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Fractional Derivatives and Projectile Motion

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Abstract: Projectile motion is studied using fractional calculus. Specifically, a newly defined fractional derivative (the Leibniz L-derivative) and its successor (Λ-fractional derivative) are used to describe the motion of the projectile. Experimental data were analyzed in this study, and conclusions were made. The results of well-established fractional derivatives were also compared with those of L-derivative and Λ-fractional derivative, showing the many advantages of these new derivatives.

Keywords: projectile motion; fractional calculus; L-fractional derivative; Λ-fractional derivative

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1. Introduction

Fractional calculus (FC) was coined by G.W Leibniz (1646–1716) [1] in 1695. The main issue that concerned Leibniz was to define a meaningful derivative $\frac{d^n y}{dx^n}$ of order n = ½. Ever since, the theory has been developed and implemented in many scientific fields by many well-known mathematicians (Euler in 1730, Lagrange in 1772, Laplace in 1812, Liouville [2] in 1832, and Riemann [3] in 1876).

When we define a fractional derivative (FD) $D^\gamma f(x)$ of order γ, where γ is some rational number, we replace the integer order n of the corresponding classical derivative with some rational order γ. That is when we have a classical derivative $\frac{d^n f(x)}{dx^n}$, the order n is an integer, while in a fractional derivative $D^\gamma f(x)$, the order γ is rational.

The main advantages of fractional derivatives are flexibility and non-locality. Since these derivatives are of fractional order, they can approximate real data with more flexibility than classical derivatives. Furthermore, they also take into consideration non-locality, something that classical derivatives cannot do. Therefore, they are more suitable for cases with memory (non-locality in time) and global interactions (non-locality in space). The interested reader might recur at [4–7], while for short memory models, one might wish to look up [8–10].

The applications of fractional calculus are limitless (Podlubny [8], Kilbas et al. [11], Samko et al. [12], Oldham [13]). Especially in physics and applied mathematics where non-locality is the issue, a large number of interesting articles have appeared (Tarasov [14,15], Baleanu et al. [16], Golmankhaneh et al. [17], Atanackovic [18,19], Mainardi [20,21], etc.), dealing with the description of viscoelastic problems, viscous flows, and flows in porous media.

Especially in mechanics, there is a plethora of interesting articles, showing a robust activity in the field. The articles written range from continuum mechanics (Drapaca et al. [22], Lazopoulos [23]) and elasticity theory (Di Paola et al. [24], Carpinteri et al. [25]) to Lagrangian and Hamiltonian mechanics (Baleanu et al. [26]), viscoelasticity (Lazopoulos et al. [27]), viscoplasticity (Sumelka [28]), and many other topics.

The novelties of this article, L-fractional derivative (L-FD), and Λ-fractional derivative (Λ-FD) are pointed out. These mathematical concepts were introduced in 2015 and 2019, respectively, by Lazopoulos et al. [23,29], and since then, they have been implemented in various important scientific problems, specifically viscoelasticity [27,30]. Therefore, many aspects of mechanics are studied using these fractional derivatives, mainly by

Lazopoulos et al. [23,27,29,30], especially viscoelasticity, continuum mechanics, and peridynamic deformations. The main advantage of L-fractional derivative is that it corresponds to a differential. This feature is lacking from all other fractional derivatives, and it is quite crucial from a topological point of view. Nevertheless, to define geometry, a derivative must have additional features; this is why the Λ-fractional derivative was invented (introduced in 2019 [29]). This derivative is the evolution of L-fractional derivative, which uses it as its base. Along with the Λ-transform (Λ-T) and Λ-space (Λ-S), this derivative satisfies the Leibniz and chain rule in differential topology in Λ-space. Therefore, Λ-FD tackles the main difficulties faced by all other fractional derivatives and defines a geometry in Λ-S to solve numerous problems in fractional calculus.

Fractional calculus applications are quite limited in ballistics. Although there are many interesting efforts to combine these two scientific fields, not many articles have appeared. For instance, Ebaid [31] used the Caputo derivative to describe the ballistics problem and compare theoretical with experimental results. Moreover, in problems with air resistance, we have the work of El-Sayed et al. [32] (who also used the Caputo derivative). Furthermore, in maneuvering problems, we have the work of Ye et al. [33] and Ahmad et al. [34] who used Riemann–Liouville derivatives to study the intercepting of the maneuvering target.

In this article, we used the L-fractional and Λ-fractional derivatives to describe a projectile's behavior during its motion. The necessary equations are stated, and the problem is solved. The results are compared with experimental data, and it appears as though this approach is more successful than other fractional derivatives. It seems that these derivatives provide better precision and a more natural description of the phenomenon.

2. Basic Properties of Fractional Calculus, L-Derivative, and Λ-Derivative

Fractional calculus mainly studies fractional derivatives and fractional integrals. As far as the fractional derivatives are concerned, there are many definitions. Each of them has its own advantages and disadvantages. Nevertheless, they all share a common feature: they are all non-local. This non-locality makes them flexible and unique. Therefore, they are recommended for solving problems with memory (e.g., viscoelasticity (Lazopoulos et al. [27])) or spatial dependence (e.g., fractional peridynamic deformation (Lazopoulos [30])).

The fractional derivative that we are going to consider is called L-fractional derivative. It is defined as the ratio $\frac{{}_0D_x^\gamma f(x)}{{}_0D_x^\gamma x}$, where ${}_0D_x^\gamma f(x)$ may be the Caputo, Riemann–Liouville, or the Grunwald–Letnikov fractional derivative of the function $f(x)$ of fractional order γ. Therefore, we have ${}_0^LD_x^\gamma f(x) = \frac{{}_0D_x^\gamma f(x)}{{}_0D_x^\gamma x} = \frac{\frac{d^\gamma f(x)}{dx^\gamma}}{\frac{d^\gamma x}{dx^\gamma}} = \frac{d^\gamma f(x)}{d^\gamma x}$, where

$$ {}_0D_x^\gamma f(x) = {}_0^{GL}D_x^\gamma f(x) = \frac{1}{h^\gamma}\sum_{r=0}^{m}(-1)^r \left(\begin{array}{c} \gamma \\ r \end{array} \right) f(x - rh), \tag{1} $$

and

$$ \left(\begin{array}{c} \gamma \\ r \end{array} \right) = \frac{\gamma!}{r!(\gamma - r)!} = \frac{\Gamma(\gamma+1)}{\Gamma(r+1)\Gamma(\gamma-r+1)} \tag{2} $$

($h =$ where t is time, and $\Gamma(x)$ is the Gamma function).

