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Generalized Quantum Integro-Differential Fractional Operator with Application of 2D-Shallow Water Equation in a Complex Domain

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Abstract: In this paper, we aim to generalize a fractional integro-differential operator in the open unit disk utilizing Jackson calculus (quantum calculus or q-calculus). Next, by consuming the generalized operator to define a formula of normalized analytic functions, we present a set of integral inequalities using the concepts of subordination and superordination. In addition, as an application, we determine the maximum and minimum solutions of the extended fractional 2D-shallow water equation in a complex domain.

Keywords: quantum calculus; fractional calculus; analytic function; subordination; univalent function; open unit disk; differential operator; convolution operator

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1. Introduction

Elementary series and polynomials, particularly the Mittag–Leffler functions and polynomials and their consequences, can be frequently seen in specific areas of number theory, including the theory of partitions. These functions are valuable in an extensive diversity of fields involving, for instance, finite vector spaces, combinator analysis, lie theory, nonlinear electric circuit theory, particle physics, optical studies, fluid theory, mechanical engineering, quantum mechanics, cosmology, theory of thermal conduction and measurements (see [1–6]). Quantum power series, especially the Mittag–Leffler functions, are known to have common applications, specifically in numerous areas of function theory, geometric function theory and others. As a substance of detail, q-Mittag–Leffler functions are beneficial too in a extensive diversity of arenas. In our study, we employ the definition of the q- Mittag–Leffler functions to modify a fractional integral operator of a complex variable.

The 2D-shallow water equations (SWEs) are utilized to designate flow in precipitously well mixed water figures where the straight length scales are much bigger than the fluid depth (long wavelength phenomena) [7]. The SWEs are selected by supposing a hydrostatic pressure distribution and a uniform velocity profile in the vertical direction. The SWEs can be used to study numerous physical phenomena of interest, such as storm surges, tidal variations, tsunami waves, and forces performing on off-shore assemblies, and can be joined to transport equations to formulate transport of chemical species. Most of these equations are solved by numerical techniques [8,9]. Our study is based on an approximated analytic solution given in the open unit disk.

In this study, we investigate a generalization of fractional integro-differential operators in the open unit disk formulated by the q-calculus. We employ the q-operator to describe a formulation of normalized analytic functions. We consider a set of integral inequalities indicating the notion of differential subordination and superordination. In addition, as an application, we regulate the upper and lower bound solutions of the generalized fractional 2D-shallow water equation in a complex domain. In addition, as an application, we compute the maximum and minimum solutions of the modified fractional 2D-shallow water equation in a complex domain.

2. Methods

In this section, we deal with the techniques used in this study.

2.1. Geometric Presentations

In this presentation, we give some definitions based on the geometric function theory, which are located in [10]

Definition 1. *Define the set* $\mathbb{O} := \{\chi \in \mathbb{C} : |\chi| < 1\}$*, which is the open unit disk. Two analytic functions* ϱ_1, ϱ_2 *in* $\mathbb{O}$ *are subordinated (* $\varrho_1 \prec \varrho_2$ *or* $\varrho_1(\chi) \prec \varrho_2(\chi), \chi \in \mathbb{O}$*) if an analytic function* $w, |w| \leq |\chi| < 1$ *occurs that fulfils*

$$\varrho_1(\chi) = \varrho_2(w(\chi)), \quad \chi \in \mathbb{O}.$$

Definition 2. *A class of analytic functions of the power series*

$$\varrho(\chi) = \chi + \sum_{n=2}^{\infty} \varrho_n \chi^n, \chi \in \mathbb{O}$$

denoted by Δ *and known as the class of univalent functions which is called the normalized subclass with the normalization equation* $\varrho(0) = \varrho'(0) - 1 = 0$.

Moreover, the normalized functions $\kappa, \eta \in \Delta$ *are called convoluted (* $\kappa * \eta$*) if*

$$(\kappa * \eta)(\chi) = \left(\chi + \sum_{n=2}^{\infty} \kappa_n \chi^n\right) * \left(\chi + \sum_{n=2}^{\infty} \eta_n \chi^n\right)$$
$$= \chi + \sum_{n=2}^{\infty} \kappa_n \eta_n \chi^n.$$

Definition 3. *The generalized Mittag–Leffler function is powered as follows:* [11]

$$\mathcal{E}^{\vartheta}_{\nu,\mu}(\chi) = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\nu n + \mu)} \frac{\chi^n}{n!},$$

where $(\vartheta)_n$ indicates the Pochhammer operator. Note that [6]

$$\mathcal{E}^{1}_{\nu,\mu}(\chi) = \mathcal{E}_{\nu,\mu}(\chi) = \sum_{n=0}^{\infty} \frac{\chi^n}{\Gamma(\nu n + \mu)}.$$

and

$$\mathcal{E}^{\vartheta}_{\nu,\mu}(-\chi^{\nu}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\vartheta)_n}{\Gamma(\nu n + \mu)} \frac{\chi^{n\nu}}{n!}.$$

