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Some Unified Integrals for Generalized Mittag-Leffler Functions

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Abstract: Here, we ascertain generalized integral formulas concerning the product of the generalized Mittag-Leffler function. These integral formulas are described in the form of the generalized Lauricella series. Some special cases are also presented in terms of the Wright hypergeometric function.

Keywords: wright hypergeometric functions ${}_p\Psi_q$; generalized Lauricella series; Mittag-Leffler function; Oberhettinger's integral formula

MSC: 33B10; 33B15; 33C05; 33C20; 33C65; 33E12

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1. Introduction and Preliminaries

There are many properties like integral formulas and differential formulas concerning a diversity of special functions; specifically, hypergeometric functions and Mittag-Leffler functions have been discussed by numerous authors [1–9]. Recently, integral formulas concerning a generalized Mittag-Leffler function have been introduced by Jain et al. [10].

Currently, we are using a product of the generalized Mittag-Leffler function to ascertain some generalized integral formulas, which was described by Prabhakar [11]:

$$\begin{aligned}\prod_{i=1}^{m}\left(E_{(\rho_i),\zeta_i}^{(\gamma_i)}(z_i)\right) &\equiv E_{(\rho_1),\zeta_1}^{(\gamma_1)}(z_1)E_{(\rho_2),\zeta_2}^{(\gamma_2)}(z_2)\cdots E_{(\rho_m),\zeta_m}^{(\gamma_m)}(z_m)\\ &= \sum_{l_1=0}^{\infty}\frac{(\gamma_1)_{l_1}z_1^{l_1}}{\Gamma(\zeta_1+\rho_1 l_1)(l_1)!}\sum_{l_2=0}^{\infty}\frac{(\gamma_2)_{l_2}z_2^{l_2}}{\Gamma(\zeta_2+\rho_2 l_2)(l_2)!}\cdots\sum_{l_m=0}^{\infty}\frac{(\gamma_m)_{l_m}z_m^{l_m}}{\Gamma(\zeta_m+\rho_m l_m)(l_m)!},\end{aligned} \tag{1}$$

where, $(\zeta_i, \gamma_i, \rho_i, z_i \in \mathbb{C}, \Re(\rho_i) > 0, i = 1, \cdots, m)$.

Remark 1. *If we set $i = 1$ in (1.1), it becomes the generalized Mittag-Leffler function, due to Prabhakar [11]:*

$$E_{\rho,\zeta}^{\gamma}(z) = \sum_{l=0}^{\infty}\frac{(\gamma)_l z^l}{\Gamma(\zeta+l\rho)l!}, \tag{2}$$

where, $(\gamma, \rho, \zeta \in \mathbb{C}, \Re(\rho), \Re(\zeta) > 0, z \in \mathbb{C})$, with $(\zeta)_m$ defined by the Pochhammer symbol as (see [12]):

$$\begin{aligned}(\zeta)_m :&= \begin{cases}1 & (m=0)\\ \zeta(\zeta+1)\ldots(\zeta+m-1) & (m\in\mathbb{N}:=\{1,2,3,\ldots\})\end{cases}\\ &= \frac{\Gamma(\zeta+m)}{\Gamma(\zeta)} \quad (\zeta\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{aligned} \tag{3}$$

and $\mathbb{Z}_0^-$ represents the set of non-positive integers.

The generalized Lauricella series is defined as [13]:

$$F^{A:B^{(1)};\cdots;B^{(m)}}_{C:D^{(1)};\cdots;D^{(m)}}\begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = F^{A:B^{(1)};\cdots;B^{(m)}}_{C:D^{(1)};\cdots;D^{(m)}}\left(\begin{matrix}[(p):\eta^{(1)},\ldots,\eta^{(m)}]: \\ [(r):\psi^{(1)},\ldots,\psi^{(m)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(q)^{(1)}:\phi^{(1)}];\ldots;[(q)^{(m)}:\phi^{(m)}]; \\ [(s)^{(1)}:\delta^{(1)}];\ldots;[(s)^{(m)}:\delta^{(m)}];\end{matrix} z_1,\ldots,z_m\right), \tag{4}$$