Or, alternatively, we can have

$$ {}_0D_x^\gamma f(x) = {}_0^{RL}D_x^\gamma f(x) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_0^t \frac{f(s)}{(x-s)^\gamma}ds \text{ (applied to this article for } \Lambda\text{-derivative)} \tag{3} $$

With

$$ {}_0^{RL}I_x^\gamma f(x) = \frac{1}{\Gamma(\gamma)}\int_0^x \frac{f(s)}{(x-s)^{1-\gamma}}ds, \tag{4} $$

where ${}_0^{RL}I_x^\gamma f(x)$ is called the Riemann–Liouville fractional integral of $f(x)$.

Of course, we can also imagine that the derivative that builds the L-derivative is the Caputo derivative:

$$ {}_0D_x^\gamma f(x) = {}_0^C D_x^\gamma f(x) = \frac{1}{\Gamma(1-\gamma)} \int_0^x \frac{f'(s)}{(x-s)^\gamma} ds \tag{5}$$

The L-derivative might be built from all the above-mentioned operators, that is, this operator can be the fraction of two Grunwald–Letnikov fractional derivatives, or two Riemann–Liouville derivatives, or two Caputo derivatives. Nevertheless, in this article, the L-derivative is considered as the fraction of two Grunwald–Letnikov fractional derivatives.

The corresponding fractional derivative for

$$ \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\frac{df(x)}{dx} \text{ is } {}_0^L D_x^\gamma {}_0^L D_x^\gamma f(x), \tag{6}$$

The main problem of the well-established fractional derivatives, a problem that L-fractional derivative does not have, is the definition of the proper differential. Lazopoulos et al. have investigated this topic thoroughly [23,29], and, in order to tackle the problem, proposed this new derivative, which shows many interesting advantages.

Another problem that well-established fractional derivatives face is that of dimensions. Dimensions become fractional, and it is very complicated for the experimentalist to translate them physically. On the other hand, the L-FD does not change dimensions.

The evolution of the L-fractional derivative is the Λ-fractional derivative. It is about the same ratio (Equation (1)), only that in this case, this ratio of strictly Riemann–Liouville derivatives is accompanied by Λ-transform and Λ-space.

Riemann–Liouville fractional derivative is essential to our methodology since Λ-derivative is defined as the fraction of strictly two such derivatives (see Lazopoulos [27,29]):

$$ {}_0^\Lambda D_x^\gamma f(x) = \frac{{}_0^{RL} D_x^\gamma f(x)}{{}_0^{RL} D_x^\gamma x} = \frac{\frac{d {}_0^{RL} I_x^{1-\gamma} f(x)}{dx}}{\frac{d {}_0^{RL} I_x^{1-\gamma} x}{dx}} = \frac{d {}_0^{RL} I_x^{1-\gamma} f(x)}{d {}_0^{RL} I_x^{1-\gamma} x} \tag{7}$$

It is clear that ${}_0^{RL} D_x^\gamma f(x)$ is the Riemann–Liouville derivative of $f(x)$, as described in FC (Equation (3)) and ${}_0^{RL} I_x^{1-\gamma} f(x)$ is the Riemann–Liouville fractional integral of real fractional dimension (Equation (4)). In this article, $0 < \gamma \leq 1$ is considered (see Podlubny [8], Samko et al. [12]).

In order for a fractional differential equation (FDE) to be solved in the initial space, the procedure in Figure 1 must be followed:

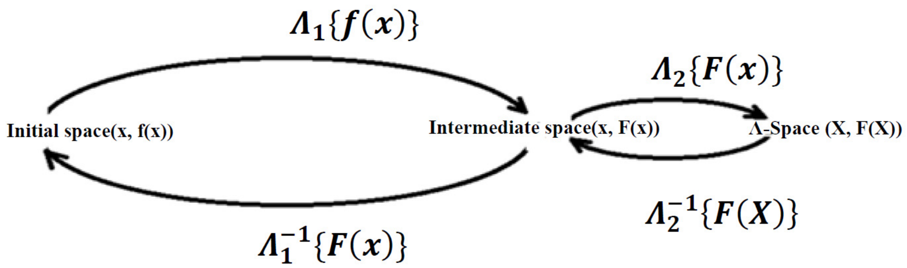

Figure 1. Λ-transform in four steps.

In this figure, the Λ-transform is depicted in four steps, analogous to Laplace transform. The main difference is that the Laplace transform needs two computation steps, while the Λ-transform needs four (as is shown, there is an intermediate space in Λ-transform).

Starting from the initial space, we pose the fractional differential equation, and using the Λ_1-transform, we convert any function $f(x)$ from the initial space to the intermediate space as follows:

$$\Lambda_1\{f(x)\} = {}_0^{RL}I_x^{1-\gamma}f(x) = \frac{1}{\Gamma(1-\gamma)}\int_0^x \frac{f(s)}{(x-s)^\gamma}ds = F(x) \tag{8}$$

As far as the derivatives are concerned, we assume that the Λ-T transforms Λ-FD of any function $f(x)$ of any order to itself in the intermediate space to the according derivative in Λ-space. Therefore, we have

$$\Lambda_1\left\{{}_0^{\Lambda}D_x^\gamma f(x)\right\} = {}_0^{\Lambda}D_x^\gamma f(x) = \frac{{}_0^{RL}D_x^\gamma f(x)}{{}_0^{RL}D_x^\gamma x} = \frac{\frac{d_0^{RL}I_x^{1-\gamma}f(x)}{dx}}{\frac{d_0^{RL}I_x^{1-\gamma}x}{dx}} = \frac{d_0^{RL}I_x^{1-\gamma}f(x)}{d_0^{RL}I_x^{1-\gamma}x} = \frac{dF(x)}{dX} \tag{9}$$

