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Polynomial-Exponential Bounds for Some Trigonometric and Hyperbolic Functions

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Abstract: Recent advances in mathematical inequalities suggest that bounds of polynomial-exponential-type are appropriate for evaluating key trigonometric functions. In this paper, we innovate in this sense by establishing new and sharp bounds of the form $(1 - \alpha x^2)e^{\beta x^2}$ for the trigonometric sinc and cosine functions. Our main result for the sinc function is a double inequality holding on the interval $(0, \pi)$, while our main result for the cosine function is a double inequality holding on the interval $(0, \pi/2)$. Comparable sharp results for hyperbolic functions are also obtained. The proofs are based on series expansions, inequalities on the Bernoulli numbers, and the monotone form of the l'Hospital rule. Some comparable bounds of the literature are improved. Examples of application via integral techniques are given.

Keywords: polynomial-exponential bounds; l'Hôpital's rule of monotonicity; Bernoulli numbers; Jordan's inequality; Kober's inequality; trigonometric functions

MSC: 26D05; 26D07; 26D20; 33B10

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1. Introduction

We know that the sinc and cosine functions, i.e., $\sin x/x$ and $\cos x$, are less than 1 for $0 < x < \pi/2$. These rude inequalities have been refined over time in several ways. In this regard, we may mention Jordan's and Kober's inequalities, which are

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1; \quad 0 < x < \frac{\pi}{2} \quad (1)$$

and

$$1 - \frac{2x}{\pi} < \cos x < 1 - \frac{x^2}{\pi}; \quad 0 < x < \frac{\pi}{2}, \quad (2)$$

respectively. Several proofs of these results exist. We refer the reader to [1–6] for more information. Due to their importance in mathematics, the inequalities (1) and (2) are sharpened and generalized in many ways by researchers. Moreover, different bounds for sine and cosine functions have been established in the literature so far. The list of references of this topic is extensive, and includes [3,4,7–30]. The obtained bounds involve polynomial functions, trigonometric functions, exponential functions, and combinations of them. In

particular, recently, Chouikha et al. in [18] obtained the polynomial-exponential bounds for the sinc and cosine functions, as follows:

$$\left(1 - \frac{x^2}{\pi^2}\right)^{\frac{\pi^4}{90}} e^{\left(\frac{\pi^2}{90} - \frac{1}{6}\right)x^2} < \frac{\sin x}{x}; \quad 0 < x < \pi, \quad (3)$$

$$\frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3} \left(1 - \frac{4x^2}{\pi^2}\right)^{\frac{\pi^4}{96}} e^{\left(\frac{\pi^2}{24} - \frac{1}{2}\right)x^2}; \quad 0 < x < \frac{\pi}{2} \quad (4)$$

and

$$\left(1 - \frac{4x^2}{\pi^2}\right)^{\frac{\pi^4}{96}} e^{\left(\frac{\pi^2}{24} - \frac{1}{2}\right)x^2} < \cos x; \quad 0 < x < \frac{\pi}{2}. \quad (5)$$

The inequalities (3)–(5) were proven by infinite product methods. They demonstrate that bounds of polynomial-exponential-type are appropriate for evaluating, with precision, these key trigonometric functions. This paper aims to provide some contributonal material on this subject.

In the first part of this work, we aim to provide new sharp bounds for polynomial-exponential types of the form $(1 - \alpha x^2)e^{\beta x^2}$ for the sinc and cosine functions. In the second part, we aim to present sharp polynomial-exponential bounds for the hyperbolic sinc and hyperbolic cosine functions. The constants are obtained such that the bounds are as sharp as possible. The proofs include series expansions, Bernoulli number inequalities, and the monotone form of the l'Hospital rule. As a result, we employ techniques that are completely different from those used by [18]. Furthermore, under some conditions on the domain of x , we improve the existing bounds of the literature, including those in (3)–(5). Another advantage of the findings is that the obtained bounds are manageable from the mathematical viewpoint. To illustrate that, some applications based on integral techniques are given to get new bounds for the cosine and hyperbolic cosine functions.