2.2. ABC-Fractional Differential Operator

Atangana and Baleanu [12] presented a new fractional operator, which is extended to the complex plane [13] :

$$^{C}\Delta^{\nu} h(\chi) = \frac{\beta(\nu)}{2\pi i(1-\nu)} \int_{\mathbb{D}} h'(\zeta) \mathcal{E}_{\nu}(-\mu_{\nu}(\chi - \zeta)^{\nu}) d\zeta, \tag{1}$$

where $\beta(\nu)$ is normalized by $\beta(0) = \beta(1) = 1$ and $\mathcal{E}_\nu(\omega)$ is the Mittag–Leffler function. Additionally, they familiarized the succeeding fractional differential operator

$$^{R}\Delta^{\nu}h(\chi) = \frac{\beta(\nu)}{2\pi i(1-\nu)}\frac{d}{d\chi}\int_{\mathbb{D}} h(\zeta)\mathcal{E}_\nu(-\mu_\nu(\chi-\zeta)^\nu)d\zeta, \tag{2}$$

$$\left(\mu_\nu = \frac{\nu}{1-\nu}, \nu \in [0,1], \mathbb{D} = \{\chi + re^{i\pi}(\chi - t) : 0 < r < 1\}\right).$$

Definition 4. *Let $\varrho \in \Delta$. Then, the ABC-fractional operators of* (1) *and* (2) *are given by the next integrals correspondingly*

$$^{C}\Delta^{\nu}_{\chi}\varrho(\chi) = \frac{\beta(\nu)}{1-\nu}\int_0^\chi \varrho'(\zeta)\mathcal{E}_{\nu,\upsilon}(-\mu_\nu\zeta^\nu)\mathcal{E}_\nu(-\mu_\nu(\chi-\zeta)^\nu)d\zeta, \tag{3}$$

and

$$^{R}\Delta^{\nu}_{\chi}\varrho(\chi) = \frac{\beta(\nu)}{1-\nu}\frac{d}{d\chi}\int_0^\chi \varrho(\zeta)\mathcal{E}_{\nu,\upsilon}(-\mu_\nu\zeta^\nu)\mathcal{E}_\nu(-\mu_\nu(\chi-\zeta)^\nu)d\zeta, \tag{4}$$

where υ designates the power of χ. Furthermore, we ensure that ϱ is analytic in simply connected region of the complex z-plane involving the origin, and the multiplicity of $(\chi-\zeta)$ is flouted by representing $\log(\chi-\zeta)$ as real when $\Re(\chi-\zeta) > 0$.

Example 1. *For instance, let $\varrho(\chi) = \chi$,; then, from Theorem 2.4* [14] *or Theorem 11.2* [15], *we arrange*

$$\begin{aligned}
^{C}\Delta^{\nu}_{\chi}(\chi) &= (\beta(\nu)/1-\nu)\int_0^\chi \mathcal{E}_\nu(-\mu_\nu\zeta^\nu)\mathcal{E}_\nu(-\mu_\nu(\chi-\zeta)^\nu)d\zeta \\
&= (\beta(\nu)/1-\nu)\chi\mathcal{E}^2_{\nu,2}(-\mu_\nu(\chi)^\nu) \\
&= (\beta(\nu)/1-\nu)\chi\sum_{k=0}^{\infty}\frac{(2)_k\,\chi^k}{k!\Gamma(k\nu+2)}.
\end{aligned}$$

$$\Big((u)_0 = 1, (u)_n = u(u+1)...(u+n-1)\Big)$$

Based on [14], *Theorem 2.2, we obtain*

$$\begin{aligned}
^{R}\Delta^{\nu}_{\chi}(\chi) &= (\beta(\nu)/1-\nu)\frac{d}{d\chi}\int_0^\chi \mathcal{E}_\nu(-\mu_\nu\zeta^\nu)\mathcal{E}_\nu(-\mu_\nu(\chi-\zeta)^\nu)\zeta\, d\zeta \\
&= (\beta(\nu)/1-\nu)\Big(\chi^2\mathcal{E}^2_{\nu,3}(-\mu_\nu(\chi)^\nu)\Big)' \\
&= (\beta(\nu)/1-\nu)\Big(\chi\mathcal{E}^2_{\nu,2}(-\mu_\nu(\chi)^\nu)\Big).
\end{aligned}$$

Obviously, we obtain

$$^{C}\Delta^{\nu}_{\chi}(\chi) = {}^{R}\Delta^{\nu}_{\chi}(\chi).$$

Generally, we obtain

$$^{C}\Delta^{\nu}_{\chi}(\chi^n) = (\beta(\nu)/1-\nu)n\chi^n\Big(\mathcal{E}^2_{\nu,1+n}(-\mu_\nu(\chi)^\nu)\Big), \quad n \geq 1,$$

$$^{R}\Delta^{\nu}_{\chi}(\chi^n) = (\beta(\nu)/1-\nu)\chi^n\Big(\mathcal{E}^2_{\nu,1+n}(-\mu_\nu(\chi)^\nu)\Big).$$

Next, we investigate some possessions of the exceeding operators.