Moreover, in the second step of the Λ-transform (Λ_2), $F(x)$ is transformed into $F(X)$ as follows:

$$\Lambda_2\{F(x)\} = F(X) \tag{10}$$

where

$$X = \Lambda_1\{x\} = {}_0^{RL}I_x^\gamma x = \frac{1}{\Gamma(\gamma)}\int_0^x \frac{x}{(x-s)^{1-\gamma}}ds = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \tag{11}$$

and

$$x = (X\cdot\Gamma(3-\gamma))^{\frac{1}{2-\gamma}} \tag{12}$$

Then, the fractional derivatives of any order are transformed in the second step (Λ_2) as

$$\Lambda_2\left\{{}_0^{\Lambda}D_x^\gamma f(x)\right\} = \Lambda_2\left\{\frac{d_0I_x^{1-\gamma}f(x)}{d_0I_x^{1-\gamma}x}\right\} = \frac{dF(X)}{dX} \tag{13}$$

Finally, we solve the Equation in Λ-space (as an ordinary differential equation (ODE)), and afterward, with the inverse Λ-transforms ${\Lambda_1}^{-1}$ and ${\Lambda_2}^{-1}$, we restore the solution $F(X)$ of the equation in Λ-space to $f(x)$ in the initial space. In our case

$$\Lambda_2^{-1}\{F(X)\} = F(x) \tag{14}$$

and

$$\Lambda_1^{-1}\{F(x)\} = {}_0^{RL}D_x^{1-\gamma}F(x) = {}_0^{RL}D_x^{1-\gamma}\left({}_0^{RL}I_x^{1-\gamma}f(x)\right) = f(x) \tag{15}$$

The beauty of this new mathematical concept is that it is local and non-local, simultaneously, in two different spaces: the initial space (non-local) and Λ-space (local). Thus Λ-fractional derivative has all the advantages of a classic local derivative (it is mainly consistent with all the requirements of differential topology for a proper derivative), as well as the advantages of a non-local one (it is affected by the phenomenon non-locally). This central feature of Λ-FD provides great mathematical value to our study since this derivative might be the only proper derivative in the whole field of FC. Furthermore, fractional differential equations in the initial space are transformed in ODEs in Λ-space. This advantage is also revolutionary, since no complex methodologies (i.e., application of Mittag–Leffler functions, etc.) are needed to solve these FDEs in the initial space.

Further information may be found in Podlubny [8], Kilbas et al. [11], Samko et al. [12], Lazopoulos et al. [27] and Lazopoulos [29] et al.

3. Analysis of Projectile Motion with Least Air Resistance via L-Fractional Derivative

Projectile motion in a void is the simplest motion of a projectile. It is described in classical ballistics by a parabola, and only demands the velocity of the projectile at the beginning of the phenomenon.

In classical ballistics, the equations describing this motion are given by implementing Newton's second law in the phenomenon:

Motion on x-axis:

$$\frac{d^2x(t)}{dt^2} = 0 \tag{16}$$

Motion on y-axis:

$$\frac{d^2y(t)}{dt^2} = -g \tag{17}$$

These two equations are written according to x-coordinates and y-coordinates of the projectile (x, y), time t, and gravity acceleration g.

In addition, the solution of the prementioned differential equation is given by

$$x(t) = U(0) \cdot cos(\varphi) \cdot t \tag{18}$$

$$y(t) = U(0) \cdot sin(\varphi) \cdot t - \frac{1}{2} \cdot g \cdot t^2 \tag{19}$$

where $U(0)$ is the initial velocity at $t = 0$ and φ is the angle of this velocity with the x-axis.

In fractional analysis, these equations become

$${}_0^L D_t^\gamma {}_0^L D_t^\gamma x = 0 \tag{20}$$

$${}_0^L D_t^\gamma {}_0^L D_t^\gamma y = -g \tag{21}$$

where ${}_0^L D_t^\gamma f(t)$ is the L-fractional derivative of function $f(t)$ expressed as a ratio of Grunwald–Letnikov derivatives of f, as stated in paragraph 2.

To be more precise, Equation (20) becomes

$${}_0^L D_t^\gamma x(t) = U_x(t) \tag{22}$$

$${}_0^L D_t^\gamma U_x(t) = 0 \tag{23}$$

With the help of Equations (1) and (2), we consider the following formulas for ${}_0^L D_t^\gamma x(t)$ and ${}_0^L D_t^\gamma U_x(t)$:

$${}_0^L D_t^\gamma x(t) = \frac{{}_0 D_t^\gamma x(t)}{{}_0 D_t^\gamma t} = \frac{\frac{1}{h^\gamma} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} \gamma \\ r \end{pmatrix} x(t - rh)}{\frac{1}{h^\gamma} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} \gamma \\ r \end{pmatrix} (t - rh)}, \text{ and} \tag{24}$$

$${}_0^L D_t^\gamma U_x(t) = \frac{{}_0 D_t^\gamma U_x(t)}{{}_0 D_t^\gamma t} = \frac{\frac{1}{h^\gamma} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} \gamma \\ r \end{pmatrix} U_x(t - rh)}{\frac{1}{h^\gamma} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} \gamma \\ r \end{pmatrix} (t - rh)}. \tag{25}$$

As far as Equation (21) is concerned, we have

$${}_0^L D_t^\gamma y(t) = U_y(t) \tag{26}$$

$${}_0^L D_t^\gamma U_y = -g \tag{27}$$

with

$${}_0^L D_t^\gamma y(t) = \frac{{}_0 D_t^\gamma y(t)}{{}_0 D_t^\gamma t} = \frac{\frac{1}{h^\gamma} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} \gamma \\ r \end{pmatrix} y(t - rh)}{\frac{1}{h^\gamma} \sum_{r=0}^{m} (-1)^r \begin{pmatrix} \gamma \\ r \end{pmatrix} (t - rh)}, \text{ and} \tag{28}$$

$$ {}_0^L D_t^\gamma U_y(t) = \frac{{}_0 D_t^\gamma U_y(t)}{{}_0 D_t^\gamma t} = \frac{\frac{1}{h^\gamma}\sum_{r=0}^{m}(-1)^r\begin{pmatrix}\gamma\\ r\end{pmatrix}U_y(t-rh)}{\frac{1}{h^\gamma}\sum_{r=0}^{m}(-1)^r\begin{pmatrix}\gamma\\ r\end{pmatrix}(t-rh)}. \tag{29}$$

