1-r}\Big\{\ln \frac{r}{1-r} \ln\Big(\frac{r\mu_f(r)}{1-r}\Big)\Big\}^{1/2+\delta} \leq$$
$$\leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}, \ \text{as } r \to 1-0,\ r \notin E,\ \int_E \frac{dr}{1-r} < +\infty,$$

i.e., the theorem about the Kővari-type inequality for analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ ([11,12]).

Regarding the statement about the Wiman inequality (2), Prof. I.V. Ostrovskii in 1995 formulated the following problem: What is the best possible description of the value of an exceptional set E? In article [7], the authors found, in a sense, the best possible description of the magnitude of the exceptional set E in inequality (2) for entire functions of one complex variable. In fact, we obtain, in a sense, the best possible description for each entire function f for $h(r) = \ln \mu_f(r)$.

The same issue was considered in a number of articles (for example, see [13–15]) in relation to many other relations obtained in the Wiman–Valiron theory.

Note, that for analytic functions $f \in \mathcal{E}_1$ such a problem is still open. Theorem 1 contains a new description of the exceptional set in the inequality (2) for analytic functions $f \in \mathcal{E}_1$. Perhaps the best possible description of an exceptional set is also obtained with $h(r) = \ln \mu_f(r)$.

3. Wiman's Type Inequality for Analytic Functions of Several Variables

Some analogues of Wiman's inequality for entire functions of several complex variables can be found in [16–22], and for analytic functions in the polydisc $\mathbb{D}^p$, $p \geq 2$, in [23,24].

In paper [25] some analogues of Wiman's inequality are proven for the analytic $f(z)$ and random analytic $f(z,t)$ functions on $\mathbb{G} = \mathbb{D}^\ell \times \mathbb{C}^{p-\ell}$, $\ell \in \mathbb{N}$, $1 \leq \ell < p$, $I = \{1,\dots,\ell\}$, $J = \{\ell+1,\dots,p\}$ of the form (1) and $f(z,t) = \sum_{\|n\|=0}^{+\infty} a_n Z_n(t) z^n$, respectively. Here, $Z = (Z_n)$ is a multiplicative system of complex random variables on the Steinhaus probability space, almost surely (a.s.) uniformly bounded by the number 1. In particular, the following statements are proven:

Theorem 2 ([25])**.** *Let $f \in \mathcal{A}^p(\mathbb{G})$, $\mathbb{G} = \mathbb{D}^\ell \times \mathbb{C}^{p-\ell}$, $\ell \in \mathbb{N}$, $1 \leq \ell < p$. For every $\delta > 0$ there exist the sets $E_1 = E_1(\delta, f)$, $E_2 = E_2(\delta, f) \subset [0,1)^l \times (1,+\infty)^{p-l}$ of asymptotically finite logarithmic measure (i.e., $\int_{\Delta_\varepsilon \cap [0,1)^\ell \times \mathbb{R}^{p-\ell}} \frac{dr_1 \cdot \ldots \cdot dr_\ell \cdot dr_{\ell+1} \cdot \ldots \cdot dr_p}{(1-r_1)\cdot \ldots \cdot (1-r_\ell)\cdot r_{\ell+1}\cdot \ldots \cdot r_p} < +\infty$ for some $\varepsilon > 0$), such that the inequalities*

$$M_f(r) \leq \mu_f(r) \prod_{i\in I} \frac{1}{(1-r_i)^{1+\delta}} \ln^{p/2+\delta}\Big(\mu_f(r) \prod_{i\in I} \frac{1}{1-r_i}\Big) \Big(\prod_{j\in J} \ln r_j\Big)^{p+\delta}, \tag{3}$$

$$M_f(r,t) \leq \mu_f(r) \prod_{i\in I} \frac{1}{(1-r_i)^{1/2+\delta}} \ln^{p/4+\delta}\Big(\mu_f(r) \prod_{i\in I} \frac{1}{1-r_i}\Big) \Big(\prod_{j\in J} \ln r_j\Big)^{p/2+\delta}.$$

hold for all $r \in |G| \setminus E_1$ and for all $r \in |D| \setminus E_2$ a.s. in t, respectively.

The sharpness of the obtained inequalities is also proven.

The main purpose of this article is to prove analogues of Theorems 1 and 2 in the class of analytic functions $f \in \mathcal{A}^p_0(\mathbb{G})$ for the arbitrary complete Reinhardt domain $\mathbb{G}$.

