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New Expressions for Sums of Products of the Catalan Numbers

Conghui Xie [1,2,*] and Yuan He [3]

[1] College of Information Science and Technology/College of Cyber Security, Jinan University, Guangzhou 510632, China
[2] Faculty of Science, Kunming University of Science and Technology, Kunming 650500, China
[3] School of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, China; hyyhe@aliyun.com
* Correspondence: xch1997@vip.163.com

Abstract: In this paper, we perform a further investigation for the Catalan numbers. By making use of the method of derivatives and some properties of the Bell polynomials, we establish two new expressions for sums of products of arbitrary number of the Catalan numbers. The results presented here can be regarded as the development of some known formulas.

Keywords: Catalan numbers; Bell polynomials; combinatorial identities

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1. Introduction

As is well known, the famous Catalan numbers C_n count the number of ways to triangulate a regular polygon with $n+2$ sides. These natural numbers are usually given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad (n \geq 0). \tag{1}$$

In fact, the Catalan numbers can be also defined by the generating function

$$\frac{2}{1+\sqrt{1-4t}} = \sum_{n=0}^{\infty} C_n t^n, \tag{2}$$

or defined recursively by

$$C_0 = 1, \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad (n \geq 0). \tag{3}$$

The Catalan numbers are probably the most frequently encountered sequence of numbers, and have been studied in depth in many papers and monographs; see, for example [1–5].

Recently, some authors have taken lively interest in dealing with some identities for the Catalan numbers and their generalizations. For example, Mahmoud and Qi [6] used the generalized hypergeometric series to find three identities for the Catalan-Qi numbers, which generalize Touchard's [7], Jonah's [8] and Koshy's [9] identities for the Catalan numbers. Qi and Cerone [10] used the properties of the Gamma function to establish some connections between the Catalan-Qi numbers and the Fuss–Catalan numbers. Kim and Kim [11] explored the family of inhomogeneous linear differential equations arising from the generating function of the Catalan–Daehee numbers, by virtue of which they obtained some new identities for the Catalan–Daehee numbers and the Catalan numbers. We here mention [12–15] for some interesting identities for the Catalan polynomials and numbers. More recently, Zhang and Chen [16] considered the calculating problem of the sums of products of arbitrary number of the Catalan numbers, and showed that for positive integer m and non-negative integer n,

$$\sum_{\substack{k_1+\cdots+k_{2m+1}=n\\ k_1,\dots,k_{2m+1}\ge 0}} C_{k_1}\cdots C_{k_{2m+1}} = \frac{1}{(2m)!}\sum_{k=0}^{m} A(m,k)\sum_{j=0}^{\min(n,k)}\binom{k}{j}(-4)^j\frac{(m+n+k-j)!}{(n-j)!}C_{m+n+k-j}, \tag{4}$$

and

$$\sum_{\substack{k_1+\cdots+k_{2m}=n\\ k_1,\dots,k_{2m}\ge 0}} C_{k_1}\cdots C_{k_{2m}} = \frac{1}{(2m-1)!}\sum_{k=0}^{m-1} B(m,k+1)\sum_{j=0}^{n}\binom{k+\frac{1}{2}}{j}(-4)^j\frac{(m+n+k-j)!}{(n-j)!} \times C_{m+n+k-j}, \tag{5}$$

where $\binom{\alpha}{k}$ are the general binomial coefficients given for complex number α and non-negative integer k by

$$\binom{\alpha}{0}=1,\quad \binom{\alpha}{k}=\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}\quad (k\ge 1),$$

$A(m,k)$ is a sequence of numbers satisfying the recurrence relation

$$A(m+1,k)=A(m,k-1)-(8k+2)A(m,k)+(4k+2)(4k+4)A(m,k+1)$$

with the initial values $A(1,0)=-2, A(m,m)=1$, and $B(m,k+1)$ is another sequence of numbers satisfying the recurrence relation

$$B(m+1,k)=B(m,k-1)-(8k-2)B(m,k)+4k(4k+2)B(m,k+1)$$

with the initial values $B(m,0)=0, B(m,m)=1$. After that, Zhang and Chen [17] used inductive hypotheses to determine the explicit expressions of $A(m,k)$ and $B(m,k+1)$ appearing in (4) and (5).