The solution of this problem provides the following formulas for $x(t)$ (x-axis coordinate of the motion of the projectile), $y(t)$ (y-axis coordinate of the motion of the projectile), $U_x(t)$ (x-axis velocity of the projectile), and $U_y(t)$ (y-axis velocity of the projectile):

$$ x(t) = -\sum_{r=1}^{m}(-1)^r\cdot\begin{pmatrix}\gamma\\ r\end{pmatrix}\cdot x(t-r\cdot h) + U_x(t)\cdot\sum_{r=0}^{m}(-1)^r\cdot\begin{pmatrix}\gamma\\ r\end{pmatrix}\cdot(t-r\cdot h) \tag{30}$$

$$ U_x(t) = -\sum_{r=1}^{m}(-1)^r\cdot\begin{pmatrix}\gamma\\ r\end{pmatrix}\cdot U_x(t-r\cdot h) \tag{31}$$

$$ y(t) = -\sum_{r=1}^{m}(-1)^r\cdot\begin{pmatrix}\gamma\\ r\end{pmatrix}\cdot y(t-r\cdot h) + U_y(t)\cdot\sum_{r=0}^{m}(-1)^r\cdot\begin{pmatrix}\gamma\\ r\end{pmatrix}\cdot(t-r\cdot h) \tag{32}$$

$$ U_y(t) = -\sum_{r=1}^{m}(-1)^r\cdot\begin{pmatrix}\gamma\\ r\end{pmatrix}\cdot U_y(t-r\cdot h) - g\cdot\sum_{r=0}^{m}(-1)^r\cdot\begin{pmatrix}\gamma\\ r\end{pmatrix}\cdot(t-rh) \tag{33}$$

In these equations $U_x(0) = U(0)\ cos\varphi$ and $U_y(0) = U(0)\ sin\varphi$, while $x(0) = y(0) = 0$.

The nature of these solutions does not allow analytical solutions; therefore, the features of the projectile motion (such as range, flight time, and maximum height) must be computed arithmetically or graphically.

4. Analysis of Projectile Motion with Least Air Resistance via Λ-Fractional Derivative

This section investigates the projectile motion of a body in the absence of air resistance, where the governing equations, which describe the motion, are supposed to be fractional over time t in initial space. In our analysis, we will use the Lazopoulos Λ-fractional derivate definition:

$$ {}_0^\Lambda D_t^\gamma(f(t)) = \frac{{}_0^{RL}D_t^\gamma(f(t))}{{}_0^{RL}D_t^\gamma(t)} \tag{34}$$

where ${}_0^{RL}D_t^\gamma(f(t))$ is the right Riemann–Liouville fractional derivative over time for $0 \le \gamma \le 1$.

According to classical differential equations for the projectile motion, the corresponding Λ-fractional differential equations are

$$ {}_0^\Lambda D_t^\gamma({}_0^\Lambda D_t^\gamma(x(t))) = 0 \tag{35}$$

$$ {}_0^\Lambda D_t^\gamma({}_0^\Lambda D_t^\gamma(y(t))) = -g \tag{36}$$

where g = the acceleration of gravity = 9.81 m/s^2 = 32.2 ft/s^2.

In order to solve the above system of fractional differential equations, we use the method of transformation from the initial space to Λ-space, using two consecutive transformations Λ_1 and Λ_2 [27,29]. First, we impose the Λ_1 transformation in the initial function, defined as

$$ \Lambda_1\{f(t)\} = \sigma^{\gamma-1}{}_0I_t^{1-\gamma}(f(t)) = F(t) \tag{37}$$

where ${}_0I_t^\gamma$ is the right Riemann–Liouville fractional integral over time for $0 \le \gamma \le 1$.

With this transformation, the function $f(t)$ is converted from initial space to an intermediate space function $F(t)$. To be consistent with dimensionality (same dimensions in initial space and in Λ-space), we introduced a parameter σ in our transformation procedure, whose dimension is in seconds.

Secondly, we impose Λ_2 transformation on $F(t)$:

$$ \Lambda_2\{F(t)\} = F(T) \tag{38}$$

where $F(t)$ is transformed to $F(T)$, a function of time T in Λ-space, while T is defined by

$$T = \Lambda_1\{t\} = \sigma^{\gamma-1} \cdot {}_0I_t^{1-\gamma}(t) = \sigma^{\gamma-1} \cdot \frac{t^{2-\gamma}}{\Gamma(3-\gamma)} \tag{39}$$

Thus,

$$t = [\Gamma(3-\gamma) \cdot T]^{\frac{1}{2-\gamma}} \cdot \sigma^{\frac{1-\gamma}{2-\gamma}} \tag{40}$$

According to the Riemann–Liouville fractional derivative definition (for $0 \leq \gamma \leq 1$), the Λ-fractional derivative can be written as

$$ {}_0^{\Lambda}D_t^{\gamma}(f(t)) = \frac{\frac{d(\sigma^{\gamma-1}{}_0I_t^{1-\gamma}(f(t)))}{dt}}{\frac{d(\sigma^{\gamma-1}{}_0I_t^{1-\gamma}(t))}{dt}} = \frac{dF(T)}{dT} \tag{41}$$

where Λ-fractional derivative in initial space is converted to a classical derivative in Λ-space.