The organization of the paper is divided into the following sections: Section 2 presents the main theorems. Preliminaries and lemmas are described in Section 3. The proofs of the main results are detailed in Section 4. Applications are given in Section 5. Final discussions and conclusions are given in Section 6.

2. Main Theorems

We begin with our new polynomial-exponential bounds of the form $(1 - \alpha x^2)e^{\beta x^2}$ for the sinc and cosine functions. Our main results are stated below.

Theorem 1. *The inequalities*

$$\left(1 - \frac{x^2}{\pi^2}\right)e^{-\frac{\ln 2}{\pi^2}x^2} < \frac{\sin x}{x} < \left(1 - \frac{x^2}{\pi^2}\right)e^{\left(\frac{1}{\pi^2} - \frac{1}{6}\right)x^2}; \quad 0 < x < \pi \quad (6)$$

hold; $\beta = -(\ln 2)/\pi^2$ and $\beta = 1/\pi^2 - 1/6$ are the best possible constants for lower and upper bounds for $\sin x/x$ of the form $(1 - \alpha x^2)e^{\beta x^2}$ with $\alpha = 1/\pi^2$, respectively.

Theorem 2. *The inequalities*

$$\left(1 - \frac{4x^2}{\pi^2}\right)e^{\frac{4\ln(\pi/4)}{\pi^2}x^2} < \cos x < \left(1 - \frac{4x^2}{\pi^2}\right)e^{\left(\frac{4}{\pi^2} - \frac{1}{2}\right)x^2}; \quad 0 < x < \frac{\pi}{2} \quad (7)$$

hold; $\beta = 4\ln(\pi/4)/\pi^2$ and $\beta = 4/\pi^2 - 1/2$ are the best possible constants for lower and upper bounds for $\cos x$ of the form $(1 - \alpha x^2)e^{\beta x^2}$ with $\alpha = 4/\pi^2$, respectively.

Now, we aim to present sharp polynomial-exponential bounds for the hyperbolic sinc and hyperbolic cosine functions. In particular, we establish hyperbolic counterparts of (6) and (7) in the following theorems.

Theorem 3. Let $r > 0$. Then the inequalities

$$\left(1 + \frac{x^2}{\pi^2}\right)e^{ax^2} < \frac{\sinh x}{x} < \left(1 + \frac{x^2}{\pi^2}\right)e^{\left(\frac{1}{6} - \frac{1}{\pi^2}\right)x^2}; \quad 0 < x < r \quad (8)$$

hold, with $a = \ln[\pi^2 \sinh r / (r(\pi^2 + r^2))] / r^2$; $\beta = a$ and $\beta = 1/6 - 1/\pi^2$ are the best possible constants for lower and upper bounds for $\sinh x / x$ of the form $(1 - \alpha x^2)e^{\beta x^2}$ with $\alpha = -1/\pi^2$, respectively.

Theorem 4. Let $r > 0$. Then, the inequalities

$$\left(1 + \frac{4x^2}{\pi^2}\right)e^{bx^2} < \cosh x < \left(1 + \frac{4x^2}{\pi^2}\right)e^{\left(\frac{1}{2} - \frac{4}{\pi^2}\right)x^2}; \quad 0 < x < r \quad (9)$$

hold with the best possible constants $b = \ln[\pi^2 \cosh r / (\pi^2 + 4r^2)] / r^2$; $\beta = b$ and $\beta = 1/2 - 4/\pi^2$ are the best possible constants for lower and upper bounds for $\cosh x$ of the form $(1 - \alpha x^2)e^{\beta x^2}$ with $\alpha = -4/\pi^2$, respectively.

The proofs of these new results, several applications, and a discussion of the significance of the findings and existing literature results are presented in the remainder of the work.

3. Preliminaries and Lemmas

We first recall the following simple geometric series expansion:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k; \quad |x| < 1, \quad (10)$$

as well as the following known power series expansions of $\cosh x$ and $\sinh x$:

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}. \quad (11)$$

Furthermore, the following series expansions:

$$\frac{\tan x}{x} = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} |B_{2k}| x^{2k-2}; \quad |x| < \frac{\pi}{2} \quad (12)$$

and

$$\frac{\cot x}{x} = \frac{1}{x^2} - \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} |B_{2k}| x^{2k-2}; \quad |x| < \pi. \quad (13)$$

can be found in ([31], 1.411). Here, B_{2k} stands for the even indexed Bernoulli number.

