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A Series Representation for the Hurwitz–Lerch Zeta Function

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Abstract: We derive a new formula for the Hurwitz–Lerch zeta function in terms of the infinite sum of the incomplete gamma function. Special cases are derived in terms of fundamental constants.

Keywords: Hurwitz–Lerch zeta function; incomplete gamma function; Apéry's constant; Catalan's constant; exponential integral function; Cauchy integral

MSC: 33-01; 33-03; 33-04; 33-33B; 33E20

1. Significance Statement

In 1887 Mathias Lerch [1] produced his famous manuscript on the Hurwitz–Lerch zeta function. Series representations of the Hurwitz–Lerch zeta function have been extensively studied in [1–3]. The Hurwitz–Lerch zeta function generalizes the Hurwitz zeta function, the polylogarithm function, and many interesting and important special functions.

The purpose of this work is to contribute to the existing literature on Hurwitz–Lerch zeta function series representations by giving a formal derivation of the Hurwitz–Lerch zeta function expressed in terms of the infinite sum of the incomplete gamma function. We expect that researchers will find this new integral formula helpful in their present and future study. Every new finding on the Hurwitz–Lerch zeta function is significant due to the function's many applications in both applied and pure mathematics. This study contains no previously published findings.

2. Introduction

We develop a novel formulation for the Hurwitz–Lerch zeta function in terms of the infinite sum of the incomplete gamma functions given by

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{a^{-m-2in-i}}\left(\frac{e^{\frac{i\pi k}{2}}a^{4in+2i}\Gamma(k+1,(-m+2in+i)\log(a))}{(-im-2n-1)^{k+1}}+\frac{\log^{k}(a)\Gamma(k+1,(-m-2in-i)\log(a))}{(i\log(a))^{k}(im-2n-1)^{k+1}}\right)\\&=\pi^{k+1}e^{\frac{\pi m}{2}}\Phi\left(-e^{m\pi},-k,\frac{\log(a)}{\pi}+\frac{1}{2}\right)\end{aligned}\tag{1}$$

where the variables k, a, m are general complex numbers and the branch cut for $\log(a)$ is given by Equation (4.2.3) in [4]. This new expression is then used to derive special cases in terms of fundamental constant and special functions. The derivations follow the method used by us in [5]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^{k}}{\Gamma(k+1)}=\frac{1}{2\pi i}\int_{C}\frac{e^{wy}}{w^{k+1}}dw,\tag{2}$$

where $y, w \in \mathbb{C}$ and C is in general an open contour in the complex plane where the bilinear concomitant [5] has the same value at the end points of the contour. The contour C is a counterclockwise oriented simple closed contour around the origin in the complex w-plane. This method involves using a form of Equation (2) then multiplies both sides by a function, then takes the definite integral of both sides. This yields a definite integral in terms of a

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contour integral. Then we multiply both sides of Equation (2) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

The Incomplete Gamma Function

The multivalued incomplete gamma functions [4], $\gamma(s,z)$ and $\Gamma(s,z)$, are defined by

$$\gamma(s,z) = \int_0^z t^{s-1}e^{-t}dt,$$

and

$$\Gamma(s,z) = \int_z^\infty t^{s-1}e^{-t}dt,$$

where $Re(a) > 0$. The incomplete gamma function has a recurrence relation given by

$$\gamma(s,z) + \Gamma(s,z) = \Gamma(s),$$

where $a \neq 0, -1, -2, \ldots$ The incomplete gamma function is continued analytically by

$$\gamma(a, ze^{2m\pi i}) = e^{2\pi mia}\gamma(a,z),$$

and

$$\Gamma(s, ze^{2m\pi i}) = e^{2\pi mis}\Gamma(s,z) + (1 - e^{2\pi mis})\Gamma(s),$$

where $m \in \mathbb{Z}$. When $z \neq 0$, $\Gamma(s,z)$ is an entire function of s and $\gamma(s,z)$ is meromorphic with simple poles at $s = -n$ for $n = 0, 1, 2, \ldots$ with residue $\frac{(-1)^n}{n!}$. These definitions are listed in Section 8.2(i) and (ii) in [4].