Before proceeding further, we want to remind the reader (as pointed out in Section 2) that we assume that the application of the transformations Λ_1 and Λ_2 on the Λ-fractional derivative converts it to itself, i.e.,

$$\begin{aligned} &\Lambda_1\{{}_0^{\Lambda}D_t^{\gamma}(f(t))\} = {}_0^{\Lambda}D_t^{\gamma}(f(t)) = \tfrac{dF(t)}{dT}, \text{ and} \\ &\Lambda_2\{{}_0^{\Lambda}D_t^{\gamma}(f(t))\} = \tfrac{dF(T)}{dT} \end{aligned} \tag{42}$$

Thus, a Λ-fractional differential equation (Λ-FDE) in initial space can be converted to one with classical derivatives in Λ-space and can be solved as a classical ordinary differential equation (ODE) with unknown function $F(T)$. When we find the solution of this ODE in Λ-space, we can return to the initial space by the inverse transformation:

$$\Lambda_2^{-1}\{F(T)\} = F(t) \tag{43}$$

$$f(t) = \Lambda_1^{-1}\{F(t)\} = \sigma^{1-\gamma} \cdot {}_0^{RL}D_t^{1-\gamma}(F(t)) \tag{44}$$

obtaining the solution $f(t)$ in the initial space.

Returning to the Equations (35) and (36) for the case of projectile motion, we apply the transformation Λ_1 and Λ_2 at the equations and take the following system of equations:

$$\frac{d(\sigma^{\gamma-1}{}_0I_t^{1-\gamma}(\frac{dX(T)}{dT}))}{dT} = 0 \tag{45}$$

$$\frac{d(\sigma^{\gamma-1} \cdot {}_0I_t^{1-\gamma}(\frac{dY(T)}{dT}))}{dT} = -g \cdot \sigma^{\gamma-1} \cdot \frac{t^{1-\gamma}}{\Gamma(2-\gamma)} \tag{46}$$

where

$$X(T) = X(t) = \sigma^{\gamma-1} \cdot {}_0I_t^{1-\gamma}(x(t)),\ Y(T) = Y(t) = \sigma^{\gamma-1} \cdot {}_0I_t^{1-\gamma}(y(t)) \tag{47}$$

We define

$$XX(T) = \sigma^{\gamma-1} \cdot {}_0I_t^{1-\gamma}(\frac{dX(T)}{dT}),\ YY(T) = \sigma^{\gamma-1} \cdot {}_0I_t^{1-\gamma}(\frac{dY(T)}{dT}) \tag{48}$$

Therefore, the Equations (45) and (46) are written as

$$\frac{dXX(T)}{dT} = 0 \tag{49}$$

$$\frac{dYY(T)}{dT} = -\frac{g \cdot [\Gamma(3-\gamma)]^{\frac{1-\gamma}{2-\gamma}}}{\Gamma(2-\gamma)} \cdot \sigma^{\frac{\gamma-1}{2-\gamma}} \cdot T^{\frac{1-\gamma}{2-\gamma}} \tag{50}$$

By integrating both sides, we obtain

$$XX(T) = A_1 \tag{51}$$

$$YY(T) = -g \cdot (\frac{2-\gamma}{3-2\gamma}) \cdot \frac{[\Gamma(3-\gamma)]^{\frac{1-\gamma}{2-\gamma}}}{\Gamma(2-\gamma)} \cdot \sigma^{\frac{\gamma-1}{2-\gamma}} \cdot T^{\frac{3-2\gamma}{2-\gamma}} + B_1 \tag{52}$$

where A_1 and B_1, are the integrations constants to be determined by the initial conditions.

Applying inverse transformation, we obtain

$$XX(t) = \Lambda_2^{-1}\{XX(T)\} = A_1 \tag{53}$$

$$\begin{aligned} YY(t) &= \Lambda_2^{-1}\{YY(T)\} \\ &= -g \cdot \left(\frac{2-\gamma}{3-2\gamma}\right) \cdot \frac{1}{\Gamma(2-\gamma)\cdot\Gamma(3-\gamma)} \cdot \sigma^{2\gamma-2} \cdot t^{3-2\gamma} \\ &+B_1 \end{aligned} \tag{54}$$

A second inverse transformation gives

$$\begin{aligned} \frac{dX(T)}{dT} &= \Lambda_1^{-1}\{XX(t)\} = \sigma^{1-\gamma}{}_0^{RL}D_t^{1-\gamma}(XX(t)) = \sigma^{1-\gamma} \cdot A_1 \cdot \frac{t^{\gamma-1}}{\Gamma(\gamma)} \\ &= A_1 \cdot \frac{[\Gamma(3-\gamma)]^{\frac{\gamma-1}{2-\gamma}}}{\Gamma(\gamma)} \cdot \sigma^{\frac{1-\gamma}{2-\gamma}} \cdot T^{\frac{\gamma-1}{2-\gamma}} \end{aligned} \tag{55}$$

$$\begin{aligned} \frac{dY(T)}{dT} &= \Lambda_1^{-1}\{YY(t)\} = \sigma^{1-\gamma} \cdot {}_0^{RL}D_t^{1-\gamma}(YY(t)) \\ &= -g \cdot \left(\frac{2-\gamma}{3-2\gamma}\right) \cdot \frac{\Gamma(4-2\gamma)}{\Gamma(2-\gamma)\cdot[\Gamma(3-\gamma)]^2} \cdot \sigma^{\gamma-1} \cdot t^{2-\gamma} + B_1 \cdot \sigma^{1-\gamma} \frac{t^{\gamma-1}}{\Gamma(\gamma)} \end{aligned} \tag{56}$$

which ends at

$$\frac{dY(T)}{dT} = g \cdot \left(\frac{2-\gamma}{3-2\gamma}\right) \cdot \frac{\Gamma(4-2\gamma)}{\Gamma(2-\gamma)\cdot\Gamma(3-\gamma)} \cdot T + B_1 \cdot \frac{[\Gamma(3-\gamma)]^{\frac{\gamma-1}{2-\gamma}}}{\Gamma(\gamma)} \cdot \sigma^{\frac{1-\gamma}{2-\gamma}} \cdot T^{\frac{\gamma-1}{2-\gamma}} \tag{57}$$