Motivated and inspired by the work of the above authors, we perform a further investigation for the Catalan numbers in this paper. By making use of the method of derivatives and some properties of the Bell polynomials, we establish the following two new expressions for sums of products of arbitrary number of the Catalan numbers.

Theorem 1. *Let m,n be non-negative integers with $m\ge 1$. Then*

$$\sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m\ge 0}} C_{k_1}\cdots C_{k_m} = (-4)^n\sum_{k=0}^{n}\binom{m+k-1}{k}\frac{1}{2^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{\frac{j}{2}}{n}. \tag{6}$$

Theorem 2. *Let m,n be positive integers. Then*

$$\sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\dots,k_m\ge 0}} C_{k_1}\cdots C_{k_m} = \sum_{k=1}^{n}\left(\frac{(m+1)(n+1-k)}{n}-1\right)(-4)^{k-1}C_{n+1-k}\sum_{i=0}^{k-1}\binom{m+i-1}{i}\frac{1}{2^i} \times\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{\frac{j}{2}}{k-1}. \tag{7}$$

The following second and third section is contributed to the detailed proof of Theorems 1 and 2, respectively.

2. The Proof of Theorem 1

We shall give two different proofs of Theorem 1 in this section.

The First Proof of Theorem 1. Clearly, from (2) and the Cauchy product, we discover that for positive integer m and non-negative integer n,

$$n! \sum_{\substack{k_1+\cdots+k_m=n \\ k_1,\ldots,k_m\geq 0}} C_{k_1}\cdots C_{k_m} = 2^m \frac{\partial^n}{\partial t^n}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right)\bigg|_{t=0}. \tag{8}$$

Let $F(u) = \frac{1}{(1+u)^m}$ and $G(t) = \sqrt{1-4t}$ with $u = G(t)$. The famous Faà di Bruno formula states that for non-negative integer n (see, e.g., ([18], pp. 137–139)),

$$F^{(n)}(G(t)) = \sum_{k=0}^{n} F^{(k)}(u) B_{n,k}\left(G^{(1)}(t), G^{(2)}(t), \ldots, G^{(n-k+1)}(t)\right), \tag{9}$$

where $f^{(n)}(x)$ denotes the n-th derivative for the function $f(x)$ with respect to x, and $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ are the partial Bell polynomials. Observe that for non-negative integer k,

$$\begin{aligned} F^{(k)}(u) &= (-m)(-m-1)\cdots(-m-k+1)(1+u)^{-m-k} \\ &= (-1)^k k! \binom{m+k-1}{k} \frac{1}{(1+u)^{m+k}}, \end{aligned}$$

which implies

$$F^{(k)}(u)\bigg|_{t=0} = (-1)^k k! \binom{m+k-1}{k} \frac{1}{2^{m+k}}. \tag{10}$$

Similarly, for positive integer k,

$$\begin{aligned} G^{(k)}(t)\bigg|_{t=0} &= \frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-k+1\right)(1-4t)^{\frac{1}{2}-k}(-4)^k\bigg|_{t=0} \\ &= -2^k(2k-3)!!, \end{aligned} \tag{11}$$

where $(2n-1)!!$ is the double factorial given for non-negative integer n by $(-1)!! = 1$ and $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$ for positive integer n. It follows from (11) that

$$\begin{aligned} & B_{n,k}\left(G^{(1)}(t), G^{(2)}(t), \ldots, G^{(n-k+1)}(t)\right)\bigg|_{t=0} \\ &= B_{n,k}\left(-2(-1)!!, -2^2 1!!, \ldots, -2^{n-k+1}(2(n-k+1)-3)!!\right) \\ &= (-1)^k 2^n B_{n,k}\left((-1)!!, 1!!, \ldots, (2(n-k+1)-3)!!\right), \end{aligned} \tag{12}$$

where we have used the formula (see, e.g., ([18], p. 135))

$$B_{n,k}(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}).$$

We next evaluate the special values of the partial Bell polynomials on the right side of (12). Since the partial Bell polynomials can be defined by the series

