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# On Some Fractional Integral Inequalities Involving Caputo–Fabrizio Integral Operator

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**Abstract:** In this paper, we deal with the Caputo–Fabrizio fractional integral operator with a nonsingular kernel and establish some new integral inequalities for the Chebyshev functional in the case of synchronous function by employing the fractional integral. Moreover, several fractional integral inequalities for extended Chebyshev functional by considering the Caputo–Fabrizio fractional integral operator are discussed. In addition, we obtain fractional integral inequalities for three positive functions involving the same operator.

**Keywords:** Caputo–Fabrizio fractional integral; fractional integral inequality

**MSC:** 26D10; 26D33



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## 1. Introduction

Fractional calculus is a generalization of traditional calculus which deals with non-negative integer order integration and differentials which have various applications in different fields of science and technology. On this vast subject, we may cite [1–6]. In order to introduce some preliminary background to our findings, let us consider the following:

$$\mathcal{T}(\phi,\varphi) := \frac{1}{r_2-r_1}\int_{r_1}^{r_2}\phi(\xi)\varphi(\xi)d\xi - \frac{1}{r_2-r_1}\left(\int_{r_1}^{r_2}\phi(\xi)d\xi\right)\left(\frac{1}{r_2-r_1}\int_{r_1}^{r_2}\varphi(\xi)d\xi\right), \quad (1)$$

where $\phi$ and $\varphi$ are two integrable functions which are synchronous on $[r_1,r_2]$, i.e., $(\phi(\xi_1)-\phi(\xi_2))(\varphi(\xi_1)-\varphi(\xi_2)\geq 0$ for any $\xi_1,\xi_2\in[r_1,r_2]$. Further development of this functional can be found in [7,8]. Now, we present the extended Chebyshev's function defined by

$$\begin{aligned}\mathcal{T}(\Phi,\Psi,\phi,\varphi) :=& \int_{r_1}^{r_2}\varphi(\xi)d\xi\int_{r_1}^{r_2}\phi(\xi)\Phi(\xi)\Psi(\xi)d\xi\\ &+\int_{r_1}^{r_2}\phi(\xi)d\xi\int_{r_1}^{r_2}\varphi(\xi)\Phi(\xi)\Psi(\xi)d\xi\\ &-\left(\int_{r_1}^{r_2}\phi(\xi)\Phi(\xi)d\xi\right)\left(\int_{r_1}^{r_2}\varphi(\xi)\Psi(\xi)d\xi\right)\\ &-\left(\int_{r_1}^{r_2}\varphi(\xi)\Phi(\xi)d\xi\right)\left(\int_{r_1}^{r_2}\phi(\xi)\Psi(\xi)d\xi\right).\end{aligned} \quad (2)$$

See [9]. In the literature, many specialists have proposed fractional integral inequalities for Chebyshev functional (1) and extended Chebyshev functional (2), see [1,9–12]. Recently, many researchers in several fields have found different results about some known fractional integral inequalities and applications by means of the generalization of the

Riemann–Liouville, Caputo, Hadamard, Erdelyi–Kober, Saigo, Katugamapola and some other fractional integral operators, see [1,9,13–22].

The main motivation of the Caputo–Fabrizio integral and derivative operator is that it is a general fractional integral and derivative. In addition, it has a non singular kernel which can be described as a real power turned into an integral by means of the Laplace transform. Consequently, an exact solution can be easily found for several problems. Nowadays, fractional integral and derivative play big role for modeling various phenomenon physics. However, in [23,24], Caputo and Fabrizio introduced new fractional derivatives and integrals without a singular kernel. Certain phenomena related to material heterogeneities cannot be well-modeled by considering the Riemann–Liouville and Caputo fractional derivatives due to the singular kernel. It stems from Caputo and Fabrizio's proposal of a new fractional integral involving the nonsingular kernel $e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-s)}, 0<\kappa<1$. Recently, many mathematicians in applied sciences are using the Caputo–Fabrizio fractional integral operator to model their problems. For more details, we refer to [25–31]. In [32], the authors presented the fundamental solutions to the Cauchy and Dirichlet problems based upon a heat conduction equation equipped with the Caputo–Fabrizio derivative, which is investigated on a line segment. The main advantage of the Caputo–Fabrizio integral operator is that the boundary condition of the fractional differential equations with Caputo–Fabrizio derivatives admits the same form as for the integer-order differential equations. In the literature, very little work has been conducted on fractional integral inequalities using Caputo and Caputo–Fabrizio integral operators. In [10,14,16–18], the authors have established some new integral inequalities for the Chebyshev and extended Chebyshev functionals using different fractional operators. Recently, in [33], the authors have investigated several new estimations of the Hermite-Hadamard type inequality via generalized convex functions of the Raina type. In [34,35], the authors established fractional integral inequalities involving the Caputo–Fabrizio operator. From the above cited work, the main objective of this paper is to obtain some fractional integral inequalities for the functionals (1) and (2) by considering the Caputo–Fabrizio fractional integral operator. In addition, we establish some fractional integral inequalities for three positive and synchronous functions. The paper is organized into the following sections. Section 2 gives some basic definitions of fractional calculus. Section 3 is devoted to the proof of some fractional inequalities for Chebyshev functionals using the Caputo–Fabrizio fractional operator. Section 4 presents some inequalities involving the extended Chebyshev fractional in the case of synchronous function by employing the Caputo–Fabrizio fractional integral operator. Finally, concluding remarks are given in Section 5.