Through integration of both sides, we have:

$$X(T) = A_1 \cdot \frac{[\Gamma(3-\gamma)]^{\frac{\gamma-1}{2-\gamma}}}{\Gamma(\gamma)} \cdot (2-\gamma) \cdot \sigma^{\frac{1-\gamma}{2-\gamma}} \cdot T^{\frac{1}{2-\gamma}} + A_2 \tag{58}$$

and

$$X(t) = \frac{A_1 \cdot (2-\gamma)}{\Gamma(\gamma)\cdot\Gamma(3-\gamma)} \cdot t + A_2 \tag{59}$$

$$\begin{aligned} Y(T) &= -g \cdot \left(\frac{2-\gamma}{3-2\gamma}\right) \cdot \frac{\Gamma(4-2\gamma)}{2\cdot\Gamma(2-\gamma)\cdot\Gamma(3-\gamma)} \cdot T^2 \\ &+B_1 \cdot (2-\gamma) \cdot \frac{[\Gamma(3-\gamma)]^{\frac{\gamma-1}{2-\gamma}}}{\Gamma(\gamma)} \cdot \sigma^{\frac{1-\gamma}{2-\gamma}} \cdot T^{\frac{1}{2-\gamma}} + B_2 \end{aligned} \tag{60}$$

which ends up as

$$\begin{aligned} Y(t) = -g \cdot \left(\frac{2-\gamma}{3-2\gamma}\right) \cdot \frac{\Gamma(4-2\gamma)}{2\cdot\Gamma(2-\gamma)\cdot[\Gamma(3-\gamma)]^3} \cdot \sigma^{2\gamma-2}t^{4-2\gamma} \\ +\frac{B_1\cdot(2-\gamma)}{\Gamma(\gamma)\cdot\Gamma(3-\gamma)} \cdot t + B_2 \end{aligned} \tag{61}$$

where A_2 and B_2 are integration constants to be determined by the initial conditions.

Finally, by imposing inverse transformation, we obtain the relations for $x(t)$ and $y(t)$ in the initial space:

$$\begin{aligned} x(t) = \sigma^{1-\gamma} \cdot {}_0^{RL}D_t^{1-\gamma}(X(t)) = \\ \frac{A_1\cdot(2-\gamma)}{\Gamma(\gamma)\cdot\Gamma(3-\gamma)\cdot\Gamma(1+\gamma)} \cdot \sigma^{1-\gamma} \cdot t^{\gamma} + \frac{A_2}{\Gamma(\gamma)} \cdot \sigma^{1-\gamma} \cdot t^{\gamma-1} \end{aligned} \tag{62}$$

$$\begin{aligned} y(t) &= \sigma^{1-\gamma}\,{}_0^{RL}D_t^{1-\gamma}(Y(t)) \\ &= -g\cdot\left(\frac{2-\gamma}{3-2\gamma}\right)\cdot\frac{\Gamma(4-2\gamma)\cdot\Gamma(5-2\gamma)}{2\cdot\Gamma(2-\gamma)\cdot[\Gamma(3-\gamma)]^3\cdot\Gamma(4-\gamma)}\cdot\sigma^{\gamma-1}\cdot t^{3-\gamma} \\ &+\frac{B_1\cdot(2-\gamma)}{\Gamma(\gamma)\cdot\Gamma(3-\gamma)\cdot\Gamma(1+\gamma)}\cdot\sigma^{1-\gamma}\cdot t^{\gamma}+\frac{B_2}{\Gamma(\gamma)}\cdot\sigma^{1-\gamma}\cdot t^{\gamma-1} \end{aligned} \tag{63}$$

For the case where $\gamma = 1$, the above fractional solutions must be identical with the ones derived with classical differential equations. For this to happen, the values of the constants must be $A_2 = B_2 = 0$ and $A_1 = v_0 \cdot \cos\theta$, $B_1 = v_0 \cdot \sin\theta$.

Thus, the solutions (62) and (63) take the form of

$$x(t) = \frac{(2-\gamma)\cdot v_0\cdot\cos\theta}{\Gamma(\gamma)\cdot\Gamma(3-\gamma)\cdot\Gamma(1+\gamma)}\cdot\sigma^{1-\gamma}\cdot t^{\gamma} \tag{64}$$

$$\begin{aligned} y(t) &= \frac{(2-\gamma)\cdot v_0\cdot\sin\theta}{\Gamma(\gamma)\cdot\Gamma(3-\gamma)\cdot\Gamma(1+\gamma)}\cdot\sigma^{1-\gamma}\cdot t^{\gamma} \\ &-\frac{g}{2}\cdot\left[\left(\frac{2-\gamma}{3-2\gamma}\right)\cdot\frac{\Gamma(4-2\gamma)\cdot\Gamma(5-2\gamma)}{\Gamma(2-\gamma)\cdot[\Gamma(3-\gamma)]^3\cdot\Gamma(4-\gamma)}\right]\cdot\sigma^{\gamma-1}t^{3-\gamma} \end{aligned} \tag{65}$$

Time of Flight and Range

The time of flight (time taken from $t = 0$ until the body hits the ground) can be evaluated by setting in relation (65), $y(t) = 0$. Solving the resulting equation for t, we have

$$t_b = \left[\left(\frac{2\cdot v_0\cdot\sin\theta}{g}\right)\cdot\frac{(3-2\gamma)\cdot\Gamma(2-\gamma)\cdot(\Gamma(3-\gamma))^2\cdot\Gamma(4-\gamma)}{\sigma\cdot\Gamma(\gamma)\cdot\Gamma(1+\gamma)\cdot\Gamma(4-2\gamma)\cdot\Gamma(5-2\gamma)}\right]^{\frac{1}{3-2\gamma}}\cdot\sigma \tag{66}$$

The range of flight can be evaluated by setting in the relation (65), $t = t_b$.

$$x_{flight} = \frac{(2-\gamma)\cdot v_0\cdot\cos\theta}{\Gamma(\gamma)\cdot\Gamma(3-\gamma)\cdot\Gamma(1+\gamma)}\cdot\left[\left(\frac{2\cdot v_0\cdot\sin\theta}{g}\right)\cdot\frac{(3-2\gamma)\cdot\Gamma(2-\gamma)\cdot(\Gamma(3-\gamma))^2\cdot\Gamma(4-\gamma)}{\sigma\cdot\Gamma(\gamma)\cdot\Gamma(1+\gamma)\cdot\Gamma(4-2\gamma)\cdot\Gamma(5-2\gamma)}\right]^{\frac{\gamma}{3-2\gamma}}\cdot\sigma \tag{67}$$

5. Application of Projectile Motion with Least Resistance

In this section, we thoroughly study the case depicted in Table 1 and compare it with various fractional analogous cases, as well as with classical ones. There are data given for certain cases of projectile motion of a mortar in [31] (these data are provided from the American Ministry of Defense [31]). We can see these data in Table 1.