3. The Hurwitz–Lerch Zeta Function

We use Equation (1.11.3) in [2] where $\Phi(z,s,v)$ is the Hurwitz–Lerch zeta function [6,7] which is a generalization of the Hurwitz zeta $\zeta(s,v)$ and Polylogarithm functions $Li_n(z)$. The Hurwitz–Lerch zeta function has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{3}$$

where $|z| < 1, v \neq 0, -1, -2, -3, \ldots$, and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{4}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$. The classic case when $z = e^{2\pi ix}$ was studied by Lerch in [1].

4. Hurwitz–Lerch Zeta Function in Terms of the Contour Integral

We use the method in [5]. The cut and contour are in the first quadrant of the complex w-plane with $0 < Re(w+m) < 1$. The cut approaches the origin from the interior of the first quadrant and goes to infinity vertically and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we first replace $y \to \log(a) + \frac{1}{2}\pi(2y+1)$ then multiply both sides by $\frac{1}{2}\pi(-1)^y e^{\frac{1}{2}\pi m(2y+1)}$ and take the infinite sum over $y \in [0,\infty)$to get

$$\begin{aligned}&\frac{\pi^{k+1}e^{\frac{\pi m}{2}}\Phi\left(-e^{m\pi},-k,\frac{\log(a)}{\pi}+\frac{1}{2}\right)}{2\Gamma(k+1)}\\&=\sum_{n=0}^{\infty}\frac{\pi^{k+1}(-1)^{y}e^{\pi my+\frac{\pi m}{2}}\left(\frac{\log(a)}{\pi}+y+\frac{1}{2}\right)^{k}}{2\Gamma(k+1)}\\&=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C}\frac{1}{2}\pi(-1)^{y}a^{w}w^{-k-1}e^{\frac{1}{2}\pi(2y+1)(m+w)}dw\\&=\frac{1}{2\pi i}\int_{C}\sum_{n=0}^{\infty}\frac{1}{2}\pi(-1)^{y}a^{w}w^{-k-1}e^{\frac{1}{2}\pi(2y+1)(m+w)}dw\\&=\frac{1}{2\pi i}\int_{C}\frac{1}{4}\pi a^{w}w^{-k-1}sech\left(\frac{1}{2}\pi(m+w)\right)dw\end{aligned}\tag{5}$$

from Equation (1.232.2) in [8].

5. Incomplete Gamma Function in Terms of the Contour Integral

Using Equation (2) we replace $y\to iy+\log(a)$ and multiply both sides by $e^{y(im+2n+1)}$ and take the definite integral over $y\in[0,\infty)$ to get

$$\begin{aligned}&\frac{(-im-2n-1)^{-k-1}e^{\frac{1}{2}i(\pi k+\log(a)(2im+4n+2))}\Gamma(k+1,-(m-i(2n+1))\log(a))}{\Gamma(k+1)}\\&=\frac{1}{2\pi i}\int_{C}\frac{ia^{w}w^{-k-1}}{m-2in+w-i}dw\end{aligned}\tag{6}$$

from Equation (8.350.2) in [8]. Again using Equation (2) we replace $y\to -iy+\log(a)$ and multiply both sides by $e^{y(-im+2n+1)}$ and take the definite integral over $y\in[0,\infty)$ to get

$$\begin{aligned}&\frac{(i\log(a))^{-k}\log^{k}(a)a^{-m-2in-i}(im-2n-1)^{-k-1}\Gamma(k+1,-(m+2in+i)\log(a))}{\Gamma(k+1)}\\&=-\frac{1}{2\pi i}\int_{C}\frac{ia^{w}w^{-k-1}}{m+2in+w+i}dw\end{aligned}\tag{7}$$

from Equation (8.350.2) in [8]. Next we add Equations (6) and (7) and take the infinite sum over $n\in[0,\infty)$ to get