4. Main Result

The aim of this paper is to prove some analogues of Wiman's inequality for the analytic functions $f \in \mathcal{A}^p_0(\mathbb{G})$ represented by the series of form (1) with the arbitrary complete Reinhardt domain of convergence $\mathbb{G}$. By $\mathcal{A}^p(\mathbb{G})$ we denote a subclass of functions $f \in \mathcal{A}^p_0(\mathbb{G})$ such that $\frac{\partial}{\partial z_j} f(z_1,\cdots,z_p) \not\equiv 0$ in $\mathbb{G}$ for any $j \in \{1,\dots,p\}$.

Let $\mathcal{H}^p$ be the class of functions $h\colon |G| \to \mathbb{R}_+$ such that h is nondecreasing with respect to each variable and $h(r) > 10$ for all $r \in |G|$ and

$$\int\limits_{\Delta_\varepsilon} \frac{h(r) dr_1 \cdots dr_p}{r_1 \cdots r_p} = +\infty$$

for every $\varepsilon \in \mathbb{R}^p_+$ such that $|G| \cap \Delta_\varepsilon$ is a nonempty domain in $\mathbb{R}^p_+$.

For $h \in \mathcal{H}^p$ we say that $E \subset |G|$ is the set of *finite h-measure on* $|G|$ if for some $\varepsilon \in \mathbb{R}^p_+$ such that $|G| \cap \Delta_\varepsilon$ is a nonempty domain in $|G| \subset \mathbb{R}^p_+$ one has

$$\nu_h(E \cap \Delta_\varepsilon) := \int\limits_{E\cap\Delta_\varepsilon} \frac{h(r)dr_1 \cdots dr_p}{r_1 \cdots r_p} < +\infty.$$

We denote a set of such sets by $\mathcal{S}_h$.

Theorem 3. *Let $f \in \mathcal{A}^p(\mathbb{G})$. Then, for every $\varepsilon \in \mathbb{R}^p_+, \delta > 0$ there exists a set $E \in \mathcal{S}_h$ such that for all $r \in \Delta_\varepsilon \setminus E$ the following inequality takes place:*

$$M_f(r) \le \mu_f(r)(h(r))^{\frac{p+1}{2}} \ln^{\frac{p}{2}+\delta} h(r) \ln^{\frac{p}{2}+\delta}\{\mu_f(r)h(r)\} \prod_{j=1}^{p}\left(\prod_{k=1,k\neq j}^{p} \ln\frac{er_k}{\varepsilon_k}\right)^{\frac{1}{2}+\delta}. \tag{4}$$

Remark 1. *Choosing $p = 1$ and $\mathbb{G} = \mathbb{D}_R$ in Theorem 3 leads to the result in Theorem 1.*

5. Auxiliary Lemmas

The proof of the main result uses the probabilistic reasoning from [17,18] (see also [20]), which has already become traditional in this topic, and differs from the proofs of similar statements in [25].

Our proof actually uses a number of lemmas (Lemmas 1–4) from article [18]. But their proofs in article [18] are not written with sufficient completeness, and also contain inaccuracies in reasoning. Therefore, we present them here along with the complete proofs.

In order to prove a Wiman's type inequality for analytic functions in $\mathbb{G}$ we need the following auxiliary results.

Let $D_f(r) = (D_{ij})$ be a $p \times p$ matrix such that

$$D_{ij} = r_i \frac{\partial}{\partial r_i}\left(r_j \frac{\partial}{\partial r_j} \ln M_f(r)\right) = \partial_i \partial_j \ln M_f(r), \quad \partial_i = r_i \frac{\partial}{\partial r_i}, \quad i, j \in \{1, \cdots, p\}.$$

Let I be an identity matrix of order p.

For the set $E \subset \mathbb{R}^p$ by $\#(E \cap \mathbb{Z}^p_+)$ we denote the quantity of the elements of set $E \cap \mathbb{Z}^p_+$.

Lemma 1. *Let B be parallelepiped in $\mathbb{R}^p$ with edges of the lengths $l_1, l_2, \ldots, l_p$ so that there exists an isometry $H\colon \mathbb{R}^p \to \mathbb{R}^p$ such that*

$$H\colon B \to \{x \in \mathbb{R}^p\colon |x_j| \le l_j/2, j \in \{1, 2, \ldots p\}\}.$$

Then,

$$\#(B \cap \mathbb{Z}^p) \le \lambda_p \prod_{j=1}^{p}(l_j + 1),$$

where λ_p is the inverse value to the volume of a sphere with the radius $\frac{1}{2}$ in $\mathbb{R}^p$, i.e., $\lambda_p = \frac{2^n \cdot \Gamma(\frac{n+2}{2})}{\pi^{n/2}}$.