Example 2. *For a function $\varrho \in \Delta$, we have the following normalized operators*

$$^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi) := \frac{^{C}\Delta^{\nu}_{\chi}\varrho(\chi)}{\flat(\nu)\mathcal{E}^2_{\nu,2}(-\mu_\nu(\chi)^\nu)} \in \Delta$$

and

$$^{\mathfrak{R}}\Delta^{\nu}_{\chi}\varrho(\chi) := \frac{^{R}\Delta^{\nu}_{\chi}\varrho(\chi)}{\flat(\nu)\mathcal{E}^{2}_{\nu,2}(-\mu_{\nu}(\chi)^{\nu})} \in \Delta,$$

where $\flat(\nu) := (\beta(\nu)/1-\nu)$.

Proof. For $\varrho \in \Delta$, a calculation gives

$$\begin{aligned}
^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi) &= \frac{^{C}\Delta^{\nu}_{\chi}\varrho(\chi)}{\flat(\nu)\mathcal{E}^{2}_{\nu,2}(-\mu_{\nu}(\chi)^{\nu})} \\
&= \frac{\flat(\nu)\mathcal{E}^{2}_{\nu,2}(-\mu_{\nu}(\chi)^{\nu})\,\chi + \sum_{n=2}^{\infty}\varrho_n \flat(\nu) n \left(\mathcal{E}^{2}_{\nu,1+n}(-\mu_{\nu}(\chi)^{\nu})\right)\chi^{n}}{\flat(\nu)\mathcal{E}^{2}_{\nu,2}(-\mu_{\nu}(\chi)^{\nu})} \\
&= \chi + \sum_{n=2}^{\infty}\varrho_n n \left(\frac{\mathcal{E}^{2}_{\nu,1+n}(-\mu_{\nu}(\chi)^{\nu})}{\mathcal{E}^{2}_{\nu,2}(-\mu_{\nu}(\chi)^{\nu})}\right)\chi^{n} \\
&\Rightarrow\ ^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi) \in \Delta.
\end{aligned}$$

Similarly, we have $^{\mathfrak{R}}\Delta^{\nu}_{\chi}\varrho(\chi) \in \Delta$. □

Note that, when $\left(\frac{\mathcal{E}^{2}_{\nu,1+n}(-\mu_{\nu}(\chi)^{\nu})}{\mathcal{E}^{2}_{\nu,2}(-\mu_{\nu}(\chi)^{\nu})}\right) \approx 1$,*, we obtain the formula*

$$^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi) = \chi + \sum_{n=2}^{\infty}\varrho_n n \chi^{n},$$

which for k-times ($^{\mathfrak{C}}\Delta^{\nu}_{\chi} * \ldots *^{\mathfrak{C}} \Delta^{\nu}_{\chi}\varrho(\chi)$)*, we obtain the Salagean derivative operator* [16].

2.3. Q-Calculus

For a number $\varpi \in \mathbb{C}$, the q-shifted factorials is formulated by the formal [17]

$$(\varpi; q)_{\ell} = \prod_{\imath=0}^{\ell-1}(1 - q^{\imath}\varpi), \quad \ell \in \mathbb{N},\ (\varpi; q)_0 = 1. \tag{5}$$

According to (5), and in terms of gamma function, we obtain the q-shifted formula

$$(q^{\varpi}; q)_{\ell} = \frac{\Gamma_q(\varpi+\ell)(1-q)^{\ell}}{\Gamma_q(\varpi)}, \quad \Gamma_q(\varpi) = \frac{(q;q)_{\infty}(1-q)^{1-\varpi}}{(q^{\varpi};q)_{\infty}} \tag{6}$$

where

$$\Gamma_q(\varpi+1) = \frac{\Gamma_q(\varpi)(1-q^{\varpi})}{1-q}, \quad q \in (0,1).$$

and

$$(\varpi; q)_{\infty} = \prod_{\imath=0}^{\infty}(1 - q^{\imath}\varpi). \tag{7}$$

Jackson derivative is formulated in the following difference operator

$$\partial_q\, h(\chi) = \frac{h(\chi) - h(q\chi)}{\chi(1-q)} \tag{8}$$

such that

$$\partial_q\,(\chi^{\upsilon}) = \left(\frac{1-q^{\upsilon}}{1-q}\right)\chi^{\upsilon-1}.$$