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2\Gamma(k+1)}((i\log(a))^{-k}\log^{k}(a)a^{-m-2in-i}(im-2n-1)^{-k-1}\Gamma(k+1,(-m-2in-i)\log(a))\\&+(-im-2n-1)^{-k-1}e^{\frac{1}{2}i(\pi k+\log(a)(2im+4n+2))}\Gamma(k+1,(-m+2in+i)\log(a)))\\&=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C}\frac{(-1)^{n}(2n+1)a^{w}w^{-k-1}}{(m+w)^{2}+(2n+1)^{2}}dw\\&=\frac{1}{2\pi i}\int_{C}\sum_{n=0}^{\infty}\frac{(-1)^{n}(2n+1)a^{w}w^{-k-1}}{(m+w)^{2}+(2n+1)^{2}}dw\\&=\frac{1}{2\pi i}\int_{C}\frac{1}{4}\pi a^{w}w^{-k-1}sech\left(\frac{1}{2}\pi(m+w)\right)dw\end{aligned}\tag{8}$$

from Equation (5.1.26.7) in [9].

6. The Hurwitz–Lerch Zeta Function in Terms of the Infinite Sum of the Incomplete Gamma Function

Theorem 1. *For all* $k,a,m\in\mathbb{C}$,

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{a^{m+2in+i}}\left(\frac{e^{\frac{i\pi k}{2}}a^{4in+2i}\Gamma(k+1,(-m+2in+i)\log(a))}{(-im-2n-1)^{k+1}}+\frac{\log^{k}(a)\Gamma(k+1,(-m-2in-i)\log(a))}{(i\log(a))^{k}(im-2n-1)^{k+1}}\right)\\&=\pi^{k+1}e^{\frac{\pi m}{2}}\Phi\left(-e^{m\pi},-k,\frac{\log(a)}{\pi}+\frac{1}{2}\right)\end{aligned}\tag{9}$$

Proof. Observe the right-hand sides of Equations (5) and (8) are equal so we can equate the left-hand sides and simplify the gamma function to get the stated result. □

7. Special Cases

In the following section we will evaluate Equation (9) and simplify both the incomplete gamma function $\Gamma(s,z)$ and the Hurwitz–Lerch zeta function $\Phi(k,a,m)$ in terms of special functions and fundamental constants. The functions and constants we will use are; Hurwitz zeta function $\zeta(k,a)$ given in (64:4:1), entry (2) in table below (64:7) and (64:12:1) in [10], Exponential integral function $E_n(z)$ given in (8.4.13) in [4], Catalan's constant C given in

(25.11.40) in [4], Loggamma function $\log(\Gamma(x))$ given in (64:10:1) in [10], Glaisher's constant A given in (5.17.7) in [4], and Apéry's constant $\zeta(3)$ given in (5.16.2) in [4].

Example 1. *The degenerate case.*

$$\sum_{n=0}^{\infty} \frac{m^2(-1)^n}{(2n+1)(m^2+(2n+1)^2)} = \frac{1}{4}\left(\pi - \pi sech\left(\frac{\pi m}{2}\right)\right) \tag{10}$$

Proof. Use Equation (9) and set $k = 0$ and simplify using entry (2) in Table below (64:12:7) in [10]. □

Example 2.

$$\begin{array}{l} \sum_{n=0}^{\infty}(-1)^{n+1}\Gamma(k+1)\left(e^{\frac{i\pi k}{2}}(-im-2n-1)^{-k-1}+(im-2n-1)^{-k-1}\right) \\ = \pi^{k+1}e^{\frac{\pi m}{2}}\Phi\left(-e^{m\pi},-k,\frac{1}{2}\right) \end{array} \tag{11}$$

Proof. Use Equation (9) and set $a = 1$ and simplify. □

Example 3.

$$\begin{array}{l} \sum_{n=0}^{\infty}(-1)^n e^{\pi(-2n-1)}\left(E_{-2}((-2n-(1-i))\pi)-e^{4\pi n+2\pi}E_{-2}((2n+(1+i))\pi)\right) \\ = \frac{1}{16}sech^3\left(\frac{\pi}{2}\right)(7-4i\sinh(\pi)+3\cosh(\pi)) \end{array} \tag{12}$$