Proof. Denote

$$B' = \left\{x \in \mathbb{R}^p\colon |x_j| \le \frac{l_j}{2}, \, j \in \{1, \ldots, p\}\right\},$$
$$B^* = \left\{x \in \mathbb{R}^p\colon |x_j| \le \frac{l_j + 1}{2}, \, j \in \{1, \ldots, p\}\right\} \supset B.$$

Let $S(n)$ be an open sphere with a center at $n \in \mathbb{Z}_+^p$ with radius $\frac{1}{2}$. Note that

$$\bigcup_{n \in \mathbb{Z}_+^p \cap B} S(n) \subseteq B^*.$$

By the monotony of the Lebesgue measure μ in $\mathbb{R}^p$ we obtain

$$\mu\left(\bigcup_{n \in \mathbb{Z}_+^p \cap B} S(n)\right) \leq \mu(B^*).$$

Finally, by the additivity of this measure, we obtain

$$\mu(S(1/2)) \cdot \#\{B \cap \mathbb{Z}_+^p\} \leq \prod_{j=1}^{p}(l_j+1), \;\; \#\{B \cap \mathbb{Z}_+^p\} \leq \frac{1}{\mu(S(1/2))}\prod_{j=1}^{p}(l_j+1).$$

□

By A^+ we denote the Moore–Penrose inverse matrix of A ([18,26]), i.e.,

$$A^+ = \lim_{\delta \to 0} A^T(AA^T + \delta I)^{-1}.$$

Lemma 2. *Let $\alpha \in \mathbb{R}^p, C > 0$, A be a $p \times p$ nonnegative matrix $0 < m = \operatorname{rank} A \leq p$ and*

$$E = \{x \in \mathbb{R}^m \colon (x-\alpha)A^+(x-\alpha)^T \leq C\}.$$

There exists a constant $\delta = \delta(C, p) > 0$ that does not depend on A and α such that

$$\#\{E \cap \mathbb{Z}_+^m\} \leq \delta(\det(A+I))^{1/2}.$$

Proof. Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ be positive eigenvalues of the matrix A. Then, $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_m}$ are eigenvalues of the matrix A^+. Thus, there exists an isometry $H \colon \mathbb{R}^m \to \mathbb{R}^m$, such that

$$H \colon E \to \left\{x \in \mathbb{R}^m \colon \sum_{j=1}^{m} \frac{x_j^2}{\lambda_j} \leq C\right\}, \;\; E \subset H^{-1}\{x \in \mathbb{R}^m \colon x \leq \sqrt{C\lambda_j}, j \in \{1, 2, \ldots, m\}\}.$$

By Lemma 1 there exists a constant $\delta' > 0$ such that

$$\#\{E \cap \mathbb{Z}_+^m\} \leq \delta' \prod_{j=1}^{m}(2(C\lambda_j)^{1/2}+1) = \delta' \prod_{j=1}^{p}(2(C\lambda_j)^{1/2}+1).$$

It remains to remark that

$$\prod_{j=1}^{p}(\lambda_j+1) = \det(A+I), \;\; \#\{E \cap \mathbb{Z}_+^p\} \leq \delta'(\sqrt{2C})^p \left(\prod_{j=1}^{p}(\lambda_j+1)\right)^{1/2} \leq \delta''(\det(A+I))^{1/2}.$$

□

Lemma 3. *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_p)^T$ be a random vector, $\alpha = M\xi = (M\xi_1, M\xi_2, \ldots, M\xi_p)^T$, A covariance matrix of ξ, $\delta > 0$, $0 < m = \operatorname{rank} A \leq p$. Then,*

$$P\{\omega \colon (\xi(\omega)-\alpha)A^+(\xi(\omega)-\alpha)^T \leq \delta\} \geq 1 - \frac{m}{\delta}.$$

Proof. Let us consider the random variable

$$Z(\omega) = (\xi(\omega) - \alpha)^T A^+ (\xi(\omega) - \alpha).$$

As A is non-negative, then $\forall \omega \in \Omega\colon Z(\omega) \geq 0$. Moreover, as A is also symmetric, there exists an orthogonal matrix G such that $GG^T = G^T G = I$ and $G^T A G = Q$. Here, I is the identity matrix of order p and $Q = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m, 0, \ldots, 0)$ is the diagonal matrix with the ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0$, $0 < m = \operatorname{rank} A \leq p$. Then (see, for example [26,27]),

$$G^T A G = Q,\ \ GG^T AGG^T = GQG^T \implies A = GQG^T,$$
$$A^+ = (GQG^T)^+ = (G^T)^+ Q^+ G^+ = (G^T)^{-1} Q^+ G^{-1} = GQ^+G^T,$$