$$= \sum_{l_1,\ldots,l_m=0}^{\infty} \Delta(l_1,\ldots,l_m)\,\frac{z_1^{l_1}}{l_1!}\cdots\frac{z_m^{l_m}}{l_m!},$$

where, for convenience,

$$\Delta(l_1,\ldots,l_m) = \frac{\prod_{i=1}^{A}(p_i)_{l_1\eta_i^{(1)}+\cdots+l_m\eta_i^{(m)}}\prod_{i=1}^{B^{(1)}}(q_i^{(1)})_{l_1\phi_i^{(1)}}\cdots\prod_{i=1}^{B^{(m)}}(q_i^{(m)})_{l_m\phi_i^{(m)}}}{\prod_{i=1}^{C}(r_i)_{l_1\psi_i^{(1)}+\cdots+l_n\psi_i^{(m)}}\prod_{i=1}^{D^{(1)}}(s_i^{(1)})_{l_1\delta_i^{(1)}}\cdots\prod_{i=1}^{D^{(m)}}(s_i^{(m)})_{l_n\delta_i^{(m)}}}. \tag{5}$$

Here, we recommend [13,14], for the convergence conditions of the generalized Lauricella series.

The Oberhettinger's integral formula is given by [15]:

$$\int_0^{\infty} y^{\mu-1}\left(y+b+\sqrt{y^2+2by}\right)^{-\zeta} dy = 2\zeta\, b^{-\zeta}\left(\frac{b}{2}\right)^{\mu}\frac{\Gamma(2\mu)\,\Gamma(\zeta-\mu)}{\Gamma(1+\zeta+\mu)}, \tag{6}$$

here $0 < \Re(\mu) < \Re(\zeta)$.

The unified integral is defined by Edward as [16]:

$$\int_0^1\int_0^1 (v')^{\nu}\left(1-u'\right)^{\nu-1}\left(1-v'\right)^{\omega-1}\left(1-u'v'\right)^{1-\nu-\omega}du'dv' = \frac{\Gamma(\nu)\Gamma(\omega)}{\Gamma(\nu+\omega)}, \tag{7}$$

here, $\Re(\nu) > 0, \Re(\omega) > 0$.

We also require the generalized hypergeometric function ${}_r\psi_s[z]$ defined as [17]:

$${}_r\psi_s[z] = \sum_{t=0}^{\infty}\frac{\prod_{i'=1}^{r}\Gamma(a_{i'}+\alpha_{i'}t)}{\prod_{j'=1}^{s}\Gamma(b_{j'}+\beta_{j'}t)}\frac{z^t}{t!}, \tag{8}$$

provided that,
$r,s \in N_0 = N\bigcup\{0\}$; $a_{i'}, b_{j'} \in \mathbb{C}$; $\alpha_{i'}, \beta_{j'} \in \mathbb{R}$; $\alpha_{i'}, \beta_{j'} \neq 0$; $i' = 1, \ldots, r$; $j' = 1, \ldots, s$.

The generalized Mittag-Leffler function is [18]:

$$E^{\gamma_i}_{\rho_i,\zeta}\left(\frac{a_j y'(1-x')(1-y')}{(1-x'y')^2}\right) = \frac{(\gamma_1)_{l_1}\cdots(\gamma_m)_{l_m}}{\Gamma(\zeta+\rho_1 l_1+\cdots\rho_m l_m)}\left(\left(\frac{a_1 y'(1-x')(1-y')}{(1-x'y')^2}\right)^{l_1}\frac{1}{l_1!}\cdots\left(\frac{a_m y'(1-x')(1-y')}{(1-x'y')^2}\right)^{l_m}\frac{1}{l_m!}\right), \tag{9}$$

here, $i = 1, \ldots, m$.

2. Main Results

In the present paper, first, we introduce two main Theorems by using the product of the generalized Mittag-Leffler function in Equation (1). These Theorems are defined in the form of the series in Equation (4). We insert the product of the generalized Mittag-Leffler function into the integrand of Equation (6), to establish our main results.

