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Decomposition, Mapping, and Sum Theorems of ω-Paracompact Topological Spaces

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Abstract: As a weaker form of ω-paracompactness, the notion of σ-ω-paracompactness is introduced. Furthermore, as a weaker form of σ-ω-paracompactness, the notion of feebly ω-paracompactness is introduced. It is proven hereinthat locally countable topological spaces are feebly ω-paracompact. Furthermore, it is proven hereinthat countably ω-paracompact σ-ω-paracompact topological spaces are ω-paracompact. Furthermore, it is proven hereinthat σ-ω-paracompactness is inverse invariant under perfect mappings with countable fibers, and as a result, is proven hereinthat ω-paracompactness is inverse invariant under perfect mappings with countable fibers. Furthermore, if $\mathcal{A}$ is a locally finite closed covering of a topological space (X, τ) with each $A \in \mathcal{A}$ being ω-paracompact and normal, then (X, τ) is ω-paracompact and normal, and as a corollary, a sum theorem for ω-paracompact normal topological spaces follows. Moreover, three open questions are raised.

Keywords: ω-open set; ω-paracompactness; sum theorem

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1. Introduction

Generalizing the properties of the bounded and closed subsets of $\mathbb{R}^n$ is the main motivation for introducing compactness into the topology. Compactness and metrizability are the heartbeat of general topology. Furthermore, for applications, these two notions are very efficient, where metric notions are used almost everywhere in mathematical analysis, and compactness is used in many parts of analysis and also in mathematical logic. As a generalization of both metrizable topological spaces and compact topological spaces, paracompact topological spaces were defined by Dieudonné [1] in 1944; although defined much later than the two later classes, paracompact topological spaces became popular among topologists and analysts, and are now considered to be one of the most important classes of topological spaces. Due to the introduction of paracompactness, many theorems in topology and analysis have been generalized, and many proofs have been simplified. Furthermore, it turns out that the concept of local finiteness and its related concepts are very efficient and natural tools for studying topological spaces. In general topology, as in many other parts of mathematics, successful notions tend to become generalized. One motivation for such generalizations is the attempt to 'push results to their limits'. Therefore, many generalizations of the concept of paracompactness have been made by several authors. Dowker [2] generalized paracompact topological spaces by introducing the class of countably paracompact spaces. Al Ghour [3] introduced the concepts of ω-paracompactness and countable ω-paracompactness as generalizations of paracompactness and countable paracompactness, respectively.

In the present paper, we introduce the notions of σ-ω-paracompactness and feebly ω-paracompactness, where σ-ω-paracompactness is a weaker form of ω-paracompactness, and feebly ω-paracompactness is a weaker form of σ-ω-paracompactness. We prove that locally countable topological spaces are feebly ω-paracompact. Furthermore, we prove that countably ω-paracompact σ-ω-paracompact topological spaces are ω-paracompact. Furthermore, we prove that σ-ω-paracompactness is inverse invariant under perfect mappings with countable fibers, and as a result, ω-paracompactness is inverse invariant under perfect mappings with countable fibers. Furthermore, if $\mathcal{A}$ is a locally finite closed covering of a topological space (X, τ) with each $A \in \mathcal{A}$ being ω-paracompact and normal, then (X, τ) is ω-paracompact and normal, and as a corollary, a sum theorem for ω-paracompact normal topological spaces is introduced. In addition to these, three open questions are raised.

2. Preliminaries

In this paper, we follow the notions and conventions of [3,4]. Let (X, τ) be a topological space and A be a subset of X. A point $x \in X$ is called a condensation point of A [4] if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called an ω-closed subset of (X, τ) [5] if it contains all its condensation points. A is called an ω-open subset of (X, τ) [5] if $X - A$ is ω-closed. It is well known that A is ω-open in (X, τ) if and only if for each $x \in A$, there exists $U \in \tau$ and a countable set $C \subseteq X$ such that $x \in U - C \subseteq A$. It is well known that the family of all ω-open subsets of (X, τ) forms a topology on X finer than τ. If $\mathcal{A}$ and $\mathcal{B}$ are two covers of X, then $\mathcal{A}$ is called a refinement of $\mathcal{B}$ if for every $A \in \mathcal{A}$, there exists $B \in \mathcal{B}$ such that $A \subseteq B$. A family $\{A_\alpha : \alpha \in \Lambda\}$ of subsets X is called locally finite (resp. ω-locally finite [3]) in (X, τ) if for every point $x \in X$ there exists an open (resp. ω-open) set U containing x such that $\{\alpha \in \Lambda : U \cap A_\alpha \neq \emptyset\}$ is finite. Research via ω-closed sets and ω-open sets is still a significantly popular area of research in topological structures [6–20].

A Hausdorff topological space (X, τ) is called paracompact (resp. countably paracompact) if each open covering (resp. countable open) covering of X admits a locally finite open refinement. Al Ghour [3], defined ω-paracompactness and countable ω-paracompactness as weaker forms of paracompactness and countable paracompactness, respectively, as follows: a Hausdorff topological space (X, τ) is ω-paracompact (resp. countably ω-paracompact) if each open (resp. countable open) covering of X admits an ω-locally finite open refinement.

Throughout this paper, for a subset A of a topological space (X, τ); $\underline{A}$ will denote the intersection of all ω-closed sets that contain A. Furthermore, for a function $f : X \longrightarrow Y$, the sets $f^{-1}(y) = \{x \in X : f(x) = y\}$ where $y \in Y$ are called the fibers of f.

The following definitions and results will be used in the sequel:

Definition 1. *A function $f : (X, \tau) \longrightarrow (Y, \mu)$ is called:*

(a) Ref. [5] ω-closed if it maps closed sets onto ω-closed sets;
(b) Ref. [21] ω-continuous if the inverse image of each open set is an ω-open set.