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \frac{t^n}{n!}, \tag{13}$$

by taking $x_m = (2m-3)!!$ in (13), we have

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty}(2m-3)!!\frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}\big((-1)!!, 1!!, \ldots, (2(n-k+1)-3)!!\big)\frac{t^n}{n!}. \tag{14}$$

It is easy to check that the following Taylor series are complete

$$\sum_{m=1}^{\infty}(2m-3)!!\frac{t^m}{m!} = 1-\sqrt{1-2t}. \tag{15}$$

It follows from (14) and (15) that

$$B_{n,k}\big((-1)!!, 1!!, \ldots, (2(n-k+1)-3)!!\big) = \frac{1}{k!}\frac{\partial^n}{\partial t^n}(1-\sqrt{1-2t})^k\bigg|_{t=0}. \tag{16}$$

Notice that for complex number α,

$$(1+t)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}t^n, \tag{17}$$

which together with the binomial theorem gives

$$\begin{aligned}(1-\sqrt{1-2t})^k &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j(1-2t)^{\frac{j}{2}} \\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{i=0}^{\infty}\binom{\frac{j}{2}}{i}(-2)^i t^i. \end{aligned} \tag{18}$$

Hence, we get from (18) that

$$\frac{\partial^n}{\partial t^n}(1-\sqrt{1-2t})^k\bigg|_{t=0} = (-2)^n n! \sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{\frac{j}{2}}{n}. \tag{19}$$

Combing (16) and (19) gives

$$\begin{aligned}&B_{n,k}\big((-1)!!, 1!!, \ldots, (2(n-k+1)-3)!!\big) \\ &= \frac{(-2)^n n!}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{\frac{j}{2}}{n}.\end{aligned} \tag{20}$$

It follows from (9), (10), (12) and (20) that

$$F^{(n)}(G(t)) = \frac{(-4)^n n!}{2^m}\sum_{k=0}^{n}\binom{m+k-1}{k}\frac{1}{2^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{\frac{j}{2}}{n}. \tag{21}$$

Thus, by equating (8) and (21), we complete of the proof of Theorem 1.

We here refer the interested reader to [19–24] for the special values and applications for the partial Bell polynomials. It is worth noticing that two equivalent versions of (20) have been obtained by Qi et al. ([21], Theorems 1 and 2). We rediscover (20) that stems from the following alternate proof of Theorem 1. □

The Second Proof of Theorem 1. It is obvious from (17) and the binomial theorem that for positive integer m,

$$\begin{aligned} \frac{1}{(1+\sqrt{1-4t})^m} &= \sum_{j=0}^{\infty}\binom{-m}{j}(1+\sqrt{1-4t}-1)^j \\ &= \sum_{j=0}^{\infty}\binom{-m}{j}\sum_{i=0}^{j}\binom{j}{i}(\sqrt{1-4t}-1)^i. \end{aligned} \tag{22}$$

Changing the order of the summations on the right side of (22) gives

$$\frac{1}{(1+\sqrt{1-4t})^m} = \sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\binom{-m}{j}\binom{j}{i}(\sqrt{1-4t}-1)^i. \tag{23}$$