Theorem 1. *For the product of the generalized Mittag-Leffler function, the following integral holds:*

$$\begin{aligned}
&\int_0^\infty y^{\mu-1}\left(y+b+\sqrt{y^2+2by}\right)^{-\zeta}\prod_{i=1}^{m}\left(E^{(\gamma_i)}_{(\rho_i),\zeta_i}\left(\frac{z_i}{y+b+\sqrt{y^2+2by}}\right)\right)dy\\
=\;& 2^{1-\mu}\,b^{\mu-\zeta}\Gamma(2\mu)\frac{\Gamma(1+\zeta)}{\Gamma(\zeta)}\frac{\Gamma(\zeta-\mu)}{\Gamma(1+\zeta+\mu)}\frac{1}{\Gamma(\zeta_1)\cdots\Gamma(\zeta_m)}\times\\
&F^{2:1;\ldots;1}_{2:1;\ldots;1}\left[\begin{matrix}\Big[1+\zeta:1,\ldots,1\Big],\Big[\zeta-\mu:1,\ldots,1\Big] & : & [\gamma_1:1];\ldots;[\gamma_m;1] & ;\dfrac{z_1}{b},\ldots,\dfrac{z_m}{b}\\ \Big[\zeta:1,\ldots,1\Big],\Big[1+\zeta+\mu:1,\ldots,1\Big] & : & \overline{[\zeta_1,\rho_1]};\ldots;\overline{[\zeta_m,\rho_m]} & ;\end{matrix}\right],
\end{aligned}\tag{10}$$

where, $(\mu,\zeta,z_i\in\mathbb{C},y>0,(\,i=1,\cdots,m)$ and $\Re(\mu)>0,\Re(\zeta)>0,\Re(\mu)<\Re(\zeta))$.

Proof of Theorem 1. Let us denote the left-hand side of Equation (10), by $\mathcal{I}$. Then use Equation (1), in the integrand of Equation (10).

We have:

$$\begin{aligned}
\mathcal{I}=&\int_0^\infty y^{\mu-1}\left(y+b+\sqrt{y^2+2by}\right)^{-\zeta}\times\\
&\sum_{l_1=0}^{\infty}\frac{(\gamma_1)_{l_1}}{\Gamma(\zeta_1+\rho_1 l_1)}\left(\frac{z_1}{(y+b+\sqrt{y^2+2by})}\right)^{l_1}\frac{1}{l_1!}\cdots\times\\
&\cdots\sum_{l_m=0}^{\infty}\frac{(\gamma_m)_{l_m}}{\Gamma(\zeta_m+\rho_m l_m)}\left(\frac{z_m}{(y+b+\sqrt{y^2+2by})}\right)^{l_n}\frac{1}{l_m!}dy
\end{aligned}\tag{11}$$

By change of the order of summation and integration:

$$\begin{aligned}
\mathcal{I}=&\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\gamma_1)_{l_1}\cdots(\gamma_l)_{l_m}}{\Gamma(\zeta_1+\rho_1 l_1)\Gamma(\zeta_2+\rho_2 l_2)\cdots\Gamma(\zeta_m+\rho_m l_m)}\frac{z_1^{l_1}}{l_1!}\cdots\frac{z_m^{l_m}}{l_m!}\times\\
&\int_0^\infty y^{\mu-1}\left(y+b+\sqrt{y^2+2by}\right)^{-(\zeta+l_1+\cdots+l_m)}dy.
\end{aligned}\tag{12}$$

By using (6), in (12), we get:

$$\begin{aligned}
\mathcal{I}=&\sum_{l_1,\ldots,l_m=0}^{\infty}\frac{(\gamma_1)_{l_1}\cdots(\gamma_m)_{l_m}}{\Gamma(\zeta_1+\rho_1 l_1)\Gamma(\zeta_2+\rho_2 l_2)\cdots\Gamma(\zeta_m+\rho_m l_m)}\times\\
&2(\zeta+l_1+\ldots+l_m)b^{-(\zeta+l_1+\ldots+l_m)}\left(\frac{b}{2}\right)^{(\mu)}\frac{\Gamma(2\mu)\Gamma(\zeta+l_1+\ldots+l_m-\mu)}{\Gamma(1+\zeta+\mu+l_1+\ldots+l_m)}\frac{(z_1)^{l_1}}{l_1!}\cdots\frac{(z_m)^{l_m}}{l_m!}.
\end{aligned}\tag{13}$$