It is known that every closed (resp. continuous) function is ω-closed (resp. ω-continuous), but not conversely.

Definition 2 ([22])**.** *A topological space (X, τ) is called countably metacompact if every countable open cover of X has a point finite open refinement.*

Proposition 1 ([3])**.**

(a) Every countably paracompact topological space is countably ω-paracompact but not conversely;
(b) Every countably ω-paracompact topological space is countably metacompact but not conversely;
(c) Every ω-paracompact topological space is countably ω-paracompact but not conversely.

Proposition 2 ([4])**.** *For every normal topological space (X, τ), the following are equivalent:*

(a) (X, τ) is countably paracompact;
(b) (X, τ) is countably metacompact.

Proposition 3. *For every normal topological space (X, τ), the following are equivalent:*

(a) (X, τ) is countably paracompact;
(b) (X, τ) is countably ω-paracompact;

(c) (X, τ) *is countably metacompact.*

Proof. It follows from Propositions 1 (b) and 2. □

Proposition 4 ([23]). *The closed continuous image of a countably paracompact normal topological space is a countably paracompact normal topological space.*

Proposition 5 ([3]). *For every Hausdorff topological space* (X, τ)*, the following are equivalent:*

(a) (X, τ) *is countably* ω*-paracompact;*
(b) For every countable open cover $\{A_n : n \in \mathbb{N}\}$ *of X, there exists an* ω*-locally finite open cover* $\{B_n : n \in \mathbb{N}\}$ *of X such that for all* $n \in \mathbb{N}$*,* $A_n \subseteq B_n$*.*

Proposition 6 ([3]). *Let* $f : (X, \tau) \longrightarrow (Y, \mu)$ *be an* ω*-closed function in which its fibers are finite subsets of X. If* $\mathcal{A}$ *is an* ω*-locally finite family in* (X, τ)*, then* $f(\mathcal{A}) = \{f(A) : A \in \mathcal{A}\}$ *is an* ω*-locally finite in* (Y, μ)*.*

Proposition 7 ([4]). *A continuous function* $f : (X, \tau) \longrightarrow (Y, \mu)$ *is closed if and only if for every* $B \subseteq Y$ *and every open set* $A \subseteq X$ *with* $f^{-1}(B) \subseteq A$*, there exists an open set* $C \subseteq Y$ *with* $B \subseteq C$ *and* $f^{-1}(C) \subseteq A$*.*

Definition 3 ([4]). *A continuous function* $f : (X, \tau) \longrightarrow (Y, \mu)$ *is perfect if* (X, τ) *is a Hausdorff topological space, f is a closed function, and all fibers of f are compact subsets of X.*

Proposition 8 ([3]). *If f is a continuous closed map of a Hausdorff topological space* (X, τ) *onto a countably* ω*-paracompact space* (Y, μ) *on which its fibers are countable and countably compact, then* (X, τ) *is countably* ω*-paracompact.*

Definition 4 ([24]). *Let* ***P*** *be any topological property. We say that the locally finite sum theorem holds for* ***P*** *if the following is satisfied:*

If $\{F_\alpha : \alpha \in \Delta\}$ *is a locally finite closed covering of a topological space* (X, τ) *such that each* F_α *possesses the property* ***P****, then* (X, τ) *possesses the property* ***P****.*

Proposition 9 ([24]). *Let* ***P*** *be a property satisfying the following:*

(a) The disjoint sum of topological spaces possessing the property ***P*** *possesses* ***P****;*
(b) ***P*** *is preserved under closed continuous mappings with finite fibers.*

Then, the locally finite sum theorem holds for ***P****.*

Proposition 10 ([4]). *Disjoint sum of normal topological spaces is normal.*

3. Results

Definition 5. *A family* $\mathcal{A}$ *of subsets of a topological space* (X, τ) *is called* σ*-*ω*-locally finite if it can be represented as a countable union of* ω*-locally finite families.*

Definition 6. *A topological space* (X, τ) *is called:*

(a) σ*-*ω*-paracompact if every open cover has an open* σ*-*ω*-locally finite refinement;*
(b) Feebly ω*-paracompact if every open cover of X has an* ω*-locally finite refinement.*

Proposition 11. *Every* ω*-paracompact topological space is* σ*-*ω*-paracompact.*

Proof. It follows since every ω-locally finite family of subsets of a topological space is obviously σ-ω-locally finite. □

Theorem 1. *Every* σ*-*ω*-paracompact topological space is feebly* ω*-paracompact.*

Proof. Let (X, τ) be σ-ω-paracompact and let $\mathcal{A}$ be an open cover of X. Since (X, τ) is σ-ω-paracompact, then $\mathcal{A}$ has a σ-ω-locally finite open refinement $\mathcal{B} = \cup_{n=1}^{\infty} \mathcal{B}_n$ such that each $\mathcal{B}_n$ is ω-locally finite. For each $n \in \mathbb{N}$, set $C_n = \cup\{B : B \in \mathcal{B}_n\}$. Then, $\{C_n : n \in \mathbb{N}\}$ is an open cover of X. Put $D_1 = C_1$ and for each $n \in \mathbb{N} - \{1\}$, put $D_n = C_n - \cup_{k=1}^{n-1} C_k$ and let $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$. □

Claim 1. *$\mathcal{D}$ is locally finite.*

Proof of Claim 1. Let $x \in X$. Let n_x be the smallest natural number such that $x \in C_{n_x}$. Then, we have $C_{n_x} \in \tau$ and $C_{n_x} \cap D_m = \emptyset$ for all $m > n_x$ which means that C_{n_x} intersects at most $D_1, D_2, \ldots, D_{n_x}$. This ends the proof of Claim 1. □

Now, for each $n \in \mathbb{N}$, take $\mathcal{E}_n = \{B \cap D_n : B \in \mathcal{B}_n\}$ and let $\mathcal{E} = \cup_{n=1}^{\infty} \mathcal{E}_n$.