i.e., $A = GQG^T,\ A^+ = GQ^+G^T$. Therefore,

$$Z = (\xi - \alpha)^T A^+ (\xi - \alpha) = (\xi - \alpha)^T GQ^+G^T (\xi - \alpha) =$$
$$= (\xi - \alpha)^T GQ^{-1/2} Q^{-1/2} G^T (\xi - \alpha) = (Q^{-1/2} G^T (\xi - \alpha))^T (Q^{-1/2} G^T (\xi - \alpha)) = Y^T Y,$$

where $Y = Q^{-1/2} G^T (\xi - \alpha)$, $Q^{-1/2} = \operatorname{diag}(\lambda_1^{-1/2}, \lambda_2^{-1/2}, \ldots, \lambda_m^{-1/2}, 0, \ldots, 0)$. The expected value and covariance of the random vector Y satisfy the equations

$$MY = M(Q^{-1/2} G^T (\xi - \alpha)) = Q^{-1/2} G^T M(\xi - \alpha) = 0,$$
$$\operatorname{cov} Y = \operatorname{cov}(Q^{-1/2} G^T (\xi - \alpha)) = \operatorname{cov}(Q^{-1/2} G^T \xi) = Q^{-1/2} G^T \operatorname{cov}(\xi) (Q^{-1/2} G^T)^T =$$
$$= Q^{-1/2} G^T A G Q^{-1/2} = Q^{-1/2} Q Q^{-1/2} = \operatorname{diag}(\underbrace{1, \ldots, 1}_{m \text{ times}}, \underbrace{0, \ldots, 0}_{p-m \text{ times}}).$$

Therefore,

$$MZ = M(Y^T Y) = M\left(\sum_{j=1}^{p} Y_j^2 \right) = \sum_{j=1}^{p} M(Y_i^2) = \sum_{j=1}^{p} D(Y_j) = m.$$

Finally, using Markov's inequality we obtain

$$P\{\omega\colon Z(\omega) \geq \delta\} = P\{\omega\colon (\xi(\omega) - \alpha) A^+ (\xi(\omega) - \alpha)^T \leq \delta\} \leq \frac{MZ}{\delta} = \frac{m}{\delta}.$$
$$P\{\omega\colon (\xi(\omega) - \alpha) A^+ (\xi(\omega) - \alpha)^T \leq \delta\} \geq 1 - \frac{m}{\delta}.$$

□

Lemma 4 (Theorem 3.1, [18])**.** *Let $f \in \mathcal{A}^p$. There exists a constant $C_0(p)$ such that*

$$\mathfrak{M}_f(r) \leq C_0(p) \mu_f(r) (\det(D_f(r) + I))^{1/2},$$

where I is the identity $p \times p$ matrix.

Proof. Let us consider random vector $X(\omega) = (X_1(\omega), X_2(\omega), \ldots, X_p(\omega))$ such that

$$P\{\omega\colon X_j(\omega) = n_j, j \in \{1, \ldots, p\}\} = \frac{1}{\mathfrak{M}_f(r)} |a_{n_1 \ldots n_p}| r_1^{n_1} \ldots r_p^{n_p},\ k \in \mathbb{Z}_+.$$

Then for $j \in \{1, 2, \ldots, p\}$ we obtain

$$MX_j = \frac{1}{\mathfrak{M}_f(r)} \sum_{\|n\|=0}^{+\infty} n_j |a_n| r^n = r_j \frac{\partial}{\partial r_j} \ln \mathfrak{M}_f(r).$$

$D_f(r)$ is covariance matrix of random vector $X(\omega)$.
One can choose $\delta = 2p$ in Lemma 3. We then obtain

$$\frac{1}{2} \le 1 - \frac{m}{2p} \le P\{\omega\colon (x-\alpha)D_f^+(x-\alpha)^T \le 2p\} \le$$

$$\le \frac{\mu_f(r)}{\mathfrak{M}_f(r)} \cdot \#\{x \in \mathbb{R}_+^p\colon (x-\alpha)D_f^+(x-\alpha)^T \le 2p\} \le 2p\frac{\mu_f(r)}{\mathfrak{M}_f(r)}(\det(D_f(r)+I))^{1/2},$$

$$\mathfrak{M}_f(r) \le 4p\mu_f(r)(\det(D_f(r)+I))^{1/2}.$$

□

Lemma 5. *Let $f \in \mathcal{A}^p$. Then for $\varepsilon \in \mathbb{R}_+^p, \delta > 0$ there exists a set $E \in \mathcal{S}_h$ such that for all $r \in \Delta_\varepsilon \setminus E$ the inequalities*

$$\det(D_f(r)+I) \le$$

$$\le h(r)\prod_{j=1}^{p}\Big(r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r) + \ln\Big(\frac{er_j}{\varepsilon_j}\Big)\Big)\prod_{j=1}^{p}\ln^{1+\delta}\Big(r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r) + \ln\Big(\frac{er_j}{\varepsilon_j}\Big)\Big), \tag{5}$$

$$r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r) \le h(r)\ln^{1+\delta}\mathfrak{M}_f(r)\prod_{k=1,k\ne j}^{p}\ln^{1+\delta}\Big(\frac{er_k}{\varepsilon_k}\Big), \quad j \in \{1,\cdots,p\} \tag{6}$$

hold.