Then ordering all the non variable terms and putting $\zeta + l_1 + ... + l_m = \frac{\Gamma(\zeta+l_1+...+l_m+1)}{\Gamma(\zeta+l_1+...+l_m)}$, we have:

$$\begin{aligned}\mathcal{I} = {} & 2^{1-\mu}\, b^{\mu-\zeta}\Gamma(2\mu)\times \\ & \sum_{l_1,\ldots,l_m=0}^{\infty} \frac{(\gamma_1)_{l_1}\cdots(\gamma_m)_{l_m}}{\Gamma(\zeta_1+\rho_1 l_1)\Gamma(\zeta_2+\rho_2 l_2)\cdots\Gamma(\zeta_m+\rho_m l_m)} \frac{\Gamma(\zeta+l_1+\cdots+l_m+1)}{\Gamma(\zeta+l_1+\cdots+l_m)}\times \\ & \frac{\Gamma(\zeta+l_1+\cdots+l_m-\mu)}{\Gamma(1+\zeta+\mu+l_1+\cdots+l_m)}\left(\frac{z_1}{b}\right)^{l_1}\frac{1}{l_1!}\cdots\left(\frac{z_m}{b}\right)^{l_m}\frac{1}{l_m!}.\end{aligned} \tag{14}$$

Now, multiply and divide the above equation with $\Gamma(\zeta+1)$, $\Gamma(\zeta)$,$\Gamma(\zeta-\mu)$, $\Gamma(1+\zeta+\mu)$, $\Gamma(\zeta_1)\cdots\Gamma(\zeta_m)$ and using Gamma function property as $(1+\zeta)_{l_1+\cdots+l_m} = \frac{\Gamma(1+\zeta+l_1+\cdots+l_m)}{\Gamma(1+\zeta)}$, we have:

$$\begin{aligned}\mathcal{I} = {} & 2^{1-\mu}\, b^{\mu-\zeta}\Gamma(2\mu)\frac{\Gamma(1+\zeta)}{\Gamma(\zeta)}\frac{1}{\Gamma(\zeta)}\frac{\Gamma(\zeta-\mu)}{\Gamma(1+\zeta+\mu)}\frac{1}{\Gamma(\zeta_1)\cdots\Gamma(\zeta_m)}\times \\ & \sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(1+\zeta)_{l_1+\cdots+l_m}(\zeta-\mu)_{l_1+\cdots+l_m}}{(\zeta)_{l_1+\cdots+l_m}(1+\zeta+\mu)_{l_1+\cdots+l_m}}\times \\ & \frac{(\gamma)_{l_1}\cdots(\gamma)_{l_m}}{(\zeta_1)_{\rho_1 l_1}\cdots(\zeta_m)_{\rho_m l_m}}\frac{(z_1/b)^{l_1}}{l_1!}\cdots\frac{(z_m/b)^{l_m}}{l_m!}.\end{aligned} \tag{15}$$

By using Equation (4), in the above equation and rearranging the terms, we get our desired result, Equation (10). □

Theorem 2. *For the product of the generalized Mittag-Leffler function, the following integral holds:*