Claim 2. *(i) $\mathcal{E}$ covers X;*
(ii) $\mathcal{E}$ refines $\mathcal{A}$;
(iii) $\mathcal{E}$ is ω-locally finite.

Proof of Claim 2.
(i): Let $x \in X$. Let n_x be the smallest natural number such that $x \in C_{n_x}$. Since $C_{n_x} = \cup\{B : B \in \mathcal{B}_{n_x}\}$, then there exists $B_x \in \mathcal{B}_{n_x}$ such that $x \in B_x$. Thus, $x \in D_{n_x} \cap B_x \in \mathcal{E}_{n_x} \subseteq \mathcal{E}$.
(ii): Let $E \in \mathcal{E}$. Then, there exists $n_0 \in \mathbb{N}$ and $B_0 \in \mathcal{B}_{n_0}$ such that $E = B_0 \cap D_{n_0}$. Since $\mathcal{B}$ refines $\mathcal{A}$ and $B_0 \in \mathcal{B}_n$, then there exists $A_0 \in \mathcal{A}$ such that $B_0 \subseteq A_0$, and thus $E \subseteq A_0$.
(iii): Let $x \in X$. By Claim 1, there exists $O_x \in \tau$ such that $x \in O_x$ and O_x intersects at most $D_{n_1}, D_{n_2}, \ldots, D_{n_k}$ of $\mathcal{D}$. For each $i = 1, 2, \ldots, k$, we have $\mathcal{B}_{n_i}$ is ω-locally finite and so $\mathcal{E}_{n_i}$ is ω-locally finite. Thus, for each $i = 1, 2, \ldots, k$, there is an ω-open set O_i such that $x \in O_i$ and O_i intersects at most finitely many elements of $\mathcal{E}_{n_i}$. Let $O = O_x \cap \left(\cap_{i=1}^{k} O_i\right)$. Then, O is ω-open, $x \in O$, and O intersects at most finitely many elements of $\mathcal{E}$. □

By Claim 2, it follows that (X, τ) is feebly ω-paracompact.
As an application of Theorem 1, we introduce the following example:

Example 1. *Consider the topological space (X, τ) as in Example 6.2 of [3]. It is shown in [3] that (X, τ) is ω-paracompact. Hence, by Proposition 11 and Theorem 1, it follows that (X, τ) is feebly ω-paracompact.*

Recall that a topological space (X, τ) is locally countable if for each $x \in X$, there exists $U \in \tau$ such that $x \in U$ and U is countable.
It is well known that if (X, τ) is a locally countable topological space, then the topology of ω-open subsets of (X, τ) is the discrete topology on X.

Proposition 12. *Every locally countable topological space is feebly ω-paracompact.*

Proof. Let (X, τ) be locally countable and let $\mathcal{A}$ be an open cover of X. Let $\mathcal{B} = \{\{x\} : x \in X\}$. Then, $\mathcal{B}$ is a cover of X. □

Claim 3.
(i) $\mathcal{B}$ refines $\mathcal{A}$.
(ii) $\mathcal{B}$ is ω-locally finite.

Proof of Claim 3.
(i) Let $B \in \mathcal{B}$, say $B = \{y\}$ for some $y \in X$. Since $\mathcal{A}$ is a cover of X, then there exists $A \in \mathcal{A}$ such that $y \in A$. Thus, we have $A \in \mathcal{A}$ with $B \subseteq A$. It follows that $\mathcal{B}$ refines $\mathcal{A}$.

(ii) Let $y \in X$. Let $O = \{y\}$. Since (X, τ) is locally countable, then O is ω-open. Thus, we have $y \in O$, O is ω-open, and $\{B : O \cap B \neq \emptyset\} = \{\{y\}\}$ which is finite. It follows that $\mathcal{B}$ is ω-locally finite. $\square$

Therefore, (X, τ) is feebly ω-paracompact.

The next example shows that the converse of Theorem 1 is not true in general:

Example 2. *Let X be an uncountable set and let $p \in X$ be a fixed point. Let $\tau = \{\emptyset\} \cup \{U \subseteq X : p \in U\}$. Then (X, τ) is locally countable. Thus, by Proposition 12, (X, τ) is feebly ω-paracompact. Let $\mathcal{A} = \{\{p, x\} : x \in X - \{p\}\}$. Then, $\mathcal{A}$ is an open cover of X. We are going to show that every open cover of X which refines $\mathcal{A}$ is not σ-ω-locally finite. Let $\mathcal{B}$ be open cover of X which refines $\mathcal{A}$.*

Claim 4.
(i) $\mathcal{B} - \{\emptyset, \{p\}\} = \mathcal{A}$.
(ii) $\mathcal{B}$ is not σ-ω-locally finite.