Since $\sqrt{1-4t}-1$ can be written as

$$\sqrt{1-4t}-1 = \sum_{n=1}^{\infty} g_n t^n,$$

where g_n is a sequence of numbers, $(\sqrt{1-4t}-1)^i$ is divisible by t^i for non-negative integer i. It follows that for non-negative integer n,

$$\begin{aligned} &\frac{\partial^n}{\partial t^n}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right)\Bigg|_{t=0} \\ &= \frac{\partial^n}{\partial t^n}\left(\sum_{i=0}^{n}\sum_{j=i}^{\infty}\binom{-m}{j}\binom{j}{i}(\sqrt{1-4t}-1)^i\right)\Bigg|_{t=0} \\ &= \frac{\partial^n}{\partial t^n}\left(\sum_{i=0}^{n}\sum_{j=i}^{\infty}\binom{-m}{j}\binom{j}{i}\sum_{k=0}^{i}\binom{i}{k}(-1)^{i-k}(1-4t)^{\frac{k}{2}}\right)\Bigg|_{t=0} \\ &= (-4)^n n!\sum_{i=0}^{n}\sum_{j=i}^{\infty}\binom{-m}{j}\binom{j}{i}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{k}\binom{\frac{k}{2}}{n}. \end{aligned} \tag{24}$$

Observe that for non-negative integers i, j with $j \geq i$,

$$\binom{-m}{j} = (-1)^j\binom{m+j-1}{j}, \tag{25}$$

and

$$\binom{m+j-1}{j}\binom{j}{i} = \binom{m+i-1}{i}\binom{m+j-1}{j-i}. \tag{26}$$

So from (25) and (26), we have

$$\begin{aligned} \sum_{j=i}^{\infty}\binom{-m}{j}\binom{j}{i} &= \binom{m+i-1}{i}\sum_{j=i}^{\infty}(-1)^j\binom{m+j-1}{j-i} \\ &= (-1)^i\binom{m+i-1}{i}\sum_{j=0}^{\infty}(-1)^j\binom{m+j+i-1}{j} \\ &= (-1)^i\binom{m+i-1}{i}\sum_{j=0}^{\infty}\binom{-m-i}{j} \\ &= (-1)^i\binom{m+i-1}{i}\frac{1}{2^{m+i}}. \end{aligned} \tag{27}$$

By putting (27) to the right side of (24), we obtain

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right)\bigg|_{t=0}\\&=\frac{(-4)^n n!}{2^m}\sum_{i=0}^{n}\binom{m+i-1}{i}\frac{1}{2^i}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{\frac{k}{2}}{n}.\end{aligned}\tag{28}$$

Thus, equating (8) and (28) gives

$$\sum_{\substack{k_1+\cdots+k_m=n\\k_1,\ldots,k_m\geq 0}}C_{k_1}\cdots C_{k_m}=(-4)^n\sum_{i=0}^{n}\binom{m+i-1}{i}\frac{1}{2^i}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{\frac{k}{2}}{n},$$

as desired. This completes the proof of Theorem 1. □

3. The Proof of Theorem 2

It is easily seen that for positive integer n,

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left(\frac{1}{(1+\sqrt{1-4t})^{m+1}}\right)\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left\{\frac{\partial}{\partial t}\left(\frac{1}{(1+\sqrt{1-4t})^{m+1}}\right)\right\}\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left\{\frac{m+1}{(1+\sqrt{1-4t})^m}\frac{\partial}{\partial t}\left(\frac{1}{1+\sqrt{1-4t}}\right)\right\}.\end{aligned}\tag{29}$$

It follows from (29) and the Leibniz rule that

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left(\frac{1}{(1+\sqrt{1-4t})^{m+1}}\right)\\&=(m+1)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\partial^{k+1}}{\partial t^{k+1}}\left(\frac{1}{1+\sqrt{1-4t}}\right)\\&\quad\times\frac{\partial^{n-1-k}}{\partial t^{n-1-k}}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right)\\&=(m+1)\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{\partial^{k}}{\partial t^{k}}\left(\frac{1}{1+\sqrt{1-4t}}\right)\\&\quad\times\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right).\end{aligned}\tag{30}$$

On the other hand, for positive integer n,

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left(\frac{1}{(1+\sqrt{1-4t})^{m+1}}\right)\\&=\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^{k}}{\partial t^{k}}\left(\frac{1}{1+\sqrt{1-4t}}\right)\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right).\end{aligned}\tag{31}$$