Proof. Let $E_0 \subset |G|$ be a set for which inequality (5) does not hold. Now we prove that $E_0 \in \mathcal{S}_h$. Since $r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r) > 0$, there for any $r \in \Delta_\varepsilon$ we have

$$r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r) + \ln\Big(\frac{er_j}{\varepsilon_j}\Big) > 1,\ j \in \{1,\cdots,p\}.$$

Then,

$$\nu_h(E_0 \cap \Delta_\varepsilon) = \int\limits_{E_0\cap\Delta_\varepsilon}\cdots\int\frac{h(r)dr_1\cdots dr_p}{r_1\cdots r_p} \le$$

$$\le \int\limits_{E_0\cap\Delta_\varepsilon}\cdots\int\frac{\det(D_f(r)+I)dr_1\cdots dr_p}{\prod\limits_{j=1}^{p}r_j\prod\limits_{j=1}^{p}\Big(r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)+\ln r_j\Big)\prod\limits_{j=1}^{p}\ln^{1+\delta}\Big(r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)+\ln r_j\Big)}.$$

Let $U\colon |G| \to \mathbb{R}_+^p$ be a mapping such that $U = (u_1(r), u_2(r), \cdots, u_p(r))$ and $u_j(r) = r_j\frac{\partial}{\partial r_j}\ln M_f(r) + \ln(\frac{er_j}{\varepsilon_j}),\ j \in \{1,\cdots,p\},\ r = (r_1, r_2, \cdots, r_p)$. Then for $i,j \in \{1,2,\cdots,p\}$ we obtain

$$\frac{\partial u_i}{\partial r_i} = \frac{\partial}{\partial r_i}\Big(r_i\frac{\partial}{\partial r_i}\ln\mathfrak{M}_f(r) + \ln\Big(\frac{er_j}{\varepsilon_j}\Big)\Big) = \frac{1}{r_i}\partial_i\partial_i\ln\mathfrak{M}_f(r) + \frac{1}{r\,i},$$

$$\frac{\partial u_i}{\partial r_j} = \frac{\partial}{\partial r_j}\Big(r_i\frac{\partial}{\partial r_i}\ln\mathfrak{M}_f(r) + \ln\Big(\frac{er_j}{\varepsilon_j}\Big)\Big) = \frac{1}{r_j}\partial_i\partial_j\ln\mathfrak{M}_f(r),\ i \ne j.$$

Hence, the Jacobian

$$J_1 := \frac{D(u_1,u_2,\cdots,u_p)}{D(r_1,r_2,\ldots,r_p)} = \begin{vmatrix} \frac{\partial u_1}{\partial r_1} & \cdots & \frac{\partial u_1}{\partial r_p} \\ \cdots & \ddots & \cdots \\ \frac{\partial u_p}{\partial r_1} & \cdots & \frac{\partial u_p}{\partial r_p}\end{vmatrix} = \prod_{j=1}^{p}\frac{1}{r_j}\cdot\det(D_f(r)+I).$$

Therefore,

$$\nu_h(E_0 \cap \Delta_\varepsilon) \le \int\limits_{U(E_0\cap\Delta_\varepsilon)} \cdots \int \frac{du_1 du_2 \cdots du_p}{u_1 \ln^{1+\delta} u_1 \dots u_p \ln^{1+\delta} u_p} \le$$
$$\le \int\limits_1^{+\infty} \cdots \int\limits_1^{+\infty} \frac{du_1 du_2 \cdots du_p}{u_1 \ln^{1+\delta} u_1 \dots u_p \ln^{1+\delta} u_p} < +\infty.$$

Let $E_1 \subset |G|$ be a set for which inequality (6) does not hold for $j = 1$. Now we prove that $E_1 \cap \Delta_\varepsilon \in \mathcal{S}_h$.