## 2. Preliminaries

Here, we provide some basic definitions of fractional calculus related to the Caputo–Fabrizio fractional integral operator.

**Definition 1** ([24,34])**.** *Let $\kappa \in \mathbb{R}$ such that $0<\kappa\leq 1$. The Caputo–Fabrizio fractional integral of order $\kappa$ of a function $\phi$ is defined by*

$$\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)] = \frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-s)}\phi(s)ds. \quad (3)$$

For $\kappa=1$, it is reduced to

$$\mathcal{I}^{1}_{0,\xi}[\phi(\xi)] = \int_0^{\xi}\phi(s)ds.$$

The above defintion may be extended to any $\kappa>0$.

**Definition 2** ([24,34])**.** *Let $\kappa, a \in \mathbb{R}$ such that $0 < \kappa < 1$. The Caputo–Fabrizio fractional derivative of order $\kappa$ of a function $\phi$ is defined by*

$$\mathcal{J}^{\kappa}_{a,\xi}[\phi(\xi)] = \frac{1}{1-\kappa}\int_a^{\xi} e^{-\left(\frac{\kappa}{1-\kappa}\right)(\xi-s)}\phi'(s)ds. \tag{4}$$

In this study, the focus is put on the Caputo–Fabrizio fractional integral operator, aiming to demonstrate some new inequalities involving it.

## 3. Fractional Inequalities for Chebyshev Functional

Here, we obtain inequalities for the Chebyshev functional using the Caputo–Fabrizio fractional operator.

**Theorem 1.** *Let $\phi$ and $\varphi$ be two synchronous functions on $[0,\infty)$. Then for all $\xi, \kappa > 0$, we have*

$$\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)] \geq \frac{1}{\mathcal{I}^{\kappa}_{0,\xi}[1]}\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)], \tag{5}$$

*where* $\mathcal{I}^{\kappa}_{0,\xi}[1] = \frac{1}{1-\kappa}\left[1 - e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]$.

**Proof.** Since $\phi$ and $\varphi$ are synchronous on $[0,\infty)$ for all $\mu, \theta \geq 0$, we have

$$\big(\phi(\mu) - \phi(\theta)\big)\big(\varphi(\mu) - \varphi(\theta)\big) \geq 0. \tag{6}$$

From (6), we get

$$\phi(\mu)\varphi(\mu) + \phi(\theta)\varphi(\theta) \geq \phi(\mu)\varphi(\theta) + \phi(\theta)\varphi(\mu). \tag{7}$$

By multiplying (7) by $\frac{1}{\kappa}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}$, which is positive, and then integrating the resulting identity with respect to $\mu$ from 0 to $\xi$, we have

$$\begin{aligned}&\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\mu)d\mu + \frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\theta)\varphi(\theta)d\mu \\ &\geq \frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\theta)d\mu + \frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\theta)\varphi(\mu)d\mu.\end{aligned} \tag{8}$$

Hence,

$$\begin{aligned}&\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi](\xi) + \phi(\theta)\varphi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}d\mu \\ &\geq \varphi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)d\mu + \phi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\varphi(\mu)d\mu,\end{aligned} \tag{9}$$

which implies that

$$\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)] + \phi(\theta)\varphi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[1] \geq \varphi(\theta)\mathcal{I}^{\kappa}_{0,\xi}\phi(\xi) + \phi(\theta)\mathcal{I}^{\kappa}_{0,\xi}\varphi(\xi). \tag{10}$$