Table 1. Features of the projectile motion of a mortar (American Ministry of Defense).

U(0) (ft/s)	Range (ft)	Flight Time (s)	φ (Degrees)
334	3189	14.4	45°
368	3804	15.7	45°
400	4425	17.0	45°
431	5049	18.2	45°

In this table, “range” is the maximum distance that the projectile runs before ending its motion. At that point, we have $y = 0$, and the corresponding time is called “flight time.”

Due to classical ballistics equations, the corresponding results to this data are given in Table 2.

Table 2. Results for the projectile motion of a mortar in a void (classical ballistics model).

U(0) (ft/s)	Range (ft)	Flight Time (s)	φ (Degrees)
334	3464.67	14.67	45°
368	4205.71	16.17	45°
400	4968.944	17.57	45°
431	5768.975	18.93	45°

We can observe that these two sets of data do not converge completely. The reason is that the experimental data are not given for complete void, but there is the least air resistance. Moreover, the gravity acceleration is not steady during the phenomenon. Furthermore, there might be other minor factors that influence this phenomenon but were silenced.

The main issue here is to express all these negligible factors successfully, respecting the fractional void model. This is done by regarding fractional derivatives of various order α. Firstly, we would like to present some results from a very interesting article [31] that tried to model this motion with well-established fractional derivatives (specifically, the well-known Caputo derivative). These results are presented in Table 3 with various values of γ.

Table 3. Range and time of flight for various γ, with φ = *45°* and g = *32.2* ft/s^2 [31] (Caputo derivative).

U_0(ft/s)	Range of Projectile (ft)			Time of Flight (s)		
	γ = 1.95	**γ = 1.97**	**γ = 1.99**	**γ = 1.95**	**γ = 1.97**	**γ = 1.99**
334	3803.31	3659.29	3526.79	16.1	15.5	14.9
368	4640.66	4455.55	4285.56	17.8	17.1	16.5
400	5506.94	5277.71	5067.55	19.4	18.6	17.9
431	6418.76	6141.62	5887.89	21.1	20.2	19.3

It is so interesting that by closely examining the results, we can conclude that they are quite close to the ones found by classical ballistics, especially for γ = 1.99. Nevertheless, they show a quite non-negligible difference with data in Table 1.

At this point, it would be most appropriate to define the measure of proximity of two sets of data points we are going to use: the Euclidian norm, which is given by the following formula:

$$d = \sqrt{\sum_{i=1}^{n} \left(x'_{experimental_i} - x'_i \right)^2}$$

where $x'_{experimental_i}$ represents the range according to U_{0i} in Table 1, and x'_i is the range that responds to the same U_{0i} for any particular case studied (e.g., L-derivative, Λ-derivative, classical case). The measure d of the experimental case compared to the classical case was found to be 5768.975, which is considered relatively high. The minimum measure of the case we studied with this projectile motion through using the Caputo derivative was found to be 1209.38 for γ = 1.99. Nevertheless, this measure is also quite high.

This fact must make us consider an alternative approach, an approach that uses L-derivative. In Table 4, we can see the results extracted by using Equations (17)–(20) for the analogous cases.

Table 4. Range and time of flight for various γ, with φ = 45° and g = 32.2 ft/s^2 (L-fractional derivative).

U_0 (ft/s)	Range of Projectile (ft)			Time of Flight (s)		
	γ = 1.95/2	γ = 1.97/2	γ = 1.99/2	γ = 1.95/2	γ = 1.97/2	γ = 1.99/2
334	2545.7	2874.44	3249.3	12.42	13.26	14.16
368	3289.21	3506.41	3957.91	13.68	14.6	15.64
400	3662.23	4129.17	4663.22	14.88	15.88	16.96
431	4245.81	4791.27	5413.79	16.04	17.12	18.28

From Tables 1 and 2, we can conclude that the L-fractional derivative is very efficient. The least measure of proximity of a range of the experimental case and the L-fractional case is even lower than the according ones of the prementioned cases: it is only about 466, and it holds for $\gamma = 1.99/2$. Moreover, this fractional derivative shows a better behavior than Caputo fractional derivative and classical void equations. It seems that the definition of the differential for the L-FD is very important and supports very realistic results.

Finally, an even better picture is depicted in Table 5, where we use Λ-fractional derivative, along with Λ-FT. Here, the least measure of range compared to experimental data was found to be 115.923, which corresponds to $\gamma = 1.97/2$ (we used for σ parameter the constant value σ = 1 s).

Table 5. Range and time of flight for various γ, with φ = 45° and g = 32.2 ft/s^2 (Λ-fractional derivative).

U_0 (ft/s)	Range of Projectile (ft)			Time of Flight (s)		
	γ = 1.95/2	γ = 1.97/2	γ = 1.99/2	γ = 1.95/2	γ = 1.97/2	γ = 1.99/2
334	2903.88	3112	3341	12.99	13.62	14.31
368	3500.86	3762	4050	14.24	14.97	15.75
400	4111.61	4429	4779	15.42	16.23	17.1
431	4748.23	5125	5543	16.6	17.45	18.42

6. Conclusions

Projectile motion was studied thoroughly, using classical, experimental, and fractional methodologies. Using the appropriate experimental data, we compared them with the classical model and fractional models using the Caputo, L-fractional derivative, and Λ-fractional derivative. Emphasis is given to Λ-fractional derivative, a derivative that satisfies all topological perquisites in Λ-space while showing non-local behavior in the initial space. It turns out that the Λ-derivative approach is most precise and provides more accurate results than the other two.