Then,

$$\nu_h(E_1 \cap \Delta_\varepsilon) = \int\limits_{E_1\cap\Delta_\varepsilon} \cdots \int \frac{h(r) dr_1 \cdots dr_p}{r_1 \cdots r_p} \le \int\limits_{E_1\cap\Delta_\varepsilon} \cdots \int \frac{r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) dr_1 \cdots dr_p}{\left(\prod\limits_{j=1}^{p} r_j\right) \ln^{1+\delta} \mathfrak{M}_f(r) \prod\limits_{j=2}^{p} \left(\ln^{1+\delta}\left(\frac{er_j}{\varepsilon_j}\right)\right)}.$$

Let $V\colon |G| \to \mathbb{R}^p_+$ be a mapping such that $V = (v_1(r), v_2(r), \cdots, v_p(r))$ and $v_1(r) = \ln \mathfrak{M}_f(r)$, $v_j = \ln(\frac{er_j}{\varepsilon_j})$ $j \in \{2, \cdots, p\}$, $r = (r_1, r_2, \cdots, r_p)$. Therefore, the Jacobian

$$J_2 := \frac{D(v_1, v_2, \cdots, v_p)}{D(r_1, r_2, \dots, r_p)} =$$
$$= \begin{vmatrix} \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) & \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) & \cdots & \frac{\partial}{\partial r_p} \ln \mathfrak{M}_f(r) \\ 0 & \frac{1}{r_2} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \frac{1}{r_p} \end{vmatrix} = \prod_{j=1}^{p} \frac{1}{r_j} \cdot r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r).$$

Therefore,

$$\nu_h(E_1 \cap \Delta_\varepsilon) \le \int\limits_{U(E_0\cap\Delta_\varepsilon)} \cdots \int \frac{du_1 du_2 \cdots du_p}{(u_1 u_2 \dots u_p)^{1+\delta}} \le \int\limits_1^{+\infty} \cdots \int\limits_1^{+\infty} \frac{du_1 du_2 \cdots du_p}{(u_1 u_2 \dots u_p)^{1+\delta}} < +\infty.$$

Let $E_j \subset |G|$ be a set for which inequality (6) does not hold for $j \in \{2, \dots, p\}$. Similarly, $E_j \cap \Delta_\varepsilon \in \mathcal{S}_h$ for $j \in \{2, \dots, p\}$. It remains to remark that the set $E = \bigcup_{j=0}^{p} E_j$ is also a set of finite h-measure in $|G|$. $\square$

6. Proof of the Main Theorem

Proof of Theorem 2. Let E be the exceptional set from Lemma 2. Then, using Lemma 1 we obtain for all $r \in \Delta_\varepsilon \setminus E$

$$M_f(r) \le \mathfrak{M}_f(r) \le C_0\mu_f(r)(\det(D_f(r)+I))^{1/2} \le C_0\mu_f(r)\sqrt{h(r)}\times$$
$$\times\prod_{j=1}^{p}\Big(r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)+\ln\Big(\frac{er_j}{\varepsilon_j}\Big)\Big)^{1/2}\prod_{j=1}^{p}\ln^{(1+\delta)/2}\Big(r_j\frac{\partial}{\partial r_j}\ln\mathfrak{M}_f(r)+\ln\Big(\frac{er_j}{\varepsilon_j}\Big)\Big)\le$$
$$\le C_0\mu_f(r)\sqrt{h(r)}\prod_{j=1}^{p}\Big(h(r)\ln^{1+\delta}\mathfrak{M}_f(r)\prod_{k=1,k\ne j}^{p}\ln^{1+\delta}\Big(\frac{er_k}{\varepsilon_k}\Big)+\ln\Big(\frac{er_j}{\varepsilon_j}\Big)\Big)^{1/2}\times$$
$$\times\prod_{j=1}^{p}\ln^{(1+\delta)/2}\Big(h(r)\ln^{1+\delta}\mathfrak{M}_f(r)\prod_{k=1,k\ne j}^{p}\ln^{1+\delta}\Big(\frac{er_k}{\varepsilon_k}\Big)+\ln\Big(\frac{er_j}{\varepsilon_j}\Big)\Big)\le$$
$$\le\mu_f(r)(h(r))^{\frac{p+1}{2}}\ln^{\frac{p}{2}+p\delta}h(r)\ln^{\frac{p}{2}}\mathfrak{M}_f(r)\ln^{\frac{p}{2}+p\delta}\ln\mathfrak{M}_f(r)\prod_{j=1}^{p}\Bigg(\prod_{k=1,k\ne j}^{p}\ln\frac{er_k}{\varepsilon_k}\Bigg)^{1/2+\delta}\le$$
$$\le\mu_f(r)(h(r))^{\frac{p+1}{2}}\ln^{\frac{p}{2}+p\delta}h(r)\ln^{\frac{p}{2}}\mathfrak{M}_f(r)\ln^{\frac{p}{2}+p\delta}\ln\mathfrak{M}_f(r)\prod_{j=1}^{p}\Big(\ln\frac{er_j}{\varepsilon_j}\Big)^{\frac{p-1}{2}(1+4\delta)},$$
$$\ln\mathfrak{M}_f(r)\le$$
$$\le\ln\mu_f(r)+\Big(\frac{p+1}{2}+\delta\Big)\ln h(r)+\Big(\frac{p}{2}+\delta\Big)\ln\ln\mathfrak{M}_f(r)+\Big(\frac{p-1}{2}(1+4\delta)\Big)\sum_{j=1}^{p}\ln^+\ln\frac{er_j}{\varepsilon_j}.$$