Proof of Claim 4. (i) Let $B \in \mathcal{B} - \{\emptyset, \{p\}\}$. Since $B \in \tau$, then there is $x \in X - \{p\}$ such that $\{p, x\} \subseteq B$. Since $\mathcal{B}$ refines $\mathcal{A}$, then there is $A \in \mathcal{A}$ such that $B \subseteq A$. Therefore, $A = \{p, x\} = B$, and hence $B \in \mathcal{A}$. This shows that $\mathcal{B} - \{\emptyset, \{p\}\} \subseteq \mathcal{A}$. To see that $\mathcal{A} \subseteq \mathcal{B} - \{\emptyset, \{p\}\}$, let $A \in \mathcal{A}$. Then, there exists $x \in X - \{p\}$ such that $\{p, x\} = A$. Since $\mathcal{B}$ is a cover of X, then there is $B \in \mathcal{B}$ such that $x \in B$. Since $B \in \tau$, then $\{p, x\} \subseteq B$. Since $\mathcal{B}$ refines $\mathcal{A}$, then there is $A_0 \in \mathcal{A}$ such that $B \subseteq A_0$. Therefore, $A = \{p, x\} = B = A_0$ and hence $A \in \mathcal{B} - \{\emptyset, \{p\}\}$.
(ii) Suppose to the contrary that $\mathcal{B}$ is σ-ω-locally finite, then $\mathcal{B} = \cup_{n=1}^{\infty} \mathcal{B}_n$ where $\mathcal{B}_n$ is ω-locally finite for all $n \in \mathbb{N}$. Since X is uncountable, then there are $n_0 \in \mathbb{N}$ and $Y \subseteq X$ such that Y is uncountable and $\mathcal{B}_{n_0} - \{\emptyset, \{p\}\} = \{\{p, y\} : y \in Y\}$. Since $\mathcal{B}_{n_0}$ is ω-locally finite, then there is an ω-open set O in X such that $p \in O$ and $\{B : O \cap B \neq \emptyset\}$ is finite which is a contradiction. $\square$

It follows that (X, τ) is not σ-ω-paracompact.

The next example shows that the converse of Proposition 11 is not true in general:

Example 3. *Let X be a countable infinite set and let $p \in X$ be a fixed point. Let $\tau = \{\emptyset\} \cup \{U \subseteq X : p \in U\}$. To show that (X, τ) is σ-ω-paracompact, let $\mathcal{A}$ be an open cover of X. Let $\mathcal{B} = \{\{p\}\} \cup \{\{p, x\} : x \in X - \{p\}\}$. Then, $\mathcal{B}$ is an open cover of X.*

Claim 5.
(i) $\mathcal{B}$ refines $\mathcal{A}$.
(ii) $\mathcal{B}$ is σ-ω-locally finite.

Proof of Claim 5.
(i) Let $B \in \mathcal{B}$. If $B = \{p\}$, then choose $A \in \mathcal{A}$ such that $p \in A$, and thus we have $A \in \mathcal{A}$ with $B \subseteq A$. If $B = \{p, x\}$ for some $x \in X - \{p\}$, then choose $A \in \mathcal{A}$ such that $x \in A$, and thus we have $A \in \mathcal{A}$ with $B = \{p, x\} \subseteq A$. It follows that $\mathcal{B}$ refines $\mathcal{A}$.
(ii) Since X is a countable infinite set, then we can write $X = \{x_n : n \in \mathbb{N}\}$ with $x_1 = p$ and $x_n \neq x_m$ for $n \neq m$. Let $\mathcal{B}_1 = \{\{p\}\}$, and for every $n \in \mathbb{N} - \{1\}$ let $\mathcal{B}_n = \{\{p, x_n\}\}$. Then, $\mathcal{B} = \cup_{i=1}^{\infty} \mathcal{B}_n$. Since $\mathcal{B}_n$ is clearly ω-locally finite for all $n \in \mathbb{N}$, then $\mathcal{B}$ is σ-ω-locally finite. $\square$

It follows that (X, τ) is σ-ω-paracompact.

To show that (X, τ) is not ω-paracompact, let $\mathcal{A} = \{\{p, x\} : x \in X - \{p\}\}$. Then, $\mathcal{A}$ is an open cover of X. We are going to show that every open cover of X which refines $\mathcal{A}$ is not ω-locally finite. Let $\mathcal{B}$ be the open cover of X which refines $\mathcal{A}$.

Claim 6.
(i) $\mathcal{B}-\{\emptyset, \{p\}\} = \mathcal{A}$;
(ii) $\mathcal{B}$ *is not* ω*-locally finite.*

Proof of Claim 6.
(i) Let $B \in \mathcal{B}-\{\emptyset, \{p\}\}$. Since $B \in \tau$, then there exists $x \in X - \{p\}$ such that $\{p, x\} \subseteq B$. Since $\mathcal{B}$ refines $\mathcal{A}$, then there exists $A \in \mathcal{A}$ such that $B \subseteq A$. Therefore, $A = \{p, x\} = B$ and so $B \in \mathcal{A}$. This shows that $\mathcal{B}-\{\emptyset, \{p\}\} \subseteq \mathcal{A}$. To see that $\mathcal{A} \subseteq \mathcal{B}-\{\emptyset, \{p\}\}$, let $A \in \mathcal{A}$. Then, there exists $x \in X - \{p\}$ such that $\{p, x\} = A$. Since $\mathcal{B}$ is a cover of X, then there exists $B \in \mathcal{B}$ such that $x \in B$. Since $B \in \tau$, then $\{p, x\} \subseteq B$. Since $\mathcal{B}$ refines $\mathcal{A}$, then there exists $A_0 \in \mathcal{A}$ such that $B \subseteq A_0$. Therefore, $A = \{p, x\} = B = A_0$ and so $A \in \mathcal{B}-\{\emptyset, \{p\}\}$.
(ii) Suppose to the contrary that $\mathcal{B}$ is ω-locally finite. Since $\mathcal{B}$ is ω-locally finite, then there is an ω-open set O in X such that $p \in O$ and $\{B : O \cap B \neq \emptyset\}$ is finite which is a contradiction. □

It follows that (X, τ) is not σ-ω-paracompact.