$$\begin{aligned}&\int_0^\infty y^{\mu-1}\left(y+b+\sqrt{y^2+2by}\right)^{-\zeta}\prod_{i=1}^{m}\left(E^{(\gamma_i)}_{(\rho_i),\zeta_i}\left(\frac{yz_i}{y+b+\sqrt{y^2+2by}}\right)\right)dy \\ &= 2^{1-\mu}b^{\mu-\zeta}\Gamma(\zeta-\mu)\frac{\Gamma(1+\zeta)}{\Gamma(\zeta)}\frac{\Gamma(2\mu)}{\Gamma(1+\zeta+\mu)}\frac{1}{\Gamma(\zeta_1)\cdots\Gamma(\zeta_m)}\times \\ & F^{2:1;\ldots;1}_{2:1;\ldots;1}\left[\begin{matrix}[1+\zeta:1,\ldots,1],[2\mu:2,\ldots,2] & : \\ [\zeta:1,\ldots,1],[1+\zeta+\mu:2,\ldots,2] & :\end{matrix}\right. \\ & \qquad\qquad \left.\frac{[\gamma_1:1];\ldots;[\gamma_m;1]}{[\zeta_1,\rho_1];\ldots;[\zeta_m,\rho_m]}\;\begin{matrix};\\;\end{matrix}\frac{z_1}{2},\ldots,\frac{z_m}{2}\right],\end{aligned} \tag{16}$$

where, ($\mu, \zeta, z_i \in \mathbb{C}$, $y>0$,($i=1,\cdots,m$) and $\Re(\mu)>0, \Re(\zeta)>0, \Re(\mu)<\Re(\zeta)$*).*

Proof of Theorem 2. By following the same rule that lead to the result in Equation (10), we get our desired result, Equation (16). □

Taking $i=1$ in Equations (10) and (16), following the same procedure as above, and then using Equation (8), we have the following result, which holds true for the Prabhakar-type function [11].

Corollary 1. *For the Prabhakar-type function [11], the following integral holds:*

$$\int_0^\infty y^{\mu-1}\left(y+b+\sqrt{y^2+2by}\right)^{-\zeta} E^{\gamma}_{\rho,\zeta}\left(\frac{z}{y+b+\sqrt{y^2+2by}}\right)dy = \frac{\Gamma(2\mu)}{\Gamma(\gamma)}2^{(1-\mu)}b^{(\mu-\zeta)}{}_3\psi_3\left[\begin{matrix}(\gamma,1),\,(\zeta-\mu,1),\,(\zeta+1,1)\,; \\ (\zeta,\rho),\,(1+\zeta+\mu,1),\,(\zeta,1)\,;\end{matrix}\; z/b\right]. \tag{17}$$

Corollary 2. *For the Prabhakar-type function [11], the following other integral also holds:*

$$\int_0^\infty y^{\mu-1}\left(y+b+\sqrt{y^2+2by}\right)^{-\zeta} E^{\gamma}_{\rho,\zeta}\left(\frac{yz}{y+b+\sqrt{y^2+2by}}\right)dy = \frac{\Gamma(\zeta-\mu)}{\Gamma(\gamma)}2^{(1-\mu)}b^{(\mu-\zeta)}{}_3\psi_3\left[\begin{matrix}(\gamma,1),\,(\zeta+1,1),\,(2\mu,2)\,; \\ (\zeta,\rho),\,(1+\zeta+\mu,2),\,(\zeta,1)\,;\end{matrix}\; z/2\right]. \tag{18}$$

Theorem 3. *The unified integral associated with the generalized Mittag-Leffler-type function holds:*

$$\begin{aligned}&\int_0^1\int_0^1 (y')^{\nu}\,(1-x')^{\nu-1}(1-y')^{\omega-1}(1-x'y')^{1-\nu-\omega}E^{\gamma_i}_{\rho_i,\zeta}\left(\frac{a_i y(1-x')(1-y')}{(1-x'y')^2}\right)dx'dy' \\ &= \frac{\Gamma(\nu)\Gamma(\omega)}{\Gamma(\nu+\omega)\Gamma(\zeta)}F^{2:1;\dots;1}_{2:0;\dots;0}\left[\begin{matrix}\left[\nu:1,\dots,1\right],\left[\omega:1,\dots,1\right] & : \\ \left[\nu+\omega:2,\dots,2\right],\left[\zeta:\rho_1,\dots,\rho_m\right] & :\end{matrix}\right. \\ &\qquad\left.\begin{matrix}[\gamma_1:1];\dots;[\gamma_m;1] & ; \\ -;\dots;- & ;\end{matrix}\; b_1,\dots,b_m\right],\end{aligned} \tag{19}$$

here, ($\Re(\nu)>0$, $\Re(\omega)>0$, ζ, $\gamma,\rho,z_i\in\mathbb{C}$, $y>0$,($i=1,\cdots,m$) and $0<\Re(\nu)<\Re(\omega)$).