So from (30) and (31), we obtain that for positive integer n,

$$\begin{aligned}&(m+1)\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{\partial^{k}}{\partial t^{k}}\left(\frac{1}{1+\sqrt{1-4t}}\right)\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right)\\&=\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^{k}}{\partial t^{k}}\left(\frac{1}{1+\sqrt{1-4t}}\right)\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right).\end{aligned}\tag{32}$$

It is obvious that (32) can be rewritten as

$$\begin{aligned}
&\frac{1}{1+\sqrt{1-4t}}\frac{\partial^n}{\partial t^n}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right) \\
&= (m+1)\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{\partial^k}{\partial t^k}\left(\frac{1}{1+\sqrt{1-4t}}\right)\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right) \\
&\quad -\sum_{k=1}^{n}\binom{n}{k}\frac{\partial^k}{\partial t^k}\left(\frac{1}{1+\sqrt{1-4t}}\right)\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right) \\
&= \sum_{k=1}^{n}\binom{n}{k}\left(\frac{k(m+1)}{n}-1\right)\frac{\partial^k}{\partial t^k}\left(\frac{1}{1+\sqrt{1-4t}}\right) \\
&\quad \times\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1+\sqrt{1-4t})^m}\right).
\end{aligned} \tag{33}$$

If we multiply both sides of (33) by 2^m and then take $t=0$, we have

$$\begin{aligned}
&\frac{\partial^n}{\partial t^n}\left(\frac{2^m}{(1+\sqrt{1-4t})^m}\right)\bigg|_{t=0} \\
&= \sum_{k=1}^{n}\binom{n}{k}\left(\frac{k(m+1)}{n}-1\right)\frac{\partial^k}{\partial t^k}\left(\frac{2}{1+\sqrt{1-4t}}\right)\bigg|_{t=0} \\
&\quad \times\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{2^m}{(1+\sqrt{1-4t})^m}\right)\bigg|_{t=0}.
\end{aligned} \tag{34}$$

By applying (8) to both sides of (34), with the help of Theorem 1, we arrive at

$$\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_m=n\\ k_1,\ldots,k_m\geq 0}} C_{k_1}\cdots C_{k_m} \\
&= \sum_{k=1}^{n}\left(\frac{k(m+1)}{n}-1\right)C_k\sum_{\substack{k_1+\cdots+k_m=n-k\\ k_1,\ldots,k_m\geq 0}} C_{k_1}\cdots C_{k_m} \\
&= \sum_{k=1}^{n}\left(\frac{(m+1)(n+1-k)}{n}-1\right)C_{n+1-k}\sum_{\substack{k_1+\cdots+k_m=k-1\\ k_1,\ldots,k_m\geq 0}} C_{k_1}\cdots C_{k_m} \\
&= \sum_{k=1}^{n}\left(\frac{(m+1)(n+1-k)}{n}-1\right)(-4)^{k-1}C_{n+1-k}\sum_{i=0}^{k-1}\binom{m+i-1}{i}\frac{1}{2^i} \\
&\quad \times\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{\frac{j}{2}}{k-1}.
\end{aligned} \tag{35}$$

This concludes the proof of Theorem 2.

4. Conclusions

In this paper, we have used some analytic and combinatorial methods to establish two new expressions for sums of products of arbitrary number of the Catalan numbers. One can be regarded as the closed formula for sums of products of arbitrary number of the Catalan numbers, and another is very analogous to the results shown in [16,17]. The methods presented here may be applied to explore the sum relations for some families of special polynomials, for example, Dickson polynomials, Laguerre polynomials, Hermite polynomials, and so on. However, it seems that it is difficult to establish some analogous expressions for sums of products of arbitrary number of the q-Bernoulli polynomials introduced in [25,26] and the q-Daehee polynomials considered in [27,28]. From what have been discussed in the second section, the studies on the above-mentioned polynomials probably enrich the topics on the special values of the partial Bell polynomials. This is left to the interested reader for further exploration.

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