Note that we can chose set E such that $\forall r\in(\Delta_\varepsilon\cap|G|)\setminus E$

$$\mathfrak{M}_f(r)>C_p^*,\ \ \mu_f(r)>1,$$

where C_p^* is some constant such that $C_p^*\ge\ln^{2p}C_p^*$. Then,

$$\mathfrak{M}_f(r)\ge\ln^{2p}\mathfrak{M}_f(r),\ \ \ln\mathfrak{M}_f(r)\ge 2p\ln\ln\mathfrak{M}_f(r),$$
$$\frac{1}{2}\ln\mathfrak{M}_f(r)\le\ln\mathfrak{M}_f(r)-\Big(\frac{p}{2}+\delta\Big)\ln\ln\mathfrak{M}_f(r)\le$$
$$\le\ln\mu_f(r)+\Big(\frac{p+1}{2}+\delta\Big)\ln h(r)+\Big(\frac{p-1}{2}(1+4\delta)\Big)\sum_{j=1}^{p}\ln^+\ln\frac{er_j}{\varepsilon_j},$$
$$\ln\mathfrak{M}_f(r)\le(1+p+2\delta)\ln\Big\{\mu_f(r)h(r)\prod_{j=1}^{p}\ln\frac{er_j}{\varepsilon_j}\Big\}.$$
$$M_f(r)\le\mu_f(r)(h(r))^{\frac{p+1}{2}}\ln^{\frac{p}{2}+\delta}h(r)\ln^{\frac{p}{2}}\Big\{\mu_f(r)h(r)\prod_{j=1}^{p}\ln\frac{er_j}{\varepsilon_j}\Big\}\times$$
$$\times\ln^{1/2+\delta}\ln\Big\{\mu_f(r)h(r)\prod_{j=1}^{p}\ln\frac{er_j}{\varepsilon_j}\Big\}\prod_{j=1}^{p}\Big(\ln\frac{er_j}{\varepsilon_j}\Big)^{\frac{p-1}{2}(1+5\delta)}\le$$
$$\le\mu_f(r)(h(r))^{\frac{p+1}{2}}\ln^{\frac{p}{2}+\delta}h(r)\ln^{\frac{p}{2}+\delta_1}\{\mu_f(r)h(r)\}\prod_{j=1}^{p}\Big(\ln\frac{er_j}{\varepsilon_j}\Big)^{\frac{p-1}{2}+\delta_1},\quad\delta_1=5p\delta.$$

□

7. Corollaries Hypotheses.

Let us consider the case when domain G is bounded. Then there exists $R>0$ such that $G\subset C_p(R):=\{z\in\mathbb{C}^p\colon |z_i|<R,\ i\in\{1,\dots,p\}\}$. Therefore we have for all $r\in\Delta_\varepsilon\setminus E$

$$\prod_{j=1}^{p}\Bigg(\prod_{k=1,k\ne j}^{p}\ln\frac{er_k}{\varepsilon_k}\Bigg)^{\frac{1}{2}+\delta}\le\prod_{k=1}^{p}\Big(\ln\frac{eR}{\varepsilon_k}\Big)^{\frac{p}{2}+p\delta}.$$

Denote

$$K := \left\{ z \in G \colon \ln^{\delta} h(r) \le \prod_{k=1}^{p} \Big(\ln \frac{eR}{\varepsilon_k} \Big)^{\frac{p}{2} + p\delta} \right\}.$$

In addition, $\nu_h(E \cap \Delta_\varepsilon)$ is finite when

$$\nu_h^*(E \cap \Delta_\varepsilon) = \int\limits_{E \cap \Delta_\varepsilon} h(r) dr_1 \cdots dr_p < +\infty.$$

Note that

$$\nu_h^*(K \cap \Delta_\varepsilon) = \int\limits_{K \cap \Delta_\varepsilon} h(r) dr_1 \cdots dr_p \le \exp\left\{ \prod_{k=1}^{p} \Big(\ln \frac{eR}{\varepsilon_k} \Big)^{\frac{p}{2\delta} + p} \right\} \int\limits_{K \cap \Delta_\varepsilon} dr_1 \cdots dr_p \le$$
$$\le \exp\left\{ \prod_{k=1}^{p} \Big(\ln \frac{eR}{\varepsilon_k} \Big)^{\frac{p}{2\delta} + p} \right\} \int\limits_{G} dr_1 \cdots dr_p < +\infty.$$