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References

1. Leibnitz, G.W. Letter to G. A. L'Hospital. *Leibnitzen Math. Schr.* **1849**, *2*, 301–302.
2. Liouville, J. Sur le calcul des differentielles a indices quelconques. *J. Ec. Polytech.* **1832**, *13*, 71–162.
3. Riemann, B. Versuch einer allgemeinen Auffassung der Integration and Differentiation. In *Gesammelte Werke*; Dover: New York, NY, USA, 1876.
4. Wang, H.; Du, N. Fast solution methods for space-fractional diffusion equations. *J. Comput. Appl. Math.* **2014**, *255*, 376–383. [CrossRef]
5. Rahimi, Z.; Sumelka, W.; Yang, X.J. A new fractional non-local model and its applications in free vibration of Timoshenko and Euler beam. *Eur. Phys. J. Plus* **2017**, *132*, 1–10. [CrossRef]
6. Ding, W.; Patnaik, S.; Sidhardh, S.; Semperlotti, F. Applications of Distributed-Order Fractional Operators: A Review. *Entropy* **2021**, *23*, 110. [CrossRef]
7. Aydinlik, S.; Kiris, A.; Sumelka, W. Nonlocal vibration analysis of microstretch plates in the framework of space-fractional mechanics—theory and validation. *Eur. Phys. J. Plus* **2021**, *136*, 169. [CrossRef]
8. Podlubny, I. *Fractional Differential Equations (An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications)*; Academic Press: San Diego, CA, USA; Boston, MA, USA; New York, NY, USA; London, UK; Tokyo, Japan; Toronto, ON, Canada, 1999.
9. Zhou, Y.; Zhang, Y. Noether symmetries for fractional generalized Birkhoffian systems in terms of classical and combined Caputo derivatives. *Acta Mech.* **2020**, *231*, 3017–3029. [CrossRef]
10. Sumelka, W.; Łuczak, B.; Gajewski, T.; Voyiadjis, G.Z. Modelling of AAA in the framework of time-fractional damage hyperelasticity. *Int. J. Solids Struct.* **2020**, *206*, 30–42. [CrossRef]
11. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
12. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: Amsterdam, The Netherlands, 1993.
13. Oldham, K.B.; Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA; London, UK, 1974.
14. Tarasov, V.E. Fractional vector calculus and fractional Maxwell's equations. *Ann. Phys.* **2008**, *323*, 2756–2778. [CrossRef]
15. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*; Springer: Berlin, Germany, 2010.
16. Baleanu, D.; Golmankhaneh, A.K.; Golmankhaneh, A.K.; Baleanu, M.C. Fractional electromagnetic equations using fractional forms. *Int. J. Theor. Phys.* **2009**, *48*, 3114–3123. [CrossRef]
17. Golmankhaneh, A.K.; Golmankhaneh, A.K.; Baleanu, D. About Schrodinger Equation on Fractals Curves Imbedding in R^3. *Int. J. Theor. Phys.* **2015**, *54*, 1275–1282. [CrossRef]
18. Atanackovic, T.M.; Konjik, S.; Philipovic, S. Variational problems with fractional derivatives. Euler–Lagrange equations. *J. Phys. A Math. Theor.* **2008**, *41*, 095201. [CrossRef]
19. Atanackovic, T.M. A generalized model for the uniaxial isothermal deformation of a viscoelastic body. *Acta Mech.* **2002**, *159*, 77–86. [CrossRef]
20. Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity*; Imperial College Press: London, UK, 2010.
21. Mainardi, F.; Spada, G. Creep, relaxation and viscosity properties for basic fractional models in rheology. *Eur. Phys. J.* **2011**, *193*, 133–160. [CrossRef]
22. Drapaca, C.S.; Sivaloganathan, S.A. Fractional model of continuum mechanics. *J. Elast.* **2012**, *107*, 107–123. [CrossRef]
23. Lazopoulos, K.A. Fractional Vector Calculus and Fractional Continuum Mechanics. In Proceedings of the Conference «Mechanics though Mathematical Modelling», Celebrating the 70th Birthday of Prof. T. Atanackovic, Novi Sad, Serbia, 6–11 September 2015; p. 40.
24. Di Paola, M.; Failla, G.; Zingales, M. Physically-based approach to the mechanics of strong non-local linear elasticity theory. *J. Elast.* **2009**, *97*, 103–130. [CrossRef]
25. Carpinteri, A.; Cornetti, P.; Sapora, A. A Fractional calculus approach to non-local elasticity. *Eur. Phys. J. Spec. Top.* **2011**, *193*, 193–204. [CrossRef]
26. Baleanu, D.; Muslih, S.; Rabei, E. On Fractional Euler-Lagrange and Hamilton Equations and the Fractional generalization of total time Derivatives. *Nonlinear Dyn.* **2008**, *53*, 67–74. [CrossRef]
27. Lazopoulos, A.K.; Karaoulanis, D. On Λ-Fractional viscoelastic models. *Axioms* **2021**, *10*, 22. [CrossRef]
28. Sumelka, W. Fractional Viscoplasticity. *Mech. Res. Com.* **2014**, *56*, 31–36. [CrossRef]
29. Lazopoulos, K.A.; Lazopoulos, A.K. On the Mathematical Formulation of Fractional Derivatives. *Prog. Fract. Differ. Appl.* **2019**, *5*, 261–267.
30. Lazopoulos, A.K. On fractional peridynamic deformations. *Arch. Appl. Mech.* **2016**, *86*, 1987–1994. [CrossRef]
31. Ebaid, A. Analysis of projectile motion in view of fractional calculus. *Appl. Math. Model.* **2011**, *35*, 1231–1239. [CrossRef]
32. El-Sayed, A.M.A.; Nour, H.M.; Raslan, W.E.; El-Shazly, E.S. A study of projectile motion in a quadratic resistant medium via fractional differential transform method. *Appl. Math. Model.* **2015**, *39*, 2829–2835. [CrossRef]

33. Ye, J.; Lei, H.; Li, J. Novel Fractional Order Calculus Extended PN for Maneuvering Targets. *Int. J. Aerosp. Eng.* **2017**, *2017*, 5931967. [CrossRef]
34. Ahmad, B.; Batarfi, H.; Nieto, J.J.; Zarraquiρos, O.O.; Shammakh, W. Projectile Motion via Riemann Liouville Calculus. *Adv. Differ. Equ.* **2015**, *2015*, 1–14. [CrossRef]