Proof of Theorem 3. Let us denote the left-hand side of Equation (19) by $\mathcal{J}$. Then, we use the generalized Mittag-Leffler function in the integrand of unified integral (7). We have:

$$\begin{aligned}\mathcal{J} = &\int_0^1\int_0^1 (y')^{\nu}\,(1-x')^{\nu-1}\times \\ &(1-y')^{\omega-1}(1-x'y')^{1-\nu-\omega}E^{\gamma_i}_{\rho_i,\zeta}\left(\frac{a_i y'(1-x')(1-y')}{(1-x'y')^2}\right)dx'dy'.\end{aligned} \tag{20}$$

Then,

$$\begin{aligned}\mathcal{J} = &\int_0^1\int_0^1 (y')^{\nu}\,(1-x')^{\nu-1}(1-y')^{\omega-1}(1-x'y')^{1-\nu-\omega}\times \\ &\sum_{l_1,\cdots,l_m=0}^{\infty}\frac{(\gamma_1)_{l_1}\cdots(\gamma_m)_{l_m}}{\Gamma(\zeta+\rho_1 l_1+\cdots+\rho_m l_m)}\left(\frac{a_1 y'(1-x')(1-y')}{(1-x'y')^2}\right)^{l_1}\frac{1}{l_1!}\cdots\times \\ &\cdots\left(\frac{a_m y'(1-x')(1-y')}{(1-x'y')^2}\right)^{l_m}\frac{1}{l_m!}dx'dy'.\end{aligned} \tag{21}$$

Now, arranging the summation and integral part, we obtain:

$$\mathcal{J} = \sum_{l_1,\ldots,l_m=0}^{\infty} \frac{(\gamma_1)_{l_1}\cdots(\gamma_m)_{l_m}}{\Gamma(\zeta_1+\rho_1 l_1+\cdots+\rho_m l_m)} \frac{1}{l_1!}\cdots\frac{1}{l_m!} a_i^{l_1+\cdots+l_m} \times \int_0^1\int_0^1 y^{\nu+l_1+\cdots+l_m}\left(1-x'\right)^{\nu+l_1+\cdots+l_m-1} \times \left(1-y'\right)^{\omega+l_1+\cdots+l_m-1}\left(1-x'y'\right)^{1-\nu-\omega+2l_1+\cdots+2l_m} dx'dy'. \tag{22}$$

Now, using the unified integral (1.7), apparently the dean has now asked about getting my thesis back...we get:

$$\mathcal{J} = \sum_{l_1,\ldots,l_m=0}^{\infty} \frac{(\gamma_1)_{l_1}\cdots(\gamma_m)_{l_m}}{\Gamma(\zeta_1+\rho_1 l_1+\cdots+\rho_m l_m)} \frac{1}{l_1!}\cdots\frac{1}{l_m!} a_i^{l_1+\cdots+l_m} \times \frac{\Gamma(\nu+l_1+\cdots+l_m)\Gamma(\omega+l_1+\cdots+l_m)}{\Gamma(\nu+2l_1+\cdots+2l_m)}. \tag{23}$$

Multiply and divide the above equation with $\Gamma(\nu)$, $\Gamma(\omega)$, $\Gamma(\nu+\omega)$, $\Gamma(\zeta)$ and using the Gamma function property as $(\nu)_{l_1+\cdots+l_m} = \frac{\Gamma(\nu+l_1+\cdots+l_m)}{\Gamma(\nu)}$.