By multiplying (10) by $\frac{1}{\kappa}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}$, which is positive, and then integrating $\theta$ from 0 to $\xi$, we have

$$\begin{aligned}&\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}d\theta + \mathcal{I}^{\kappa}_{0,\xi}[1]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\phi(\theta)\varphi(\theta)d\theta \\ &\geq \mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\varphi(\theta)d\theta + \mathcal{I}^{\kappa}_{0,\xi}\varphi(\xi)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\phi(\theta)d\theta.\end{aligned} \tag{11}$$

Therefore

$$\begin{aligned}&\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[1]+\mathcal{I}^{\kappa}_{0,\xi}[1]\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\\&\geq \mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)].\end{aligned} \tag{12}$$

It follows that

$$\mathcal{I}^{\kappa}_{0,\xi}[1]\left[2\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\right]\geq 2\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)]. \tag{13}$$

This ends the proof of Theorem 1. □

**Theorem 2.** *Let $\phi$ and $\varphi$ be two synchronous functions on $[0,\infty)$. Then, for all $\xi,\kappa,\lambda>0$, we have*

$$\begin{aligned}&\left(\frac{1}{1-\lambda}\left[1-e^{-\left(\frac{1-\lambda}{\lambda}\right)\xi}\right]\right)\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]+\left(\frac{1}{1-\kappa}\left[1-e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]\right)\mathcal{I}^{\lambda}_{0,\xi}[\phi\varphi(\xi)]\\&\geq \mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\mathcal{I}^{\lambda}_{0,\xi}[\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)]\mathcal{I}^{\lambda}_{0,\xi}[\phi(\xi)].\end{aligned} \tag{14}$$

**Proof.** To prove this theorem, first multiply the inequality (10) by $\frac{1}{\lambda}e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}$, which is positive. Then, by integrating the resulting identity with respect to $\theta$ over 0 to $\xi$, we obtain

$$\begin{aligned}&\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\frac{1}{\lambda}\int_0^{\xi}e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}d\theta+\mathcal{I}^{\kappa}_{0,\xi}[1]\frac{1}{\lambda}\int_0^{\xi}e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}\phi(\theta)\varphi(\theta)d\theta\\&\geq \mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\frac{1}{\lambda}\int_0^{\xi}e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}\varphi(\theta)d\theta+\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)]\frac{1}{\lambda}\int_0^{\xi}e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}\phi(\theta)d\theta,\end{aligned} \tag{15}$$

and this ends the proof of Theorem 2. □

**Remark 1.** *Applying Theorem 2 for $\kappa=\lambda$, we rediscover Theorem 1.*

**Theorem 3.** *Let $(\phi_i)_{i=1,2,\ldots,n}$ be positive increasing functions on $[0,\infty)$. Then, for all $\xi,\kappa>0$, we have*

$$\mathcal{I}^{\kappa}_{0,\xi}\left[\prod_{i=1}^{n}\phi_i(\xi)\right]\geq\left[\mathcal{I}^{\kappa}_{0,\xi}[1]\right]^{1-n}\prod_{i=1}^{n}\mathcal{I}^{\kappa}_{0,\xi}[\phi_i(\xi)]. \tag{16}$$

**Proof.** We prove this theorem by induction. Clearly, for $n=1$, we have $\mathcal{I}^{\alpha}_{0,\xi}[\phi_1(\xi)]\geq\mathcal{I}^{\kappa}_{0,\xi}[\phi_1(\xi)]$, for all $\xi,\kappa>0$.

For $n=2$, applying the Equation (5), we obtain

$$\mathcal{I}^{\kappa}_{0,\xi}[\phi_1\phi_2(\xi)]\geq\left[\mathcal{I}^{\kappa}_{0,\xi}[1]\right]^{-1}\mathcal{I}^{\kappa}_{0,\kappa}[\phi_1(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[\phi_2(\xi)]. \tag{17}$$