Proof. Use Equation (9) and set $m = -1, k = 2, a = 1$ and simplify using entry (4) in table below (64:12:7) in [10]. □

Example 4.

$$\begin{array}{l} \sum_{n=0}^{\infty} ie^{\pi(-1+(-2+i)n)}\left(\Gamma(0,(-2n-1)\pi)-e^{4\pi n+2\pi}\Gamma(0,2\pi n+\pi)\right) \\ = \frac{1}{2}\left(\psi^{(0)}\left(\frac{3}{4}+\frac{i}{2}\right)-\psi^{(0)}\left(\frac{1}{4}+\frac{i}{2}\right)\right) \end{array} \tag{13}$$

Proof. Use Equation (9) and set $m = 0, k = -1, a = -1$ and simplify using entry (5) in table below (64:12:7) in [10]. □

Example 5.

$$\begin{array}{l} \sum_{n=0}^{\infty} e^{\pi(-1+(-2+i)n)}(2n+1)\left(\Gamma(0,(-2n-1)\pi)+e^{4\pi n+2\pi}\Gamma(0,2\pi n+\pi)\right) \\ = \frac{\psi^{(1)}\left(\frac{1}{4}+\frac{i}{2}\right)-\psi^{(1)}\left(\frac{3}{4}+\frac{i}{2}\right)}{4\pi} \end{array} \tag{14}$$

Proof. Use Equation (9) and set $m = 0, k = 2, a = -1$ and simplify using entry (5) in table below (64:12:7) in [10]. □

Example 6.

$$\begin{array}{l} \sum_{n=0}^{\infty}(-1)^{n+\frac{5}{4}}\left(\frac{e^{-\frac{1}{4}\pi(4n+(2-2i))}\Gamma\left(\frac{3}{2},\frac{1}{2}(-2n-(1-i))\pi\right)}{(-2n-(1-i))^{3/2}}+\frac{e^{\pi\left(n+\frac{1}{2}\right)}\Gamma\left(\frac{3}{2},\left(n+\left(\frac{1}{2}+\frac{i}{2}\right)\right)\pi\right)}{(-2n-(1+i))^{3/2}}\right) \\ = e^{-\pi/2}\pi^{3/2}\Phi\left(-e^{-\pi},-\frac{1}{2},\frac{1}{2}+\frac{i}{2}\right) \end{array} \tag{15}$$

Proof. Use Equation (9) and set $m = -1, k = 1/2, a = i$ and simplify. □

Example 7.

$$\begin{aligned}&\sum_{n=0}^{\infty} -ie^{\pi-i\pi n}\left(e^{4i\pi n}\Gamma(0,2i\pi n+(1+i)\pi)-\Gamma(0,((1-i)-2in)\pi)\right)\\&=2e^{\pi/2}-2e^{\pi}\tan^{-1}\left(e^{-\pi/2}\right)\end{aligned} \tag{16}$$

Proof. Use Equatieon (9) and set $m=-1,k=1,a=e^{\pi}$ and simplify using entry (5) in table below (64:12:7) in [10]. □

Example 8.

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^{n+\frac{7}{4}}e^{\pi-2i\pi n}\left(\frac{i\Gamma\left(\frac{1}{2},((1-i)-2in)\pi\right)}{\sqrt{-2n-(1+i)}}+\frac{e^{4i\pi n}\Gamma\left(\frac{1}{2},(2in+(1+i))\pi\right)}{\sqrt{-2n-(1-i)}}\right)\\&=e^{-\pi/2}\sqrt{\pi}\Phi\left(-e^{-\pi},\tfrac{1}{2},\tfrac{3}{2}\right)\end{aligned} \tag{17}$$

Proof. Use Equation (9) and set $m=-1,k=-1/2,a=e^{\pi}$ and simplify. □

Example 9.