Lemma 1 (see [32]). The inequality

$$|B_{2k}| > \frac{2(2k)!}{(2\pi)^{2k}} \frac{2^{2k}}{2^{2k}-1}$$

holds for all integers $k \geq 1$.

The following lemma is known as a monotone form of l'Hôpital's rule ([33] p. 10) (see also [34]).

Lemma 2. Let $p, q : [a, b] \rightarrow \mathbb{R}$ be continuous functions. Moreover, let p, q be differentiable functions on (a, b) , with $q'(x) \neq 0, x \in (a, b)$. Set

$$r_1(x) := \frac{p(x) - p(a)}{q(x) - q(a)}, \quad r_2(x) := \frac{p(x) - p(b)}{q(x) - q(b)}, \quad a < x < b.$$

Then we have:

- (i) $r_1(x)$ and $r_2(x)$ are increasing (strictly increasing) on (a, b) if $p'(x)/q'(x)$ is increasing (strictly increasing) on (a, b) .
- (ii) $r_1(x)$ and $r_2(x)$ are decreasing (strictly decreasing) on (a, b) if $p'(x)/q'(x)$ is decreasing (strictly decreasing) on (a, b) .

In addition to this, we need the following lemmas, which can be proven in a scholarly manner.

Lemma 3. Let $\gamma(k) = 32\pi^4(k-2) + 16\pi^2k(2k-1)(2k-6) + 2k(2k-1)(2k-2)(2k-3)(2k-16)$. Then $\gamma(k) > 0$ for all integers $k \geq 4$.

Lemma 4. Let $\zeta(k) = \pi^4 + 2\pi^2(2k+2)(2k+3) + (2k+1)(2k+2)(2k+3)(2k-8)$. Then $\zeta(k) > 0$ for all integers $k \geq 2$.

The above preliminaries are the basis of the proofs of the main results, which are the subject of the next section.

4. Proofs of Main Results

Proof of Theorem 1. Let us set

$$f(x) = \frac{\ln\left(\frac{\sin x}{x} \frac{\pi^2}{\pi^2 - x^2}\right)}{x^2} := \frac{f_1(x)}{f_2(x)}, \quad 0 < x < \pi,$$

where $f_1(x) = \ln[(\sin x/x)(\pi^2/(\pi^2 - x^2))] = \ln(\sin x/x) + \ln[\pi^2/(\pi^2 - x^2)]$ and $f_2(x) = x^2$ with $f_1(0+) = 0 = f_2(0)$. After differentiation, we get

$$\begin{aligned} \frac{f_1'(x)}{f_2'(x)} &= \frac{1}{2x} \left(\frac{x \cos x - \sin x}{x \sin x} + \frac{2x}{\pi^2 - x^2} \right) \\ &= \frac{1}{2} \frac{\cot x}{x} - \frac{1}{2x^2} + \frac{1}{\pi^2} \frac{1}{1 - (\frac{x}{\pi})^2}. \end{aligned}$$

Using (10) and (13), we have

$$\begin{aligned} \frac{f_1'(x)}{f_2'(x)} &= - \sum_{k=1}^{\infty} \frac{2^{2k-1}}{(2k)!} |B_{2k}| x^{2k-2} + \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{x}{\pi} \right)^{2k} \\ &= \sum_{k=0}^{\infty} \frac{1}{\pi^{2k+2}} x^{2k} - \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+2)!} |B_{2k+2}| x^{2k} \\ &= \sum_{k=0}^{\infty} \left[\frac{1}{\pi^{2k+2}} - \frac{2^{2k+1}}{(2k+2)!} |B_{2k+2}| \right] x^{2k}. \end{aligned}$$