We get:

$$\mathcal{J} = \frac{\Gamma(\nu)\Gamma(\omega)}{\Gamma(\nu+\omega)\Gamma(\zeta)} \sum_{l_1,\ldots,l_m=0}^{\infty} \frac{(\gamma_1)_{l_1}\cdots(\gamma_m)_{l_m}}{\Gamma(\zeta_1+\rho_1 l_1+\cdots+\rho_m l_m)} \times \frac{(\nu)_{l_1+\cdots+l_m}(\omega)_{l_1+\cdots+l_m}}{(\nu+\omega)_{2l_1+\cdots+2l_m}} \frac{(a_1)^{l_1}}{l_1!}\cdots\frac{(a_m)^{l_m}}{l_m!}. \tag{24}$$

Then, by using the definition of the Lauricella series (4), in the above equation and rearranging the terms, we get our desired result, Theorem 3. □

Theorem 4. *The unified integral associated with the multiple generalized Mittag-Leffler-type function holds:*

$$\int_0^1\int_0^1 (y')^{\nu}\left(1-x'\right)^{\nu-1}\left(1-y'\right)^{\omega-1}\left(1-x'y'\right)^{1-\nu-\omega}\prod_{i=1}^{m} E_{\rho_i,\zeta_i}^{\gamma_i}\left(\frac{a_i y(1-x')(1-y')}{(1-x'y')^2}\right)dx'dy'$$
$$= \frac{\Gamma(\nu)\Gamma(\omega)}{\Gamma(\nu+\omega)\Gamma(\zeta_1)\cdots\Gamma(\zeta_m)} F_{1:\rho_1;\ldots;\rho_m}^{2:1;\ldots;1}\left[\begin{matrix}\left[\nu:1,\ldots,1\right],\left[\omega:1,\ldots,1\right] & : \\ \left[\nu+\omega:2,\ldots,2\right] & : \end{matrix}\right.$$
$$\left.\begin{matrix}[\gamma_1:1];\ldots;[\gamma_m;1] & ; \\ \overline{[\zeta_1,\rho_1]};\ldots;\overline{[\zeta_m,\rho_1]} & ; \end{matrix} a_1,\ldots,a_m\right], \tag{25}$$

here, ($Re(\nu) > 0$ *,* $Re(\omega) > 0$*,* ζ*,* $\gamma, \rho, z_i \in \mathbb{C}$*,* $y > 0$*,(* $i = 1,\cdots,m$*)* and $0 < \Re(\nu) < \Re(\omega)$).

Proof of Theorem 4. Let us denote the left-hand side of Equation (25) by $\mathcal{K}$. Then, we use the generalized Mittag-Leffler function in the integrand of unified integral in Equation (7). We have:

$$\mathcal{K} = \int_0^1\int_0^1 (y')^{\nu}\left(1-x'\right)^{\nu-1}\left(1-y'\right)^{\omega-1}\left(1-x'y'\right)^{1-\nu-\omega} \times \prod_{i=1}^{m} E_{\rho_i,\zeta_i}^{\gamma_i}\left(\frac{a_i y'(1-x')(1-y')}{(1-x'y')^2}\right)dx'dy'. \tag{26}$$

By the change of the order of summation and integration, we have:

$$\begin{aligned}\mathcal{K} = \int_0^1 \int_0^1 (y')^{\nu} \left(1-x'\right)^{\nu-1} \left(1-y'\right)^{\omega-1} \left(1-x'y'\right)^{1-\nu-\omega} \times \\ \sum_{l_1,\cdots,l_m=0}^{\infty} \frac{(\gamma_1)_{l_1}\cdots(\gamma_m)_{l_m}}{\Gamma(\zeta_1+l_1\rho_1)\cdots\Gamma(\zeta_m+l_m\rho_m)} \times \\ \left(\left(\frac{a_1 y'(1-x')(1-y')}{(1-x'y')^2}\right)^{l_1}\frac{1}{l_1!}\cdots\left(\frac{a_m y'(1-x')(1-y')}{(1-x'y')^2}\right)^{l_m}\frac{1}{l_m!}\right)dx'dy'.\end{aligned} \tag{27}$$