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\Gamma\left(0,\tfrac{1}{2}((1-i)-2in)\pi\right)+e^{2i\pi n}\Gamma\left(0,i\pi n+\left(\tfrac{1}{2}+\tfrac{i}{2}\right)\pi\right)\right)\\&=-\log(1-\sinh(\pi)+\cosh(\pi))\end{aligned} \tag{18}$$

Proof. Use Equation (9) and set $m=-1,k=-1,a=e^{\pi/2}$ and simplify using entry (1) in table below (64:12:7) in [10]. □

Example 10.

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1+i)(-1)^{n+1}e^{\pi-2i\pi n}(((1+i)n+i)\Gamma(-1,((1-i)-2in)\pi)\\&+e^{4i\pi n}(1+(1+i)n)\Gamma(-1,(2in+(1+i))\pi))\\&=\frac{e^{-\pi/2}\Phi\left(-e^{-\pi},2,\frac{3}{2}\right)}{\pi}\end{aligned} \tag{19}$$

Proof. Use Equation (9) and set $m=-1,k=2,a=e^{\pi}$ and simplify. □

Example 11.

$$\begin{aligned}&\sum_{n=0}^{\infty}(-1)^{n+1}e^{-2i\pi n}(\Gamma\left(0,-\tfrac{1}{2}i(4n+1)\pi\right)-\Gamma\left(0,-\tfrac{1}{2}i(4n+3)\pi\right)\\&+e^{4i\pi n}\left(\Gamma\left(0,\tfrac{1}{2}i(4n+1)\pi\right)-\Gamma\left(0,\tfrac{1}{2}i(4n+3)\pi\right)\right))\\&=2\left(\sqrt{2}-\coth^{-1}\left(\sqrt{2}\right)\right)\end{aligned} \tag{20}$$

Proof. Use Equation (9) and form a second equation by replacing $m\to p$ and add these two equations followed by setting $k=-1,a=e^{\pi},m=i/2,p=-i/2$ and simplify using entry (5) in table below (64:12:7) in [10]. □

Example 12.

$$\begin{aligned}&\sum_{n=0}^{\infty}iE_2\left(-\tfrac{1}{4}i(4n+3)\pi\right)+E_2\left(-\tfrac{1}{4}i(4\pi n+\pi)\right)\\&+e^{2i\pi n}\left(E_2\left(\tfrac{1}{4}i(4n+3)\pi\right)+iE_2\left(\tfrac{1}{4}i(4\pi n+\pi)\right)\right)\\&=\left(\tfrac{1}{96}+\tfrac{i}{96}\right)(48C-\pi^2)\end{aligned} \tag{21}$$

Proof. Use Equation (9) and form a second equation by replacing $m\to p$ and take their difference followed by setting $k=-2,a=e^{\pi},m=i/2,p=-i/2$ and simplify in terms of Catalan's constant C using entry (2) in table below (64:12:7) and Equation (64:4:1) in [10]. □

Example 13.

$$\sum_{n=0}^{\infty}(-1)^n e^{-2i\pi n}(2n+1)\Big(\Gamma(-1,-i(2n+1)\pi)+e^{4i\pi n}\Gamma(-1,i(2n+1)\pi)\Big)=\frac{4-4C}{\pi} \quad (22)$$

Proof. Use Equation (9) and set $m=0$ and simplify in terms of the Hurwitz zeta function. Next set $k=-2, a=e^{\pi}$ and simplify using entry (2) in table below (64:12:7) and Equation (64:4:1) in [10]. □

Example 14.

$$\begin{array}{l}\sum_{n=0}^{\infty}(-1)^{n+1}a^{-i(2n+1)}(-2n-1)^{-k-1}(i\log(a))^{-k}\\ \Big(e^{\frac{i\pi k}{2}}a^{4in+2i}(i\log(a))^{k}\Gamma(k+1,i(2n+1)\log(a))+\log^{k}(a)\Gamma(k+1,-i(2n+1)\log(a))\Big)\\ =2^{k}\pi^{k+1}\Big(\zeta\Big(-k,\frac{2\log(a)+\pi}{4\pi}\Big)-\zeta\Big(-k,\frac{\log(a)}{2\pi}+\frac{3}{4}\Big)\Big)\end{array} \quad (23)$$