Then

$$\left(\frac{f_1'(x)}{f_2'(x)} \right)' = \sum_{k=1}^{\infty} 2k \left[\frac{1}{\pi^{2k+2}} - \frac{2^{2k+1}}{(2k+2)!} |B_{2k+2}| \right] x^{2k-1} := \sum_{k=1}^{\infty} 2ka_k x^{2k-1},$$

where $a_k := 1/\pi^{2k+2} - [2^{2k+1}/(2k+2)!]|B_{2k+2}|$, $k \geq 1$. Lemma 1 implies that

$$|B_{2k+2}| > \frac{(2k+2)!}{2^{2k+1}\pi^{2k+2}} \frac{2^{2k+2}}{2^{2k+2}-1} > \frac{(2k+2)!}{2^{2k+1}\pi^{2k+2}},$$

giving us $a_k < 0$ for each $k \geq 1$. Therefore $f'_1(x)/f'_2(x)$ is decreasing on $(0, \pi)$ and hence $f(x)$ is also decreasing on $(0, \pi)$ by Lemma 2. So $f(0+) > f(x) > f(\pi-)$. Since $f(0+) = 1/\pi^2 - 1/6$ and $f(\pi-) = -(\ln 2)/\pi^2$, we simply obtain the required inequalities (6). $\square$

Proof of Theorem 2. We set

$$g(x) = \frac{\ln\left(\frac{\pi^2 \cos x}{\pi^2 - 4x^2}\right)}{x^2} := \frac{g_1(x)}{g_2(x)}, \quad 0 < x < \frac{\pi}{2},$$

where $g_1(x) = \ln[\pi^2 \cos x / (\pi^2 - 4x^2)] = \ln(\cos x) + \ln[\pi^2 / (\pi^2 - 4x^2)]$ and $g_2(x) = x^2$ such that $g_1(0) = g_2(0) = 0$. Upon differentiation, we get

$$\frac{g'_1(x)}{g'_2(x)} = -\frac{1}{2} \frac{\tan x}{x} + \frac{4}{\pi^2 - 4x^2} = \frac{4}{\pi^2} \frac{1}{1 - \left(\frac{2x}{\pi}\right)^2} - \frac{1}{2} \frac{\tan x}{x}.$$

Utilizing (10) and (12), we write

$$\begin{aligned} \frac{g'_1(x)}{g'_2(x)} &= \frac{4}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{2x}{\pi}\right)^{2k} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} |B_{2k}| x^{2k-2} \\ &= \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{2^{2k}}{\pi^{2k}} x^{2k} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{2^{2k+2}(2^{2k+2}-1)}{(2k+2)!} |B_{2k+2}| x^{2k} \\ &= \sum_{k=0}^{\infty} 2^{2k+1} \left[\frac{2}{\pi^{2k+2}} - \frac{2^{2k+2}-1}{(2k+2)!} |B_{2k+2}| \right] x^{2k}. \end{aligned}$$

Then

$$\begin{aligned} \left(\frac{g'_1(x)}{g'_2(x)}\right)' &= \sum_{k=1}^{\infty} 2^{2k+2} k \left[\frac{2}{\pi^{2k+2}} - \frac{(2^{2k+2}-1)}{(2k+2)!} |B_{2k+2}| \right] x^{2k-1} \\ &:= \sum_{k=1}^{\infty} 2^{2k+2} k b_k x^{2k-1}, \end{aligned}$$

where $b_k := 2/\pi^{2k+2} - [(2^{2k+2}-1)/(2k+2)!]|B_{2k+2}| < 0$ for $k \geq 1$ owing to Lemma 1. Therefore, we conclude that $g'_1(x)/g'_2(x)$ is decreasing in $(0, \pi/2)$ and hence $g(x)$ is also decreasing in $(0, \pi/2)$ by Lemma 2. So $g(0+) > g(x) > g(\pi/2-)$. The required inequalities (1.7) follow from the obvious limit equalities $g(0+) = 4/\pi^2 - 1/2$ and $g(\pi/2-) = 4 \ln(\pi/4)/\pi^2$. This ends the proof. $\square$