We obtain the following expression:

$$\begin{aligned}\mathcal{K} = \sum_{l_1,\ldots,l_m=0}^{\infty} \frac{(\gamma_1)_{l_1}\ldots(\gamma_m)_{l_m}}{\Gamma(\zeta_1+l_1\rho_1)\cdots\Gamma(\zeta_m+l_m\rho_m)}\frac{1}{l_1!}\cdots\frac{1}{l_m!}a_i^{l_1+\cdots+l_m}\times \\ \int_0^1\int_0^1 (y')^{\nu+l_1+\cdots+l_m}\left(1-x'\right)^{\nu+l_1+\cdots+l_m-1}\times \\ \left(1-y'\right)^{\omega+l_1+\cdots+l_m-1}\left(1-x'y'\right)^{1-\nu-\omega+2l_1+\cdots+2l_m}dx'dy'.\end{aligned} \tag{28}$$

Now, using the unified integral of Equation (7). We obtained the following:

$$\begin{aligned}\mathcal{K} = \sum_{l_1,\ldots,l_m=0}^{\infty} \frac{(\gamma_1)_{l_1}\ldots(\gamma_m)_{l_m}}{\Gamma(\zeta_1+l_1\rho_1)\cdots\Gamma(\zeta_m+l_m\rho_m)}\frac{1}{l_1!}\cdots\frac{1}{l_m!}a_i^{l_1+\cdots+l_m}\times \\ \frac{\Gamma(\nu+l_1+\cdots+l_m)\Gamma(\omega+l_1+\cdots+l_m)}{\Gamma(\nu+2l_1+\cdots+2l_m)}.\end{aligned} \tag{29}$$

Multiply and divide the above equation with $\Gamma(\nu)$, $\Gamma(\omega)$,$\Gamma(\nu+\omega)$, $\Gamma(\zeta_1)\cdots\Gamma(\zeta_m)$, and using the Gamma function property as $(\nu)_{l_1+\cdots+l_m} = \frac{\Gamma(\nu+l_1+\cdots+l_m)}{\Gamma(\nu)}$.

We have:

$$\begin{aligned}\mathcal{K} = \frac{\Gamma(\nu)\Gamma(\omega)}{\Gamma(\nu+\omega)\Gamma(\zeta_1)\cdots\Gamma(\zeta_m)} \\ \sum_{l_1,\ldots,l_m=0}^{\infty}\frac{(\gamma_1)_{l_1}\ldots(\gamma_m)_{l_m}}{(\zeta_1)_{l_1\rho_1}\cdots(\zeta_m)_{l_m\rho_m}}\frac{(\nu)_{l_1+\cdots+l_m}(\omega)_{l_1+\cdots+l_m}}{(\nu+\omega)_{2l_1+\cdots+2l_m}}\frac{(a_1)^{l_1}}{l_1!}\cdots\frac{(a_m)^{l_m}}{l_m!}.\end{aligned} \tag{30}$$

Then, we use the definition of the Lauricella series (Equation (4)) in the above equation, and, rearranging the terms, we get our desired result, Theorem 4. □

Remark 2. *Taking $i = 1$ in Equation (19), following the same procedure as above, and using Equation (8), we have the following result, which holds true for the Prabhakar-type function [11], as follows:*

$$\begin{aligned}\int_0^1\int_0^1 (y')^{\nu}\left(1-x'\right)^{\nu-1}\left(1-y'\right)^{\omega-1}\left(1-x'y'\right)^{(1-\nu-\omega)}E^{\gamma}_{\rho,\zeta}\left(\frac{ay'(1-x)(1-y')}{(1-x'y')^2}\right)dx'dy' \\ = \frac{1}{\Gamma(\gamma)}{}_3\psi_2\left[\begin{matrix}(\gamma,1),\,(\nu,1),\,(\omega,1)\,; \\ (\nu+\omega,2),\,(\zeta,\rho)\,;\end{matrix}\;a\right].\end{aligned} \tag{31}$$

3. Concluding Remark

We conclude our analysis by remarking that the results presented in this article are new and important for the class of Mittag-Leffler functions. By choosing different values of parameters, we can extract several sub-results from our main results. Further research will focus on basic applications and examples of these results for various research areas.

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