Question 1. *Is every regular feebly* ω*-paracompact topological space* σ*-*ωa*-paracompact space?*

Question 2. *Is every regular* T_1 σ*-*ω*-paracompact space a topological space* ω*-paracompact space?*

Question 3. *If* (X, τ) *is a regular and feebly* ω*-paracompact topological space, then does every open cover of* X *have an* ω*-locally finite* ω*-closed refinement?*

Proof. Suppose that (X, τ) is regular and feebly ω-paracompact, and let $\mathcal{A}$ be an open cover of X. For each $x \in X$, choose $A_x \in \mathcal{A}$ such that $x \in A_x$. By regularity, for every $x \in X$, there exists $B_x \in \tau$ such that $x \in B_x \subseteq \overline{B_x} \subseteq A_x$. Let $\mathcal{B} = \{B_x : x \in X\}$. Since (X, τ) is feebly ω-paracompact, then $\mathcal{B}$ has an ω-locally finite refinement, say $\mathcal{C} = \{C_\alpha : \alpha \in \Delta\}$. It is not difficult to see that $\{\underline{C_\alpha} : \alpha \in \Delta\}$ is also ω-locally finite. □

Claim 7. $\{\underline{C_\alpha} : \alpha \in \Delta\}$ *refines* $\mathcal{A}$.

Proof of Claim 7. Let $\alpha \in \Delta$. Since $\mathcal{C}$ refines $\mathcal{B}$, there exists $x_0 \in X$ such that $C_\alpha \subseteq B_{x_0}$. Thus, we have:

$$C_\alpha \subseteq B_{x_0} \subseteq \underline{B_{x_0}} \subseteq \overline{B_{x_0}} \subseteq A_{x_0},$$

and hence $\underline{C_\alpha} \subseteq A_{x_0}$.

Therefore, $\{\underline{C_\alpha} : \alpha \in \Delta\}$ is an ω-locally finite ω-closed refinement of $\mathcal{A}$. This ends the proof. □

Question 4. *Let* (X, τ) *be a regular topological space with the property that every open cover of* X *has an* ω*-locally finite* ω*-closed refinement. Is it true that* (X, τ) *is feebly* ω*-paracompact?*

Theorem 2. *Every* σ*-*ω*-paracompact countably* ω*-paracompact topological space is* ω*-paracompact.*

Proof. Let (X, τ) be σ-ω-paracompact and countably ω-paracompact, and let $\mathcal{A}$ be an open cover of X. Since (X, τ) is σ-ω-paracompact, then there exists an open σ-ω-locally finite refinement $\mathcal{B} = \cup_{n=1}^{\infty} \mathcal{B}_n$, where $\mathcal{B}_n$ is ω-locally finite for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, set $C_n = \cup\{G : G \in \mathcal{B}_n\}$. Since $\{C_n : n \in \mathbb{N}\}$ is a countable open cover of X and (X, τ) is countably ω-paracompact, then by Proposition 5, there exists an ω-locally finite open cover $\{D_n : n \in \mathbb{N}\}$ of X such that for all $n \in \mathbb{N}$, $D_n \subseteq C_n$. Define $\mathcal{E} = \{G \cap D_n : G \in \mathcal{B}_n, n \in \mathbb{N}\}$. □

Claim 8. $\mathcal{E}$ *is* ω*-locally finite.*

Proof of Claim 8. Let $x \in X$. Since $\{D_n : n \in \mathbb{N}\}$ is a cover of X, there exists $m \in \mathbb{N}$ such that $x \in D_m$. Since $\{D_n : n \in \mathbb{N}\}$ is ω-locally finite, there exists an ω-open set O_x such that $x \in O_x$ and O_x meets at most finitely many members of $\{D_n : n \in \mathbb{N}\}$. Thus, there exists a natural number $k \geq m$ such that $O_x \cap D_n = \emptyset$ for all $n > k$. For each natural number $n \leq k$, $\mathcal{B}_n$ is ω-locally finite, and so there is an ω-open set O_n such that $x \in O_n$ and O_n meets at most finitely many members of $\mathcal{B}_n$. Let $U = O_x \cap \left(\cap_{i=1}^{n} O_n\right)$. Then, U is an ω-open set such that $x \in U$ and U meet at most finite members of $\mathcal{E}$. □

Claim 9. *$\mathcal{E}$ refines $\mathcal{A}$.*

Proof of Claim 9. Let $H \in \mathcal{E}$. Say, $H = G \cap D_{n_0}$, where $n_0 \in \mathbb{N}$ and $G \in \mathcal{B}_{n_0}$. Since $\mathcal{B}$ refines $\mathcal{A}$ and $G \in \mathcal{B}_{n_0} \subseteq \mathcal{B}$, then there exists $A_0 \in \mathcal{A}$ such that $G \subseteq A_0$, and hence $H \subseteq A_0$. This ends the proof. □

By Claims 8 and 9, it follows that (X, τ) is ω-paracompact.

Theorem 3. *Let (X, τ) be a topological space. Then, the following are equivalent:*
(a) (X, τ) is ω-paracompact;
(b) (X, τ) σ-ω-paracompact and countably ω-paracompact.

Proof.
(a) $\Longrightarrow$ (b) follows from Propositions 1 (c) and 11.
(b) $\Longrightarrow$ (a) follows from Theorem 2. □

Lemma 1. *Let $f : (X, \tau) \longrightarrow (Y, \mu)$ be an ω-continuous function in which its fibers are countable. If $\mathcal{A}$ is an ω-locally finite family of (Y, μ), then $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$ is an ω-locally finite family of (X, τ).*