Proof. Use Equation (9) and set $m=0$ and simplify in terms of the Hurwitz zeta function using entry (4) in table below (64:12:7) in [10]. □

Example 15.

$$\sum_{n=0}^{\infty} i(-1)^n e^{-2i\pi n}\Big(\Gamma(0,-i(2n+1)\pi)-e^{4i\pi n}\Gamma(0,i(2\pi n+\pi))\Big)=2-\frac{\pi}{2} \quad (24)$$

Proof. Use Equation (23) and apply L'Hopital's rule as $k\to -1$ and simplify using Equation (64:9:2) in [10]. □

Example 16.

$$\begin{array}{l}\sum_{n=0}^{\infty}\frac{e^{i\pi n}\left(e^{\pi(-2n-1)}\Gamma(0,(-2n-1)\pi)+e^{2\pi n+\pi}\Gamma(0,2\pi n+\pi)+i\pi+2\log(\pi)\right)}{2n+1}\\ =\frac{1}{2}\pi\Big(-2\log\Gamma\Big(-\frac{3}{4}+\frac{i}{2}\Big)+2\log\Gamma\Big(-\frac{1}{4}+\frac{i}{2}\Big)+\log\Big(\Big(\frac{66}{169}-\frac{112i}{169}\Big)\pi\Big)\Big)\end{array} \quad (25)$$

Proof. Use Equation (23) and take the first partial derivative with respect to k and set $k=0, a=1$ and simplify using Equation (64:10:2) in [10]. □

Example 17.

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}e^{-2i\pi n}\Big(E_1(-i(2\pi n+\pi))+e^{4i\pi n}E_1(i(2\pi n+\pi))\Big)}{2n+1}=\frac{1}{2}\pi\log\left(\frac{9\Gamma\Big(-\frac{3}{4}\Big)^2}{8\Gamma\Big(-\frac{1}{4}\Big)^2}\right) \quad (26)$$

Proof. Use Equation (23) and take the first partial derivative with respect to k and set $k=0, a=e^{\pi}$ and simplify using Equation (64:10:2) in [10]. □

Example 18.

$$\sum_{n=0}^{\infty}\frac{i(-1)^n e^{-2i\pi n}\left(e^{4i\pi n}E_1(i(2\pi n+\pi))-E_1(-i(2\pi n+\pi))\right)}{(2n+1)^2}=\pi C-\frac{1}{2}\pi^2\log(2) \quad (27)$$

Proof. Use Equation (23) and take the first partial derivative with respect to k and set $k=1, a=e^{\pi}$ and simplify. □

Example 19.

$$-\sum_{n=0}^{\infty}\frac{i\left(E_1\left(-\frac{1}{2}i(2\pi n+\pi)\right)-e^{2i\pi n}E_1\left(\frac{1}{2}i(2\pi n+\pi)\right)\right)}{2n+1}=\pi\left(\log(2)-\frac{\log(\pi)}{2}\right) \quad (28)$$

Proof. Use Equation (23) and take the first partial derivative with respect to k and set $k=0, a=e^{\pi/2}$ and simplify. □

Example 20.

$$\sum_{n=0}^{\infty}\frac{E_1\left(-\frac{1}{2}i(2\pi n+\pi)\right)+e^{2i\pi n}E_1\left(\frac{1}{2}i(2\pi n+\pi)\right)}{(2n+1)^2}=\pi^2\log\left(\frac{2^{7/12}\sqrt[4]{e}}{A^3}\right) \quad (29)$$

Proof. Use Equation (9) and take the first partial derivative with respect to k and set $m=0, k=1, a=e^{\pi/2}$ and simplify. □

Example 21.