Proof of Theorem 3. Let us consider the function

$$h(x) = \frac{\ln\left(\frac{\sinh x}{x} \frac{\pi^2}{\pi^2 + x^2}\right)}{x^2} := \frac{h_1(x)}{h_2(x)}, \quad x > 0,$$

where $h_1(x) = \ln[(\sinh x/x)(\pi^2/(\pi^2 + x^2))] = \ln(\sinh x/x) + \ln(\pi^2/(\pi^2 + x^2))$ and $h_2(x) = x^2$ that satisfy $h_1(0+) = h_2(0) = 0$. By differentiation, we get

$$\frac{h'_1(x)}{h'_2(x)} = \frac{1}{2} \frac{x \cosh x - \sinh x}{x^2 \sinh x} - \frac{1}{\pi^2 + x^2} := \frac{1}{2} h_3(x),$$

where $h_3(x) = (x \cosh x - \sinh x) / (x^2 \sinh x) - 2 / (\pi^2 + x^2) = \coth x / x - 1 / x^2 - 2 / (\pi^2 + x^2)$. From this, we obtain

$$\begin{aligned} h_3'(x) &= \frac{-x \operatorname{cosech}^2 x - \coth x}{x^2} + \frac{2}{x^3} + \frac{4x}{(\pi^2 + x^2)^2} \\ &= \frac{4x}{(\pi^2 + x^2)^2} + \frac{2}{x^3} - \frac{1}{x \sinh^2 x} - \frac{\cosh x}{x^2 \sinh x} \\ &= \frac{4x}{(\pi^2 + x^2)^2} + \frac{2}{x^3} - \frac{2}{x(\cosh 2x - 1)} - \frac{\sinh 2x}{x^2(\cosh 2x - 1)} \\ &:= -\frac{h_4(x)}{x^3(\pi^2 + x^2)(\cosh 2x - 1)}, \end{aligned}$$

where $h_4(x) = -4x^4(\cosh 2x - 1) - 2(\pi^2 + x^2)^2(\cosh 2x - 1) + 2x^2(\pi^2 + x^2)^2 + x \sinh 2x (\pi^2 + x^2)$. We simplify $h_4(x)$ as follows:

$$\begin{aligned} h_4(x) &= (\pi^4 x + 2\pi^2 x^3 + x^5)(2x + \sinh 2x) - (2\pi^4 + 4\pi^2 x^2 + 6x^4)(\cosh 2x - 1) \\ &= 2\pi^4 + (4\pi^2 + 2\pi^4)x^2 + (4\pi^2 + 6)x^4 + 2x^6 + \pi^4 x \sinh 2x \\ &\quad + 2\pi^2 x^3 \sinh 2x + x^5 \sinh 2x - 2\pi^4 \cosh 2x - 4\pi^2 x^2 \cosh 2x - 6x^4 \cosh 2x. \end{aligned}$$

By (11), it follows

$$\begin{aligned} h_4(x) &= 2\pi^4 + (4\pi^2 + 2\pi^4)x^2 + (4\pi^2 + 6)x^4 + 2x^6 \\ &\quad + \pi^4 \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} x^{2k+2} + 2\pi^2 \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} x^{2k+4} + \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k+1)!} x^{2k+6} \\ &\quad - 2\pi^4 \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k} - 4\pi^2 \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+2} - 6 \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+4} \\ &= 2\pi^4 + (4\pi^2 + 2\pi^4)x^2 + (4\pi^2 + 6)x^4 + 2x^6 \\ &\quad + \pi^4 \sum_{k=1}^{\infty} \frac{2^{2k-1}}{(2k-1)!} x^{2k} + 2\pi^2 \sum_{k=2}^{\infty} \frac{2^{2k-3}}{(2k-3)!} x^{2k} + \sum_{k=3}^{\infty} \frac{2^{2k-5}}{(2k-5)!} x^{2k} \\ &\quad - 2\pi^4 \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k} - 4\pi^2 \sum_{k=1}^{\infty} \frac{2^{2k-2}}{(2k-2)!} x^{2k} - 6 \sum_{k=2}^{\infty} \frac{2^{2k-4}}{(2k-4)!} x^{2k} \\ &= \frac{2}{45}(2\pi^4 - 180)x^6 + \sum_{k=4}^{\infty} \frac{2^{2k-5}}{(2k-5)!} c_k x^{2k}, \end{aligned}$$