Proof. Let $x \in X$. Since $\mathcal{A}$ is ω-locally finite, then there exists an ω-open set V of (Y, μ) such that $f(x) \in V$ and V meets only finitely many members of $\mathcal{A}$. Choose $U \in \mu$ and a countable subset $C \subseteq X$ such that $f(x) \in U - C \subseteq V$. Then, $U - C$ meets only finitely many members of $\mathcal{A}$, and $x \in f^{-1}(U) - f^{-1}(C) \subseteq f^{-1}(V)$. Since f is ω-continuous, then $f^{-1}(U)$ is ω-open. Furthermore, by assumption, $f^{-1}(C)$ is countable, and hence ω-closed. Set $G = f^{-1}(U) - f^{-1}(C)$. Then, G is ω-open in (X, τ) and $x \in G$. If for some $A \in \mathcal{A}$ we have $G \cap f^{-1}(A) \neq \emptyset$, then:

$$G \cap f^{-1}(A) = f^{-1}(U - C) \cap f^{-1}(A) = f^{-1}((U - C) \cap A) \neq \emptyset,$$

and hence $(U - C) \cap A \neq \emptyset$. Therefore, G meets only finitely many members of $f^{-1}(\mathcal{A})$. It follows that $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$ is ω-locally finite. □

Theorem 4. *Let f be a perfect mapping from (X, τ) onto (Y, μ) in which its fibers are countable subsets of X. If (Y, μ) is σ-ω-paracompact, then so is (X, τ).*

Proof. Let (Y, μ) be σ-ω-paracompact and let $\mathcal{A}$ be any open covering of X. For every $y \in Y$, $f^{-1}(y)$ is compact, and so there exists a finite subfamily $\mathcal{A}_y$ of $\mathcal{A}$ such that $f^{-1}(y) \subseteq \cup\{A : A \in \mathcal{A}_y\}$. Since f is a closed function, then by Proposition 7, for every $y \in Y$, there exists $B_y \in \mu$ such that $y \in B_y$ and $f^{-1}(B_y) \subseteq \cup\{A : A \in \mathcal{A}_y\}$. Since $\{B_y : y \in Y\}$ is an open cover of Y and (Y, μ) is σ-ω-paracompact, then $\{B_y : y \in Y\}$ has a σ-ω-locally finite open refinement $\mathcal{C} = \cup_{n=1}^{\infty} \mathcal{C}_n$ such that each $\mathcal{C}_n$ is ω-locally finite. For each $n \in \mathbb{N}$, set $\mathcal{D}_n = \{f^{-1}(C) : C \in \mathcal{C}_n\}$ and $\mathcal{D} = \cup_{n=1}^{\infty} \mathcal{D}_n$. Since f is continuous, then each $\mathcal{D}_n \subseteq \tau$ and so $\mathcal{D} \subseteq \tau$. Since $\mathcal{C}$ covers Y, then $\mathcal{D}$ covers X. Furthermore, by Lemma 1, each $\mathcal{D}_n$ is ω-locally finite, and hence $\mathcal{D}$ is σ-ω-locally finite. Since $\mathcal{C}$ refines $\{B_y : y \in Y\}$, then for every $C \in \mathcal{C}$, there exists $y(C) \in Y$ such that $C \subseteq B_{y(C)}$. For each $n \in \mathbb{N}$, choose:

$$\mathcal{E}_n = \left\{f^{-1}(C) \cap A : A \in \mathcal{A}_{y(C)}, C \in \mathcal{C}_n\right\}, \text{ and let } \mathcal{E} = \cup_{n=1}^{\infty} \mathcal{E}_n$$

□

Then, $\mathcal{E} \subseteq \tau$.

Claim 10.
(i) $\mathcal{E}$ covers X;
(ii) $\mathcal{E}$ refines $\mathcal{A}$;
(iii) Each $\mathcal{E}_n$ is ω-locally finite;

Proof of Claim 10.
(i) Let $x \in X$. Since $\mathcal{D}$ covers X, there exist $n_x \in \mathbb{N}$ and $C_x \in \mathcal{C}_{n_x}$ such that $x \in f^{-1}(C_x) \subseteq f^{-1}\left(B_{y(C_x)}\right) \subseteq \cup\left\{A : A \in \mathcal{A}_{y(C_x)}\right\}$. Choose $A_x \in \mathcal{A}_{y(C_x)}$ such that $x \in A_x$. Thus, we have $x \in f^{-1}(C_x) \cap A_x$ where $f^{-1}(C_x) \cap A_x \in \mathcal{E}$. It follows that $\mathcal{E}$ covers X.
(ii) Obvious.
(iii) Let $x \in X$. Since $\mathcal{D}_n$ is ω-locally finite, then there exists an ω-open set O_x with $x \in O_x$, and there exists a finite subcollection $\mathcal{C}_x \subseteq \mathcal{C}_n$ such that for all $C \in \mathcal{C} - \mathcal{C}_x, O_x \cap f^{-1}(C) = \emptyset$. It follows that O_x meets at most the finite subcollection $\left\{f^{-1}(C) \cap A : A \in \mathcal{A}_{y(C)}, C \in \mathcal{C}_x\right\}$ of $\mathcal{E}_n$. □

By the above Claim, it follows that $\mathcal{E}$ is a σ-ω-locally finite open refinement of $\mathcal{A}$. Hence, (X, τ) is σ-ω-paracompact.

Corollary 1 ([18]). *Let f be a perfect mapping from (X, τ) onto (Y, μ) in which its fibers are countable subsets of X. If (Y, μ) is ω-paracompact, then so is (X, τ).*

Proof. By Theorem 3, (Y, μ) is σ-ω-paracompact and countably ω-paracompact. Then, by Proposition 8 and Theorem 4, we have (X, τ) is countably ω-paracompact and σ-ω-paracompact. Thus, again by Theorem 3, we have (X, τ) is ω-paracompact. □

Theorem 5. *Let f be a perfect mapping from (X, τ) onto (Y, μ) in which its fibers are finite subsets of X. If (X, τ) is ω-paracompact and normal, then so is (Y, μ).*

Proof. By Proposition 1 (c), (X, τ) is countably ω-paracompact. By Propositions 3 and 4, (Y, μ) is normal and countably paracompact. Therefore, by Theorem 3, it is sufficient to see that (Y, μ) is σ-ω-paracompact. □