Moreover, the notion of q-binomial formula achieves the equality

$$(\vartheta - \upsilon)_b = \vartheta^b \left(\frac{-\upsilon}{\vartheta}; q \right)_b. \tag{9}$$

In [18], the authors presented the q-Mittag–Leffler function as follows:

$$\mathcal{E}^{\vartheta}_{\nu,\mu}(\chi; q) = \sum_{n=0}^{\infty} \frac{(q^{\vartheta}; q)_n}{(q; q)_n} \frac{\chi^n}{\Gamma_q(\nu\, n + \mu)} \tag{10}$$

Based on q- Mittag–Leffler function, we have the q-ABC fractional operator acting on $\varrho \in \Delta$

$$\begin{aligned} [{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q &= \frac{\flat(\nu)\mathcal{E}^2_{\nu,2}(-\mu_\nu(\chi)^\nu)\,\chi + \sum_{n=2}^{\infty} \varrho_n \flat(\nu)[n]_q \left(\mathcal{E}^2_{\nu,1+n}(-\mu_\nu(\chi)^\nu; q)\right)\chi^n}{\flat(\nu)\mathcal{E}^2_{\nu,2}(-\mu_\nu(\chi)^\nu; q)} \\ &= \chi + \sum_{n=2}^{\infty} \varrho_n [n]_q \left(\frac{\mathcal{E}^2_{\nu,1+n}(-\mu_\nu(\chi)^\nu; q)}{\mathcal{E}^2_{\nu,2}(-\mu_\nu(\chi)^\nu; q)} \right)\chi^n \\ &:= \chi + \sum_{n=2}^{\infty} \varrho_n [n]_q [\Xi_n]_q \chi^n \\ &\Rightarrow\ [{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q \in \Delta, \end{aligned}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad [\Xi_n]_q = \left(\frac{\mathcal{E}^2_{\nu,1+n}(-\mu_\nu(\chi)^\nu; q)}{\mathcal{E}^2_{\nu,2}(-\mu_\nu(\chi)^\nu; q)} \right), \quad q \in (0,1).$$

More investigations and applications of q-calculus can be located in [19–22].

3. Lemmas

The results of this investigation are based on the differential subordination theory via the following preliminaries:

Lemma 1. [10] *Let two analytic functions $\varphi(\chi)$ and $\psi(\chi)$ be convex univalent defined in $\mathbb{O}$ such that $\varphi(0) = \psi(0)$. Moreover, for a constant $t \neq 0$, $\Re(t) \geq 0$, the subordination*

$$\varphi(\chi) + (1/t)\varphi'(\chi) \prec \psi(\chi)$$

implies that

$$\varphi(\chi) \prec \psi(\chi).$$

Lemma 2. [10] *Define the general class of analytic functions*

$$\mathbb{A}[a, n] = \{g : g(\chi) = a + a_n\chi^n + a_{n+1}\chi^{n+1} + \cdots\},$$

where $a \in \mathbb{C}$ and n is a positive integer. If $t \in \mathbb{R}$, then

$$\Re\{g(\chi) + t\chi g'(\zeta)\} > 0 \Rightarrow \Re(g(\chi)) > 0.$$

Moreover, if $t > 0$ and $g \in \mathbb{A}[1, n]$, then there are fixed numbers $c_1 > 0$ and $c_2 > 0$ such that the inequality

$$g(\chi) + t\chi g'(\chi) \prec \left(\frac{1 + \chi}{1 - \chi} \right)^{c_1}$$

yields

$$g(\chi) \prec \left(\frac{1 + \chi}{1 - \chi} \right)^{c_2}.$$

Lemma 3. *(See [23].) Let $\hbar, p \in \mathbb{A}[a,n]$, where p is convex univalent in Δ and for $\Bbbk_1, \Bbbk_2 \in \mathbb{C}, \Bbbk_2 \neq 0$; then,*

$$\Bbbk_1 \hbar(\chi) + \Bbbk_2 \chi \hbar'(\chi) \prec \Bbbk_1 p(\chi) + \Bbbk_2 \chi p'(\chi) \to \hbar(\chi) \prec p(\chi).$$

Lemma 4. *(See [24].) Let $g, p \in \mathbb{A}[a,n]$, where p is convex univalent in Δ such that $g(\chi) + \Bbbk \chi g'(\chi)$ is univalent; then,*

$$p(\chi) + \Bbbk \chi p'(\chi) \prec g(\chi) + \Bbbk \chi g'(\chi) \to p(\chi) \prec g(\chi).$$