where

$$\begin{aligned} c_k &= 1 - \frac{12}{(2k-4)} + \frac{8\pi^2}{(2k-3)(2k-4)} - \frac{32\pi^2}{(2k-2)(2k-3)(2k-4)} \\ &\quad + \frac{16\pi^4}{(2k-1)(2k-2)(2k-3)(2k-4)} - \frac{64\pi^4}{2k(2k-1)(2k-2)(2k-3)(2k-4)} \\ &= \frac{\gamma(k)}{2k(2k-1)(2k-2)(2k-3)(2k-4)}. \end{aligned}$$

Since $2\pi^4 - 180 \approx 14.8182 > 0$, and $c_k > 0$ for $k \geq 4$ thanks to Lemma 3, it is clear that $h_4(x) > 0$. Thus, we have $h_3'(x) < 0$. Consequently, $h_1'(x)/h_2'(x)$ is decreasing for $x > 0$. By Lemma 2, $h(x)$ is decreasing for $0 < x < r$, $r > 0$. Lastly, due to relation $h(0+) = 1/6 - 1/\pi^2 > h(x) > h(r)$, the inequalities (8) follow. $\square$

Proof of Theorem 4. Let us consider

$$j(x) = \frac{\ln\left(\frac{\pi^2 \cosh x}{\pi^2 + 4x^2}\right)}{x^2} := \frac{j_1(x)}{j_2(x)}, \quad x > 0,$$

where $j_1(x) = \ln[\pi^2 \cosh x / (\pi^2 + 4x^2)] = \ln(\cosh x) + \ln[\pi^2 / (\pi^2 + 4x^2)]$ and $j_2(x) = x^2$ with $j_1(0+) = j_2(0) = 0$. Differentiation gives

$$\frac{j_1'(x)}{j_2'(x)} = \frac{1}{2} \frac{\tanh x}{x} - \frac{4}{\pi^2 + 4x^2}$$

and

$$\begin{aligned} \left(\frac{j_1'(x)}{j_2'(x)}\right)' &= \frac{1}{2} \frac{x \operatorname{sech}^2 x - \tanh x}{x^2} + \frac{32x}{(\pi^2 + 4x^2)^2} \\ &= \frac{(-1)}{2x^2(\pi^2 + 4x^2)^2} \left[(\tanh x - x \operatorname{sech}^2 x)(\pi^2 + 4x^2)^2 - 64x^3 \right] \\ &= \frac{(-1)}{2x^2 \cosh x (\pi^2 + 4x^2)^2} \left[(\sinh x \cosh x - x)(\pi^2 + 4x^2)^2 - 64x^3 \cosh^2 x \right] \\ &:= \frac{(-1)}{4x^2 \cosh x (\pi^2 + 4x^2)^2} j_3(x), \end{aligned}$$

where $j_3(x) = (\sinh 2x - 2x)(\pi^2 + 4x^2)^2 - 64x^3(1 + \cosh 2x)$. Next we show that $j_3(x)$ is positive for $x > 0$. Using (11), we have