Let $\{A_\alpha : \alpha \in \Delta\}$ be any open cover of (Y, μ) and let $<$ be a well ordering of Δ. Then, $\{f^{-1}(A_\alpha) : \alpha \in \Delta\}$ is an open covering of X, and so there is an ω-locally finite open cover of X, $\mathcal{B}_1 = \{H_{\alpha,1} : \alpha \in \Delta\}$ such that for all $\alpha \in \Delta$, we have $H_{\alpha,1} \subseteq f^{-1}(A_\alpha)$. For each $\alpha \in \Delta$, set:

$$S_{\alpha,2} = f^{-1}(A_\alpha) - f^{-1}\big(f\big(X - \cup_{\beta \geq \alpha} H_{\beta,1}\big)\big) \subseteq f^{-1}(A_\alpha).$$

Then, $S_{\alpha,2} \in \tau$ and $S_{\alpha,2} \subseteq f^{-1}(A_\alpha)$. For every $x \in X$, denote the smallest element $\alpha \in \Delta$ such that $x \in f^{-1}(A_\alpha)$ by $\alpha(x)$. Let:

$$E_{\alpha,1} = X - \cup_{\beta \geq \alpha} H_{\beta,1}.$$

Then:

$$E_{\alpha(x),1} \subseteq \cup_{\beta < \alpha(x)} H_{\beta,1} \subseteq \cup_{\beta < \alpha(x)} f^{-1}(A_\beta).$$

So:

$$f^{-1}\left(f\left(E_{\alpha(x),1}\right)\right) \subseteq \cup_{\beta < \alpha(x)} f^{-1}\left(f\left(f^{-1}(A_\beta)\right)\right) = \cup_{\beta < \alpha(x)} f^{-1}(A_\beta).$$

Since $x \notin \cup_{\beta<\alpha(x)} f^{-1}(A_\beta)$, then $x \notin f^{-1}\Big(f\Big(E_{\alpha(x),1}\Big)\Big)$. It follows that $x \in S_{\alpha(x),2}$, and hence $\{S_{\alpha,2} : \alpha \in \Delta\}$ is a cover of X. Therefore, there is an ω-locally finite open cover of X, $\mathcal{B}_2 = \{H_{\alpha,2} : \alpha \in \Delta\}$ such that for all $\alpha \in \Delta$ we have $H_{\alpha,2} \subseteq S_{\alpha,2}$. For all $\alpha \in \Delta$, it is easy to check that:

$$H_{\alpha,2} \subseteq f^{-1}(A_\alpha) \text{ and } f(H_{\alpha,2}) \cap f(E_{\alpha,1}) = \varnothing.$$

Now, we can inductively find an ω-locally finite open cover of X, $\mathcal{B}_n = \{H_{\alpha,n} : \alpha \in \Delta\}$ satisfying the conditions:
(1) $H_{\alpha,n} \subseteq f^{-1}(A_\alpha)$ for $\alpha \in \Delta$ and $n \in \mathbb{N}$.
(2) $f(H_{\alpha,n}) \cap f(E_{\alpha,n-1}) = \varnothing$ where $E_{\alpha,n-1} = X - \cup_{\beta\geq\alpha} H_{\beta,n-1}$ for $n > 1$.

Claim 11. *For every $\alpha_0 \in \Delta$ and $n \in \mathbb{N}$ we have:*
(3) $E_{\alpha_\circ,n} = X - \cup_{\alpha\geq\alpha_0} H_{\alpha,n} \subseteq \cup_{\alpha<\alpha_0}\big(X - \cup_{\beta>\alpha} H_{\beta,n}\big)$.

Proof of Claim 11. Let $x \in E_{\alpha_0,n}$. Then, $x \notin \cup_{\alpha\geq\alpha_0} H_{\alpha,n}$ and so, there exists $\alpha < \alpha_0$ such that $x \in H_{\alpha,n}$. Denote the maximal element in Δ such that $x \in H_{\alpha,n}$ by α_1. Then, $\alpha_1 < \alpha_0$ and $x \in X - \cup_{\beta>\alpha_1} H_{\beta,n}$. Therefore, $x \in \cup_{\alpha<\alpha_0}\big(X - \cup_{\beta>\alpha} H_{\beta,n}\big)$. $\square$

As in Claim 11, one can easily see that:
(4) $X = \cup_{\alpha\in\Delta}\big(X - \cup_{\beta>\alpha} H_{\beta,n}\big)$.
Consider the open sets $V_{\alpha,n} = Y - f(X - H_{\alpha,n})$. Then, for $\alpha \in \Delta$ and $n :\in \mathbb{N}$,

$$f^{-1}(V_{\alpha,n}) = X - f^{-1}(f(X - H_{\alpha,n})) \subseteq H_{\alpha,n} \text{ and } V_{\alpha,n} \subseteq f(H_{\alpha,n}).$$

By Proposition 6, it follows that $\{f(H_{\alpha,n}) : \alpha \in \Delta\}$ is ω-locally finite for each $n \in \mathbb{N}$. Therefore, $\mathcal{V}_n = \{V_{\alpha,n} : \alpha \in \Delta\}$ is ω-locally finite for each $n \in \mathbb{N}$.

Claim 12.
(a) $\cup_{i=1}^{\infty}\mathcal{V}_i$ covers Y.
(b)$\cup_{i=1}^{\infty}\mathcal{V}_i$ refines $\{A_\alpha : \alpha \in \Delta\}$.