Lemma 5. *(See [25].) Let $\hbar, \hbar, g \in \mathbb{A}[a,n]$ and g is convex univalent in $\mathbb{O}$ such that $\hbar \prec g$ and $\hbar \prec g$; then,*

$$\Bbbk \hbar + (1-\Bbbk)\hbar \prec g, \ \Bbbk \in [0,1].$$

4. Results

Our investigation is about the following class:

Definition 5. *A function $\varrho \in \Delta$ is called in the class $[\Sigma_\sigma^\nu(p)]_q$ if it satisfies the inequality*

$$\left(\frac{1-\sigma}{\chi}\right) [^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma\, [^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]'_q \prec p(\chi) = \frac{a\chi+1}{b\chi+1}. \tag{11}$$

$$\Big(\chi \in \mathbb{O}, \nu, \sigma \in [0,1], -1 \leq b < a \leq 1\Big),$$

where p is convex univalent in $\mathbb{O}$.

For example,

$$p(\chi) = \frac{a\chi+1}{b\chi+1} = \Upsilon_{a,b}(\chi),$$

which is univalent convex in $\mathbb{O}$ and it is the extreme function in the set

$$\mathcal{P} := \{p \in \mathbb{O} : p(\chi) = 1 + \sum_{n=1}^{\infty} p_i \chi^n\}.$$

Define a functional $\Psi : \mathbb{O} \to \mathbb{O}$, as follows:

$$\Psi(\chi) := \left(\frac{1-\sigma}{\chi}\right) [^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]'_q \tag{12}$$

Shortly, by Definition 5, we have the following inequality

$$\Psi(\chi) \prec \Upsilon_{a,b}(\chi) := \frac{a\chi+1}{b\chi+1}, \quad \chi \in \mathbb{O}.$$

Theorem 1. *Suppose that $\varrho \in [\Sigma_\sigma^\nu(p)]_q$. If*

$$\Re\{\Psi(\chi)\} = \Re\Big\{1 + \sum_{n=1}^{\infty} [\varrho_{n+1}[n+1]_q [\Xi_{n+1}]_q (1+\sigma n)\chi^n]\Big\}$$
$$:= \Re\Big\{1 + \sum_{n=1}^{\infty} \Psi_n\Big\} > 0$$

then the coefficient bounds of Ψ satisfy the inequality

$$\frac{|\Psi_n|}{2} \leq \int_0^{2\pi} |e^{-in\theta}|\, d\mathfrak{M}(\theta),$$

where $d\mathfrak{M}$ is a probability measure. Additionally, if

$$\Re\left(e^{i\vartheta}\Psi(\chi)\right) > 0, \quad \chi \in \mathbb{O}, \, \vartheta \in \mathbb{R}$$

then, $\varrho \in [\Sigma_{\sigma}^{\nu}\left(\frac{a\chi+1}{b\chi+1}\right)]_q$ *that is*

$$\Psi(\chi) \approx \frac{a\chi+1}{b\chi+1}, \quad \chi \in \mathbb{O}.$$

Proof. By the assumption, we have

$$\Re(\Psi(\chi)) = \Re\left(1 + \sum_{n=1}^{\infty} \Psi_n \chi^n\right) > 0.$$

Thus, the Carathéodory positivist method implies

$$|\varrho_n| \leq 2\int_0^{2\pi} |e^{-in\theta}| \, d\mathfrak{M}(\theta),$$

where $d\mathfrak{M}$ is a probability measure. In addition, if

$$\Re\left(e^{i\vartheta}\Psi(\chi)\right) > 0, \quad \chi \in \mathbb{O}, \quad \vartheta \in \mathbb{R}$$

then according to [26], Theorem 1.6, and for fixed $\vartheta \in \mathbb{R}$, we have

$$\Psi(\chi) \approx p(\chi) = \frac{a\chi+1}{b\chi+1}, \quad \chi \in \mathbb{O}.$$

Hence, $\varrho \in [\Sigma_{\sigma}^{\nu}\left(\frac{a\chi+1}{b\chi+1}\right)]_q$. □

The next outcomes indicate the sufficient and necessary conditions for the sandwich behavior of the functional $\Psi(\chi)$.