$$\begin{aligned} j_3(x) &= -128x^3 - 64x^3 \sum_{k=1}^{\infty} \frac{(2x)^{2k}}{(2k)!} + (\pi^4 + 8\pi^2 x^2 + 16x^4) \sum_{k=1}^{\infty} \frac{(2x)^{2k+1}}{(2k+1)!} \\ &= -128x^3 - 64 \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+3} + \pi^4 \sum_{k=1}^{\infty} \frac{2^{2k+1}}{(2k+1)!} x^{2k+1} \\ &\quad + 8\pi^2 \sum_{k=1}^{\infty} \frac{2^{2k+1}}{(2k+1)!} x^{2k+3} + 16 \sum_{k=1}^{\infty} \frac{2^{2k+1}}{(2k+1)!} x^{2k+5} \\ &= \left(\frac{4\pi^4}{3} - 128\right)x^3 - 64 \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k+3} + \pi^4 \sum_{k=1}^{\infty} \frac{2^{2k+3}}{(2k+3)!} x^{2k+3} \\ &\quad + 8\pi^2 \sum_{k=1}^{\infty} \frac{2^{2k+1}}{(2k+1)!} x^{2k+3} + 16 \sum_{k=2}^{\infty} \frac{2^{2k-1}}{(2k-1)!} x^{2k+3} \\ &= \left(\frac{4\pi^4}{3} - 128\right)x^3 + \left(\frac{4\pi^4}{15} + \frac{32}{3}\pi^2 - 128\right)x^5 \\ &\quad + \sum_{k=2}^{\infty} \left(\frac{\pi^4 \cdot 2^{2k+3}}{(2k+3)!} + \frac{\pi^2 \cdot 2^{2k+4}}{(2k+1)!} + \frac{2^{2k+3}}{(2k-1)!} - \frac{2^{2k+6}}{(2k)!}\right)x^{2k+3} \\ &:= \left(\frac{4\pi^4}{3} - 128\right)x^3 + \frac{1}{15}(4\pi^4 + 160\pi^2 - 1920)x^5 + \sum_{k=2}^{\infty} \frac{2^{2k+3}}{(2k-1)!} d_k x^{2k+3}, \end{aligned}$$

where

$$\begin{aligned} d_k &= 1 - \frac{8}{2k} + \frac{2\pi^2}{2k(2k+1)} + \frac{\pi^4}{2k(2k+1)(2k+2)(2k+3)} \\ &= \frac{(2k-8)(2k+1)(2k+2)(2k+3) + 2\pi^2(2k+2)(2k+3) + \pi^4}{2k(2k+1)(2k+2)(2k+3)} \\ &= \frac{\zeta(k)}{2k(2k+1)(2k+2)(2k+3)}. \end{aligned}$$

It is worth noting that $4\pi^4/3 - 128 \approx 1.8788 > 0$, $4\pi^4 + 160\pi^2 - 1920 \approx 48.7731 > 0$ and $d_k > 0$ for $k \geq 2$ thanks to Lemma 4. Thus $j_3(x) > 0$, so that $j(x)$ is decreasing for $x > 0$ by Lemma 2. Hence $j(0+) = 1/2 - 4/\pi^2 > j(x) > j(r)$, $0 < x < r$, which simply yields the required inequalities (9). $\square$

5. Applications

The bounds established in the previous theorems have the quality of being manageable from the analytical viewpoint. In this section, we exploit this quality to propose some other sharp polynomial-exponential bounds for cosine and hyperbolic cosine functions. More precisely, these bounds will be obtained from Theorems 1 and 2, and integral developments.

Proposition 1. Let $0 < x < \pi$. Then

$$\phi_1(x) < \cos x < \phi_2(x), \quad (14)$$

where

$$\phi_1(x) = 1 - \frac{3\pi^2(\pi^2 - 12)}{(\pi^2 - 6)^2} + \frac{3\pi^2}{(\pi^2 - 6)^2} \left[(\pi^2 - 6) \left(1 - \frac{x^2}{\pi^2} \right) - 6 \right] e^{\left(\frac{1}{\pi^2} - \frac{1}{6} \right) x^2}$$

and

$$\phi_2(x) = 1 + \frac{\pi^2(1 - \ln 2)}{2(\ln 2)^2} + \frac{\pi^2}{2(\ln 2)^2} \left[\left(1 - \frac{x^2}{\pi^2} \right) \ln 2 - 1 \right] e^{-\frac{\ln 2}{\pi^2} x^2}.$$