Proof of Claim 12.
(a) Let $y \in Y$. By (4), the smallest element α in Δ such that $y \in f\big(X - \cup_{\beta>\alpha} H_{\beta,n}\big)$ for some $n \in \mathbb{N}$ exists, denote it by $\alpha(y)$ and take an integer $n(y)$ such that $y \in f\Big(X - \cup_{\beta>\alpha(y)} H_{\beta,n(y)-1}\Big)$. Now, for $\alpha > \alpha(y)$:

$$\cup_{\beta\geq\alpha} H_{\beta,n(y)-1} \subseteq \cup_{\beta>\alpha(y)} H_{\beta,n(y)-1}, \text{ and so } X - \cup_{\beta>\alpha(y)} H_{\beta,n(y)-1} \subseteq X - \cup_{\beta\geq\alpha} H_{\beta,n(y)-1}.$$

Thus:

$$f\Big(X - \cup_{\beta>\alpha(y)} H_{\beta,n(y)-1}\Big) \subseteq f\Big(X - \cup_{\beta\geq\alpha} H_{\beta,n(y)-1}\Big) = f\Big(E_{\alpha,n(y)-1}\Big)$$

and hence, $y \in f\Big(E_{\alpha,n(y)-1}\Big)$ for all $\alpha > \alpha(y)$, and by virtue of (2):

$$y \notin \cup_{\alpha>\alpha(y)} f\Big(H_{\alpha,n(y)}\Big) = f\Big(\cup_{\alpha>\alpha(y)} H_{\alpha,n(y)}\Big)$$

(5) *i.e.*, $f^{-1}(y) \cap \Big(\cup_{\alpha>\alpha(y)} H_{\alpha,n(y)}\Big) = \varnothing$.
On the other hand, by virtue of (3):

$$\begin{aligned} f\Big(X - \cup_{\alpha\geq\alpha(y)} H_{\alpha,n(y)}\Big) &\subseteq f\Big(\cup_{\alpha<\alpha(y)}\Big(X - \cup_{\beta>\alpha} H_{\beta,n(y)}\Big)\Big) \\ &= \cup_{\alpha<\alpha(y)} f\Big(X - \cup_{\beta>\alpha} H_{\beta,n(y)}\Big). \end{aligned}$$

By the minimality of $\alpha(y)$, $y \notin \cup_{\alpha<\alpha(y)} f\Big(X - \cup_{\beta>\alpha} H_{\beta,n(y)}\Big)$. Thus:

$$y \notin f\left(X - \cup_{\alpha \geq \alpha(y)} H_{\alpha,n(y)}\right)$$

and hence, we have:

(6) $f^{-1}(y) \subseteq \cup_{\alpha \geq \alpha(y)} H_{\alpha,n(y)}$.

By (5) and (6), $f^{-1}(y) \subseteq H_{\alpha(y),n(y)}$ and $y \in V_{\alpha(y),n(y)}$. This shows that $\cup_{n=1}^{\infty} \mathcal{V}_i$ covers Y.

(b) Since for all $\alpha \in \Delta$ and $n \in \mathbb{N}$, $H_{\alpha,n} \subseteq f^{-1}(A_\alpha)$ and $f^{-1}(V_{\alpha,n}) \subseteq H_{\alpha,n}$, then we have $f^{-1}(V_{\alpha,n}) \subseteq f^{-1}(A_\alpha)$, and hence $V_{\alpha,n} \subseteq A_\alpha$. □

As an application of Theorem 5, we introduce the following example:

Example 4. *Let $X = [1,2] \cup [3,4]$, $Y = [1,3]$, and τ and σ be the usual topologies on X and Y, respectively. Define $f : (X,\tau) \longrightarrow (Y,\mu)$ by $f(x) = x$ if $x \in [1,2]$ and $f(x) = x - 1$ if $x \in [3,4]$. Then, f is a perfect mapping from (X,τ) onto (Y,μ), and its fibers are finite subsets of X. Since both of (X,τ) and (Y,μ) are compact and Hausdorff, then both (X,τ) and (Y,μ) are ω-paracompact and normal.*

Proposition 13. *A disjoint sum of ω-paracompact topological spaces is ω-paracompact.*

Proof. Let $\{(X_\alpha,\tau_\alpha) : \alpha \in \Delta\}$ be a disjoint family of ω-paracompact topological spaces, and denote the disjoint sum of this family by (X,τ). For each $\alpha \in \Delta$,$\{A \cap X_\alpha : A \in \mathcal{A}\}$ is an open cover of X_α, and so it has an ω-locally finite open refinement of $\mathcal{B}_\alpha$. Since $\{X_\alpha : \alpha \in \Delta\}$ is a disjoint family and each $\mathcal{B}_\alpha$ is ω-locally finite, then $\cup_{\alpha\in\Delta}\mathcal{B}_\alpha$ is an ω-locally finite open refinement of $\mathcal{A}$. Therefore, (X,τ) is ω-paracompact. □

Theorem 6. *If $\{F_\alpha : \alpha \in \Delta\}$ is a locally finite closed covering of a topological space (X,τ) such that each F_α is ω-paracompact and normal, then (X,τ) is ω-paracompact and normal.*

Proof. It follows from Definition 4, Propositions 9, 10, and 13, and Theorem 5. □

Corollary 2. *The locally finite sum theorem holds for the property ω-paracompact normal.*

4. Conclusions

We define the notions of σ-ω-paracompactness and feebly ω-paracompactness as two new generalizations of paracompactness. We prove that feebly ω-paracompactness is strictly weaker than each of σ-ω-paracompactness and local countability. We prove the following main results: (1) countably ω-paracompact σ-ω-paracompact topological spaces are ω-paracompact; (2) ω-paracompactness is inverse invariant under perfect mappings with countable fibers; (3) if $\mathcal{A}$ is a locally finite closed covering of a topological space (X,τ) with each $A \in \mathcal{A}$ being ω-paracompact and normal, then (X,τ) is ω-paracompact. In future studies, the following topics could be considered: (1) solving the three open questions raised in this paper; and (2) investigate the behavior of our new notions under product topological spaces.

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