Theorem 2. *Let the following assumptions hold*

$$\sigma\chi[{}^{\mathfrak{C}}\Delta_{\chi}^{\nu}\varrho(\chi)]_q'' + [{}^{\mathfrak{C}}\Delta_{\chi}^{\nu}\varrho(\chi)]_q' \prec p_2(\chi) + \chi p_2'(\chi), \tag{13}$$

where $p_2(0) = 1$ and convex in $\mathbb{O}$. Moreover, let $\Psi(\chi)$ be univalent in $\mathbb{O}$ such that $\Psi \in \mathbb{H}[p_1(0), 1] \cap \mathbb{Q}$,, where $\mathbb{Q}$ represents the set of all (1-1) analytic functions f with $\lim_{\chi \in \partial\mathbb{O}} f \neq \infty$ and

$$p_1(\chi) + \chi p_1'(\chi) \prec \sigma\chi[{}^{\mathfrak{C}}\Delta_{\chi}^{\nu}\varrho(\chi)]_q'' + [{}^{\mathfrak{C}}\Delta_{\chi}^{\nu}\varrho(\chi)]_q'. \tag{14}$$

Then,

$$p_1(\chi) \prec \Psi(\chi) \prec p_2(\chi)$$

and $p_1(\chi)$ is the best sub-dominant and $p_2(\chi)$ is the best dominant.

Proof. Since,

$$\Psi(\chi) = \left(\frac{1-\sigma}{\chi}\right)[{}^{\mathfrak{C}}\Delta_{\chi}^{\nu}\varrho(\chi)]_q + \sigma[{}^{\mathfrak{C}}\Delta_{\chi}^{\nu}\varrho(\chi)]_q'$$

then a computation yields

$$\begin{aligned}\Psi(\chi)+\chi\Psi'(\chi) &= \sigma[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]'_q \\ &+ \frac{(\chi(\sigma\chi[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]''_q-(\sigma-1)[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]'_q)+(\sigma-1)[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q)}{\chi} \\ &+ \frac{((1-\sigma)[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q)}{\chi} \\ &= \sigma\chi[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]''_q+[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]'_q.\end{aligned}$$

As a consequence, we obtain the next double inequality

$$p_1(\chi)+\chi p_1'(\chi)\prec\Psi(\chi)+\chi\Psi'(\chi)\prec p_2(\chi)+\chi p_2'(\chi).$$

Thus, Lemmas 3 and 4 imply the desired assertion. □

Theorem 3. *Let p be a univalent convex function in $\mathbb{O}$ such that $v(0)=0$ and*

$$[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q\prec v(\chi), \quad [{}^{\mathfrak{R}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q\prec v(\chi).$$

Then,

$$\Bbbk\ [{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q+(1-\Bbbk)\ [{}^{\mathfrak{R}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q\prec v(\chi), \quad \Bbbk\in[0,1].$$

Proof. By the definition of $[\mathcal{Q}^{m\,\alpha}\psi(\zeta)]$ and $[\mathcal{L}^{m\,\alpha}\psi(\zeta)]$, clearly we have

$$\Bbbk\ [{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q+(1-\Bbbk)\ [{}^{\mathfrak{R}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q\in\Delta.$$

Hence, a direct application of Lemma 5, we obtain the result. □

Theorem 4. *Let $\sigma_2\le\sigma_1<0$ and $\varrho\in\Delta$. Then*

$$[\Sigma^{\nu}_{\sigma_2}(p)]_q\subset[\Sigma^{\nu}_{\sigma_1}(p)]_q.$$

Proof. Let $\varrho\in[\Sigma^{\nu}_{\sigma_2}(p)]_q$. Define the analytic function in $\mathbb{O}$, as follows:

$$\phi(\chi)=\frac{[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q}{\chi},$$

satisfying $\phi(0)=1$. A computation implies that

$$\left(\frac{1-\sigma_2}{\chi}\right)[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q+\sigma_2[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]'_q=\phi(\chi)+\sigma_2(\chi\phi'(\chi)). \tag{15}$$

This leads to

$$\phi(\chi)+\sigma_2\,(\chi\,\phi'(\chi))\prec\frac{a\chi+1}{b\chi+1}$$

Applying Lemma 1 with $\sigma_2>0$ gives

$$\phi(\chi)\prec\frac{a\chi+1}{b\chi+1}. \tag{16}$$

Since $0 < \sigma_1/\sigma_2 < 1$ and $\frac{a\chi+1}{b\chi+1}$ is convex univalent in $\mathbb{O}$, we obtain

$$\begin{aligned}
& \left(\frac{1-\sigma_1}{\chi}\right)[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma_1[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q' \\
= \; & (1-\sigma_1)\phi(\chi) + \sigma_1 [^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q' \\
= \; & (1-\sigma_1)\phi(\chi) + \sigma_1(\zeta\phi'(\chi) + \phi(\chi)), \\
= \; & (1-\sigma_1)\phi(\chi) + \sigma_1(\chi\phi'(\chi) + \phi(\chi)) + \left(\frac{\sigma_1}{\sigma_2}\phi(\chi) - \frac{\sigma_1}{\sigma_2}\phi(\chi)\right) \\
= \; & \frac{\sigma_1}{\sigma_2}(1-\sigma_2)\phi(\chi) + \sigma_2(\chi\phi'(\chi) + \phi(\chi)) + \left(1-\frac{\sigma_1}{\sigma_2}\right)\phi(\chi) \\
= \; & \frac{\sigma_1}{\sigma_2}\left[\frac{(1-\sigma_2)}{\chi}[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma_2[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q'\right] + \left(1-\frac{\sigma_1}{\sigma_2}\right)\phi(\chi) \\
\prec \; & \frac{a\chi+1}{b\chi+1} = p(\chi).
\end{aligned}$$