Finally, for all $r \in \Delta_\varepsilon \setminus (E \cup K)$ we obtain

$$M_f(r) \le \mu_f(r)(h(r))^{\frac{p+1}{2}} \ln^{\frac{p}{2}+\delta} h(r) \ln^{\frac{p}{2}+\delta}\{\mu_f(r)h(r)\} \prod_{j=1}^{p} \left(\prod_{k=1, k\neq j}^{p} \ln \frac{er_k}{\varepsilon_k} \right)^{\frac{1}{2}+\delta} \le$$
$$\le \mu_f(r)(h(r))^{\frac{p+1}{2}} \ln^{\frac{p}{2}+\delta} h(r) \ln^{\frac{p}{2}+\delta}\{\mu_f(r)h(r)\} \prod_{k=1}^{p} \Big(\ln \frac{eR}{\varepsilon_k} \Big)^{\frac{p}{2}+p\delta} \le$$
$$\le \mu_f(r)(h(r))^{\frac{p+1}{2}} \ln^{\frac{p}{2}+2\delta} h(r) \ln^{\frac{p}{2}+\delta}\{\mu_f(r)h(r)\}.$$

Thus, we prove such a statement.

Theorem 4. *Let $f \in \mathcal{A}^p(\mathbb{G})$, G is bounded. Then for every $\varepsilon \in \mathbb{R}_+^p, \delta > 0$ there exists a set $E \in \mathcal{S}_h$ such that for all $r \in \Delta_\varepsilon \setminus E$ we have*

$$M_f(r) \le \mu_f(r)(h(r))^{\frac{p+1}{2}} \ln^{\frac{p}{2}+\delta} h(r) \ln^{\frac{p}{2}+\delta}\{\mu_f(r)h(r)\}. \tag{7}$$

In the case when

$$G = \mathbb{B}_p(1) := \{z \in \mathbb{C}^p \colon |z| := \sqrt{|z_1|^2 + \ldots + |z_p|^2} < 1\}$$

one can choose $h(r) = (1 - |r|)^{-p}$, $|r| = (r_1^2 + \ldots + r_p^2)^{1/2}$.

Theorem 5. *Let $f \in \mathcal{A}^p(\mathbb{B}_p(1))$, $h(r) = (1 - |r|)^{-p}$. Then, for every $\varepsilon \in \mathbb{R}_+^p, \delta > 0$ there exists a set $E \in \mathcal{S}_h$ such that for all $r \in \Delta_\varepsilon \setminus E$ we have*

$$M_f(r) \le \frac{\mu_f(r)}{(1-|r|)^{\frac{1}{2}(p^2+p)+\delta}} \ln^{\frac{p}{2}+\delta} \frac{\mu_f(r)}{1-|r|}.$$

If we additionaly suppose that

$$h(r) = \prod_{j=1}^{p} h_j(r_j) \tag{8}$$

then (see [23]) inequality (6) from Lemma 5 can be replayced by

$$r_j \frac{\partial}{\partial r_j} \ln \mathfrak{M}_f(r) \le h^{\delta}(r) h_j^{1-\delta}(r_j) \ln^{1+\delta} \mathfrak{M}_f(r) \prod_{k=1, k\neq j}^{p} \ln^{1+\delta}\Big(\frac{er_k}{\varepsilon_k}\Big), \quad j \in \{1, \cdots, p\}.$$

We therefore have the following statement:

Theorem 6. *Let $f \in \mathcal{A}^p(\mathbb{G})$, $h \in \mathcal{H}^p$ satisfies condition (8). Then for every $\varepsilon \in \mathbb{R}_+^p$, $\delta > 0$ there exists a set $E \in \mathcal{S}_h$ such that for all $r \in \Delta_\varepsilon \setminus E$ we have*

$$M_f(r) \le \mu_f(r)(h(r))^{1+\delta} \ln^{\frac{p}{2}+\delta}\{\mu_f(r)h(r)\} \prod_{j=1}^{p} \Bigg(\prod_{k=1, k\neq j}^{p} \ln \frac{er_k}{\varepsilon_k} \Bigg)^{\frac{1}{2}+\delta}.$$

Inequality (3) follows from this statement if we choose $h(r) = \prod_{i\in I} \frac{1}{(1-r_i)}$.

8. Discussion

In view of the obtained results we can formulate the following conjectures:

Conjecture 1. The descriptions of exceptional sets in the Theorems 1–3 are in a sense the best possible.

Conjecture 2. For a given $h \in \mathcal{H}$, the inequality (4) is sharp in the general case.

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