$$\sum_{n=0}^{\infty}\left(E_3\left(-\frac{1}{2}i(2\pi n+\pi)\right)+e^{2i\pi n}E_3\left(\frac{1}{2}i(2\pi n+\pi)\right)\right)=-\frac{3\zeta(3)}{16} \quad (30)$$

Proof. Use Equation (23) and set $k=-3, a=e^{\pi/2}$ and simplify. □

Example 22.

$$\sum_{n=0}^{\infty}\left(E_4\left(-\frac{1}{2}i(2\pi n+\pi)\right)+e^{2i\pi n}E_4\left(\frac{1}{2}i(2\pi n+\pi)\right)\right)=-\frac{7\pi^4}{5760} \quad (31)$$

Proof. Use Equation (23) and set $k=-4, a=e^{\pi/2}$ and simplify. □

Example 23.

$$\begin{array}{l}\sum_{n=0}^{\infty}(-1)^{n-1}(ie^{\frac{1}{3}\pi(-m-2in-i)}\Gamma\left(0,\frac{1}{3}(-m-2in-i)\pi\right)\\ +e^{\frac{1}{2}i\left(-\pi+\frac{1}{3}\pi(2im+4n+2)\right)}\Gamma\left(0,\frac{1}{3}(-m+2in+i)\pi\right))\\ =\frac{6}{5}e^{\frac{\pi m}{2}}\,{}_2F_1\left(\frac{5}{6},1;\frac{11}{6};-e^{m\pi}\right)\end{array} \quad (32)$$

Proof. Use Equation (9) and set $a=e^{\pi/3}, k=-1$ and simplify. □

Example 24. *The degenerate case.*

$$\sum_{n=0}^{\infty}\frac{2(-1)^n(2n+1)}{m^2+(2n+1)^2}=\frac{1}{2}\pi sech\left(\frac{\pi m}{2}\right) \quad (33)$$

Proof. Use Equation (9) and set $k=0$ and simplify using entry (2) in table below (64:12:7) in [10]. □

Example 25.

$$\begin{array}{l}\sum_{n=0}^{\infty}(-1)^{n+1}\left(e^{\pi\left(a-\frac{1}{2}\right)}\right)^{-m-2i(n+1)}(e^{\frac{i\pi k}{2}}\left(e^{\pi\left(a-\frac{1}{2}\right)}\right)^{2i(2n+1)}(-2n-im)^{-k-1}\Gamma\\ \left(k+1,\left(a-\frac{1}{2}\right)(2in-m)\pi\right)+\left(i\left(a-\frac{1}{2}\right)\right)^{-k}\left(a-\frac{1}{2}\right)^{k}\\ (-2(n+1)+im)^{-k-1}\Gamma\left(k+1,-\frac{1}{2}(2a-1)(m+2i(n+1))\pi\right))\\ =i\pi^{k+1}e^{\frac{\pi m}{2}}\Phi(e^{m\pi},-k,a)\end{array} \quad (34)$$

Proof. Use Equation (9) and set $a = e^{\pi(a-1/2)}, = m + i$ and simplify. □

Example 26.

$$\sum_{n=0}^{\infty}(-1)^n e^{-3i\pi n}\left(e^{6i\pi n}E_3\left(\frac{3}{2}i(2n-1)\pi\right)+E_3\left(-\frac{3}{2}i(2n+3)\pi\right)\right)=\frac{9}{16}(4-3\zeta(3)) \quad (35)$$

Proof. Use Equation (34) and set $m = i$ and simplify in terms of the Hurwitz zeta function. Next set $k = -3, a = 2$ and simplify. □

8. Discussion

In this short note, the authors derived an expression for the Hurwitz–Lerch zeta function $\Phi(k, a, m)$ in terms of the infinite sum of the incomplete gamma function $\Gamma(s, z)$, where the constraints on the parameters are wide. The infinite sum derived allowed for large values of the parameters and the derivations involved fundamental constants and special functions. We numerically verified the results for complex values of the parameters using Wolfram's Mathematica software. We will be applying our contour integral method to other infinite sums in future work.

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