Hence, by Definition 5, we conclude that $\varrho \in [\Sigma_{\sigma_1}^\nu(p)]_q$. □

Theorem 5. *Let*

$$\Psi(\zeta) = \frac{(1-\sigma)}{\chi}[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q'$$

then

$$\frac{[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q'}{\chi}\hbar_1 + [^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q[\hbar_1 + 3\hbar_2] + \hbar_2\chi[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q'' \prec \left(\frac{1+\chi}{1-\chi}\right)^{c_1}$$
$$\Rightarrow \Psi(\chi) \prec \left(\frac{1+\chi}{1-\chi}\right)^{c_2}$$

where $c_1 > 0, c_2 > 0, \hbar_1 = 1-\sigma,\ \hbar_2 = \sigma > 0.$

Proof. A calculation implies that

$$\begin{aligned}
\Psi(\chi) + \chi\Psi'(\chi) &= \frac{(1-\sigma)}{\chi}[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q' \\
&+ \chi\left(\frac{(1-\sigma)}{\chi}[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q'\right)' \\
&= \frac{[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q'}{\chi}\hbar_1 + [^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q[\hbar_1 + 3\hbar_2] + \hbar_2\chi[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q'' \\
&\prec \left(\frac{1+\chi}{1-\chi}\right)^{c_1}
\end{aligned}$$

According to Lemma 2 with $t = 1$, we obtain

$$\Psi(\chi) \prec \left(\frac{1+\chi}{1-\chi}\right)^{c_2}.$$

□

5. Application

By employing the concept of fractional calculus, we formulate the fractional 2D-shallow water equation in view of the suggested operator q-operator $[^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q$, which is formulated in the class $[\Sigma_\sigma^\nu\left(\frac{1+\chi}{1-\chi}\right)]_q$. We investigate the upper bound of the 2D-shallow water equation of diffusive wave (this equation is measured at the level of the water). The formula is simply given as follows:

$$(\frac{1-\sigma}{\chi})\,[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma\,[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q' = \frac{a\chi+1}{b\chi+1}, \tag{17}$$

$$\Big([{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(0)]_q = 0,\, q \in (0,1),\, \sigma \in [0,1],\, \chi \in \mathbb{O}\Big)$$

where ϱ is the height deviation of the horizontal pressure surface at two-dimensional position $\chi = \alpha + i\,\beta$ and $[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q$ represents the bed slope. We have the following result describing the solution of (17).

Theorem 6. *Consider the class of analytic functions* $[\Sigma_\sigma^\nu\Big(\dfrac{1+\chi}{1-\chi}\Big)]_q, \sigma \in (0,1]$. *Then, the solution of the differential equation corresponding to this class is*

$$[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q \approx \chi\left(\frac{2\chi\;{}_2F_1(1, 1+\frac{1}{\sigma}, 2+\frac{1}{\sigma}, \chi)}{\sigma+1} + 1\right), \tag{18}$$

where ${}_2F_1(a,b,c;\chi)$ *represents the hypergeometric function.*

Proof. Suppose that $\varrho \in [\Sigma_\sigma^\nu\Big(\dfrac{1+\chi}{1-\chi}\Big)]_q$. Then, it yields the differential equation

$$(\frac{1-\sigma}{\chi})\,[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma\,[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q' = \frac{\omega(\chi)+1}{1-\omega(\chi)},$$

where $\omega(0) = 0$ and $|\omega| \le |\chi| < 1$. This implies the integral equation

$$[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q = \chi^{(\sigma-1)/\sigma}\int_0^\chi -z^{1/(\sigma-1)}\left(\frac{\omega(z)+1}{\sigma(\omega(z)-1)}\right)dz.$$

To find the upper solution, we let $\omega(\chi) = \chi$. Thus, we have the differential equation

$$\frac{(1-\sigma)}{\chi}\,[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q + \sigma[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q' = \frac{\chi+1}{1-\chi}.$$

Rewrite the above equation as follows:

$$[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q' + \frac{1-\sigma}{\sigma\chi}\,[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q = \left(\frac{1}{\sigma}\right)\left(\frac{1+\chi}{1-\chi}\right).$$