Proof. By (6), we get

$$\int_0^x \left(1 - \frac{t^2}{\pi^2} \right) t e^{-\frac{\ln 2}{\pi^2} t^2} dt < \int_0^x \sin t dt < \int_0^x \left(1 - \frac{t^2}{\pi^2} \right) t e^{\left(\frac{1}{\pi^2} - \frac{1}{6} \right) t^2} dt$$

for $0 < x < \pi$. Using integration by parts method, this simply implies

$$\begin{aligned} & 1 - \frac{18\pi^2}{(\pi^2 - 6)^2} \left[e^{\left(\frac{1}{\pi^2} - \frac{1}{6} \right) x^2} - 1 \right] + \frac{3\pi^2}{(\pi^2 - 6)} \left[\left(1 - \frac{x^2}{\pi^2} \right) e^{\left(\frac{1}{\pi^2} - \frac{1}{6} \right) x^2} - 1 \right] \\ & < \cos x < 1 - \frac{\pi^2}{2(\ln 2)^2} \left(e^{-\frac{\ln 2}{\pi^2} x^2} - 1 \right) + \frac{\pi^2}{2\ln 2} \left[\left(1 - \frac{x^2}{\pi^2} \right) e^{-\frac{\ln 2}{\pi^2} x^2} - 1 \right]. \end{aligned}$$

After rearranging the terms, we get the inequalities in (14). $\square$

Proposition 2. Let $0 < x < r$ and $r > 0$. Then

$$\psi_1(x) < \cosh x < \psi_2(x), \quad (15)$$

where

$$\begin{aligned} \psi_1(x) &= \left(1 + \frac{1 - \pi^2 a}{2\pi^2 a^2} \right) + \frac{1}{2\pi^2 a^2} \left[\pi a \left(1 + \frac{x^2}{\pi^2} \right) - 1 \right] e^{ax^2}, \\ \psi_2(x) &= \left(1 - \frac{3\pi^4}{(\pi^2 - 6)^2} \right) + \frac{3\pi^2}{(\pi^2 - 6)^2} \left[(\pi^2 - 6) \left(1 + \frac{x^2}{\pi^2} \right) - 6 \right] e^{\left(\frac{1}{6} - \frac{1}{\pi^2} \right) x^2}, \end{aligned}$$

and the constant a being defined in Theorem 3.

Proof. We skip the proof, as it is similar to the proof of Proposition 1. The inequalities (15) can be easily obtained by integrating inequalities (8). $\square$

6. Final Discussion and Conclusions

We now discuss the importance of our findings in the light of the existing results of the literature, and conclude the paper. Our first observation is that there are some limitations to the method used in [18] to obtain the polynomial-exponential bounds stated

and proven in this paper. For instance, in [18], the lower bound for the sinc function is obtained in $(0, \pi)$, whereas the upper bound is obtained in a half interval $(0, \pi/2)$, and the upper bound of the cosine function is not obtained. We have adopted different methods and established comparable upper and lower polynomial-exponential bounds for sinc and cosine functions. The same kind of bounds for hyperbolic functions have never been discussed before in the literature. It is interesting to see that the bounds for hyperbolic functions in (8), (9), and (15) are very sharp and better than the existing bounds in the literature (they are better than the corresponding bounds presented in [19,24] and the references therein). Numerical calculations and graphical comparisons via the Maple software reveal the following important points:

- There is no strict comparison between the lower bounds of the sinc function in (3) and (6). The lower bound in (6) is sharper than that in (3) in the interval (λ_1, π) , where $\lambda_1 \approx 2.5018$.
- The upper bound of the sinc function in (6) is uniformly sharper than that in (4). Moreover, it is valid in the extended interval $(0, \pi)$.
- The lower bound of the cosine function in (7) is sharper than that in (5) for the interval $(\lambda_2, \pi/2)$, where $\lambda_2 \approx 1.2221$, and we also obtained the upper bound for the cosine function in (7).
- The lower bound in (14) is uniformly sharper than that in (5). Moreover, it is valid over a larger interval $(0, \pi)$.
- If $0 < x < \pi/2$, then the upper bound of (7) is sharper than that of (14). So, considering the better bounds of the cosine function in (7) and (14) and combining them, we have the following sharp double inequality:

$$\phi_1(x) < \cos x < \left(1 - \frac{4x^2}{\pi^2}\right) e^{\left(\frac{4}{\pi^2} - \frac{1}{2}\right)x^2}; \quad 0 < x < \frac{\pi}{2},$$

where $\phi_1(x)$ is specified as in Proposition 1.

- Both the inequalities of (15) are sharper than those of (9).

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