Suppose that, by induction hypothesis,

$$\mathcal{I}^{\kappa}_{0,\xi}\left[\prod_{i=1}^{n-1}[\phi_i(\xi)]\right]\geq\left[\mathcal{I}^{\kappa}_{0,\xi}[1]\right]^{2-n}\prod_{i=1}^{n-1}\mathcal{I}^{\kappa}_{0,\xi}[\phi_i(\xi)], \tag{18}$$

for all $\xi,\kappa>0$. Now, since $(\phi_i)_{i=1,2,\ldots,n}$ are positive increasing functions, then $(\prod_{i=1}^{n-1}\phi_i)(\xi)$ is an increasing function. Therefore we can apply Theorem 1 to the functions $\prod_{i=1}^{n-1}\phi_i=\varphi$ and $\phi_n=\phi$, and we obtain

$$\begin{aligned}
\mathcal{I}^{\kappa}_{0,\xi}\prod_{i=1}^{n}[\phi_i(\xi)] &\geq \left[\mathcal{I}^{\kappa}_{0,\xi}\prod_{i=1}^{n-1}[\phi_i\phi_n(\xi)]\right] \geq \mathcal{I}^{\kappa}_{0,\xi}[\varphi\phi(\xi)] \\
&\geq \left[\mathcal{I}^{\kappa}_{0,\xi}[1]\right]^{-1}\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)] \\
&\geq \left[\mathcal{I}^{\kappa}_{0,\xi}[1]\right]^{-1}\mathcal{I}^{\kappa}_{0,\xi}\left[\prod_{i=1}^{n-1}\phi_i(\xi)\right]\mathcal{I}^{\kappa}_{0,\xi}[\phi_n(\xi)] \\
&\geq \left[\mathcal{I}^{\kappa}_{0,\xi}[1]\right]^{-1}\left[\mathcal{I}^{\kappa}_{0,\xi}[1]\right]^{2-n}\left[\prod_{i=1}^{n-1}\mathcal{I}^{\kappa}_{0,\xi}[\phi_i(\xi)]\right]\mathcal{I}^{\kappa}_{0,\xi}[\phi_n(\xi)] \\
&\geq \left[\mathcal{I}^{\kappa}_{0,\xi}[1]\right]^{1-n}\prod_{i=1}^{n}\mathcal{I}^{\kappa}_{0,\xi}[\phi_i(\xi)].
\end{aligned} \tag{19}$$

This completes the proof of Theorem 3. □

## 4. Fractional Inequalities for Extended Chebyshev Fractional

Here, we present some inequalities on extended Chebyshev fractional in the case of synchronous functions by employing the Caputo–Fabrizio fractional integral operator.

**Lemma 1.** *Let $\phi$ and $\varphi$ be two integrable and synchronous functions on $[0,\infty)$ and $u,v:[0,\infty)\to[0,\infty)$. Then, for all $\kappa,\xi>0$, we have*

$$\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[u(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[v\phi\varphi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[v(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[u\phi\varphi(\xi)] \geq \\
&\mathcal{I}^{\kappa}_{0,\xi}[u\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[v\varphi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[v\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[u\varphi(\xi)].
\end{aligned} \tag{20}$$

**Proof.** Since $\phi$ and $\varphi$ are synchronous functions on $[0,\infty)$, for all $\mu,\theta\geq 0$, we have

$$(\phi(\mu)-\phi(\theta))(\varphi(\mu)-\varphi(\theta)) \geq 0. \tag{21}$$

Owing to (21), we obtain

$$\phi(\mu)\varphi(\mu)+\phi(\theta)\varphi(\theta) \geq \phi(\mu)\varphi(\theta)+\phi(\theta)\varphi(\mu). \tag{22}$$

By multiplying (22) by $\frac{u(\mu)}{\kappa}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}$, which is positive, and then integrating with respect to $\mu$ from 0 to $\xi$, we have

$$\begin{aligned}
&\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}u(\mu)\phi(\mu)\varphi(\mu)d\mu + \frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}u(\mu)\phi(\theta)\varphi(\theta)d\mu \\
&\geq \frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}u(\mu)\phi(\mu)\varphi(\theta)d\mu + \frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}u(\mu)\phi(\theta)\varphi(\mu)d\mu.
\end{aligned} \tag{23}$$

Consequently,

$$\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[u\phi\varphi(\xi)] + \phi(\theta)\varphi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[u(\xi)] \\
&\geq \varphi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[u\phi(\xi)] + \phi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[u\varphi(\xi)].
\end{aligned} \tag{24}$$

By multiplying (24) by $\frac{v(\theta)}{\kappa}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}$, which is positive, and then integrating with respect to $\theta$ from 0 to $\xi$, we have

$$
\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[u\phi\varphi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}v(\theta)d\theta \\
&+\mathcal{I}^{\kappa}_{0,\xi}[u(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}v(\theta)\phi(\theta)\varphi(\theta)d\theta \\
&\geq \mathcal{I}^{\kappa}_{0,\xi}[u\phi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}v(\theta)\varphi(\theta)d\theta \\
&+\mathcal{I}^{\kappa}_{0,\xi}[u\varphi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}v(\theta)\phi(\theta)d\theta.
\end{aligned} \tag{25}
$$

This completes the proof of the inequality (20). □

Now, we give our main result.