Multiplying the above equation by the functional

$$\tau(\chi) = \exp\left(\int \frac{1-\sigma}{\sigma\chi}d\chi\right),$$

then, we obtain

$$\chi^{1/\sigma-1}[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q' - \frac{[{}^{\mathfrak{C}}\Delta_\chi^\nu \varrho(\chi)]_q\Big((1-\sigma)\chi^{1/\sigma-2}\Big)}{\sigma} = \left(\frac{\chi^{1/\sigma-1}}{\sigma}\right)\left(\frac{1+\zeta}{1-\zeta}\right).$$

Hence, it yields solution (18). □

Example 3. *For, $\sigma = 0.5$, and in view of Theorem 6, we have the solution (see Figure 1)*

$$
\begin{aligned}
[{}^{\mathfrak{C}}\Delta^{\nu}_{\chi}\varrho(\chi)]_q &\approx \chi\left(\frac{2\chi\ {}_2F_1(1, 1+\frac{1}{0.5}, 2+\frac{1}{0.5}, \chi)}{0.5+1} + 1\right) \\
&= \chi + 1.33333\chi^2 + \chi^3 + 0.8\chi^4 + 0.666667\chi^5 + 0.571429\chi^6 + O(\chi^7), \quad |\chi| < 1 \\
&= \chi + 1.33333\chi^2 \sum_{n=0}^{\infty} \frac{\chi^n (1)_n (3)_n}{n!(4)_n}.
\end{aligned}
$$

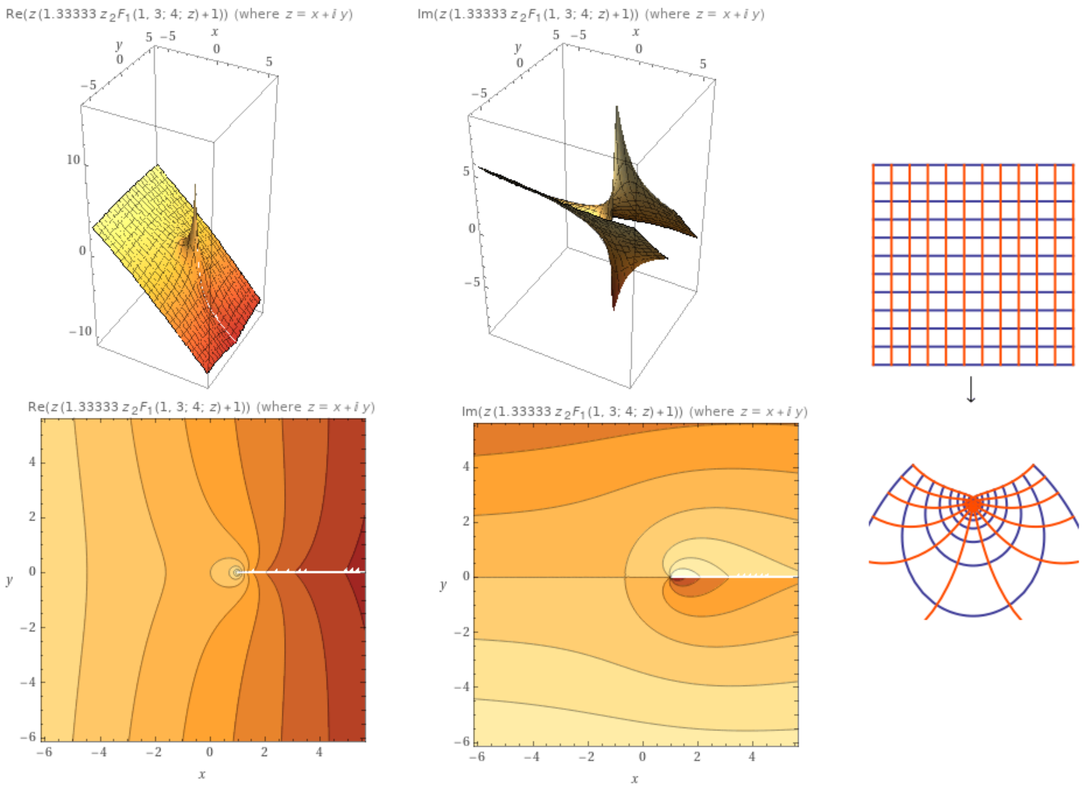

Figure 1. The plot of the solution of Equation (17) when $\sigma = 0.5$.

6. Conclusions

- The above investigation shows the extension of the ABC-fractional operator in the open unit disk and its generalization by using Jackson calculus. We expressed it in a linear convolution operator acting on a normalized analytic function. A class of analytic functions is studied involving the suggested operator. As an application, we consider the 2D-shallow water differential equation. We discovered its solution in terms of a special function-type hypergeometric function. Moreover, we indicated that the solution is also in the class of normalized analytic functions.
- For future works, we suggest modifying the operator acting on different classes of holomorphic functions including the multi-valent, meromorphic and harmonic functions in the open unit disk.

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