**Theorem 4.** *Let $\phi$ and $\varphi$ be two integrable and synchronous functions on $[0,\infty)$, and $r,p,q$: $[0,\infty)\to[0,\infty)$. Then, for all $\kappa,\xi>0$, we have*

$$
\begin{aligned}
&2\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\phi\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\phi\varphi(\xi)]\Big]+\\
&2\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\phi\varphi(\xi)]\geq\\
&\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[p\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\varphi(\xi)]\Big]+\\
&\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]\Big]+\\
&\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[p\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]\Big].
\end{aligned} \tag{26}
$$

**Proof.** To prove this theorem, put $u=p,\ v=q$, and using Lemma 1, we get

$$
\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\phi\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\phi\varphi(\xi)]\geq\\
&\mathcal{I}^{\kappa}_{0,\xi}[p\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\varphi(\xi)].
\end{aligned} \tag{27}
$$

Now , multiplying both sides in (27) by $\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]$, we have

$$
\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\phi\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\phi\varphi(\xi)]\Big]\geq\\
&\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[p\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\varphi(\xi)]\Big].
\end{aligned} \tag{28}
$$

Again, by putting $u=r, v=q$, and using Lemma 1, we get

$$
\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\phi\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\phi\varphi(\xi)]\geq\\
&\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)].
\end{aligned} \tag{29}
$$

By multiplying both sides of (29) by $\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]$, we have

$$
\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\phi\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\phi\varphi(\xi)]\Big]\geq\\
&\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[q\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[q\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]\Big].
\end{aligned} \tag{30}
$$

With the same arguments as in the inequalities (29) and (30), we can write

$$
\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\phi\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\phi\varphi](\xi)\Big]\geq\\
&\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[p\varphi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[p\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]\Big].
\end{aligned} \tag{31}
$$

Adding the inequalities (28), (30) and (31), we get the required inequality (26). □

**Lemma 2.** *Let $\phi$ and $\varphi$ be two integrable and synchronous functions on $[0,\infty)$, and $u, v$: $[0,\infty] \to [0,\infty)$. Then, for all $\xi, \kappa, \lambda > 0$, we have*

$$\begin{aligned} &\mathcal{I}^{\kappa}_{0,\xi}[u(\xi)]\mathcal{I}^{\lambda}_{0,\xi}[v\phi\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[v(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[u\phi\varphi(\xi)] \geq \\ &\mathcal{I}^{\kappa}_{0,\xi}[u\phi(\xi)]\mathcal{I}^{\lambda}_{0,\xi}[v\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[v\phi(\xi)]\mathcal{I}^{\kappa}_{0,\xi}[u\varphi(\xi)]. \end{aligned} \tag{32}$$

**Proof.** By multiplying both sides of (24) by $\frac{v(\theta)}{\lambda}e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}$, which is positive, and then integrating with respect to $\theta$ from 0 to $\xi$, we have

$$\begin{aligned} &\mathcal{I}^{\kappa}_{0,\xi}[u\phi\varphi(\xi)]\frac{1}{\lambda}\int_0^{\xi} e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}v(\theta)d\theta \\ &+ \mathcal{I}^{\kappa}_{0,\xi}[u(\xi)]\frac{1}{\lambda}\int_0^{\xi} e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}v(\theta)\phi(\theta)\varphi(\theta)d\theta \\ &\geq \mathcal{I}^{\kappa}_{0,\xi}[u\phi(\xi)]\frac{1}{\lambda}\int_0^{\xi} e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}v(\theta)\varphi(\theta)d\theta \\ &+ \mathcal{I}^{\kappa}_{0,\xi}[u\varphi(\xi)]\frac{1}{\lambda}\int_0^{\xi} e^{-\left(\frac{1-\lambda}{\lambda}\right)(\xi-\theta)}v(\theta)\phi(\theta)d\theta. \end{aligned} \tag{33}$$

This completes the proof of Lemma 2. □

**Theorem 5.** *Let $\phi$ and $\varphi$ be two integrable and synchronous functions on $[0,\infty)$, and $r, p, q$: $[0,\infty) \to [0,\infty)$. Then, for all $\xi, \kappa, \lambda > 0$, we have*

$$\begin{aligned} &\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)][\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[p\phi\varphi(\xi)] + 2\,\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\phi\varphi(\xi)] \\ &+ \mathcal{I}^{\lambda}_{0,\xi}[q(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[p\phi\varphi(\xi)]] \\ &+ \left[\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[p(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\right]\mathcal{I}^{\kappa}_{0,\xi}[r\phi\varphi(\xi)] \geq \\ &\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\left[\mathcal{I}^{\kappa}_{0,\xi}[p\phi(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[q\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[p\varphi(\xi)]\right] + \\ &\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\left[\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[q\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]\right] + \\ &\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\left[\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[p\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[pf(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]\right]. \end{aligned} \tag{34}$$

**Proof.** To prove this theorem, we put $u = p$, $v = q$ and, by using Lemma 2, we get

$$\begin{aligned} &\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\phi\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[q(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[p\phi\varphi(\xi)] \geq \\ &\mathcal{I}^{\kappa}_{0,\xi}[p\phi(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[q\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[p\varphi(\xi)]. \end{aligned} \tag{35}$$

Now, multiplying both sides of (35) by $\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]$, we obtain

$$\begin{aligned} &\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\left[\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\phi\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[q(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[p\phi\varphi(\xi)]\right] \geq \\ &\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\left[\mathcal{I}^{\kappa}_{0,\xi}[p\phi(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[q\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[p\varphi(\xi)]\right]. \end{aligned} \tag{36}$$

By putting $u = r$, $v = q$, and using Lemma 2, we get

$$\begin{aligned} &\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\phi\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[q(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[r\phi\varphi(\xi)] \geq \\ &\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\varphi(\xi)] + \mathcal{I}^{\lambda}_{0,\xi}[q\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]. \end{aligned} \tag{37}$$

By multiplying both sides of (37) by $\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]$, we have

$$\begin{aligned}&\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\phi\varphi(\xi)]+\mathcal{I}^{\lambda}_{0,\xi}[q(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[r\phi\varphi(\xi)]\Big]\geq\\&\mathcal{I}^{\kappa}_{0,\xi}[p(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[q\varphi(\xi)]+\mathcal{I}^{\lambda}_{0,\xi}[q\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]\Big].\end{aligned} \tag{38}$$

With the same argument as in the Equations (37) and (38), we obtain

$$\begin{aligned}&\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[p\phi\varphi(\xi)]+\mathcal{I}^{\lambda}_{0,\xi}[p(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[r\phi\varphi(\xi)]\Big]\geq\\&\mathcal{I}^{\kappa}_{0,\xi}[q(\xi)]\Big[\mathcal{I}^{\kappa}_{0,\xi}[r\phi(\xi)]\,\mathcal{I}^{\lambda}_{0,\xi}[p\varphi(\xi)]+\mathcal{I}^{\lambda}_{0,\xi}[p\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[r\varphi(\xi)]\Big].\end{aligned} \tag{39}$$

Adding the inequalities (36), (38) and (39), we get the inequality (34). □

**Remark 2.** *If $\phi, \varphi, r, p$ and $q$ are functions satisfying the following conditions:*
1. *The functions $\phi$ and $\varphi$ are asynchronous on $[0,\infty)$.*
2. *The functions $r$, $p$, $q$ are negative on $[0,\infty)$.*
3. *Two of the functions $r$, $p$, $q$ are positive and the third is negative on $[0,\infty)$.*

*Then, the inequalities (26) and (34) are reversed.*

Here, we give some fractional integral inequalities involving the Caputo–Fabrizio fractional integer operator.

**Theorem 6.** *Let $\phi$, $\varphi$ and $\chi$ be three positive functions on $[0,\infty)$ such that*

$$(\phi(\mu)-\phi(\theta))(\varphi(\mu)-\varphi(\theta))(\chi(\mu)+\chi(\theta))\geq 0, \tag{40}$$

*for all $\mu,\theta\geq 0$. Then*

$$\begin{aligned}&\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi\chi(\xi)]\left(\frac{1}{1-\kappa}\left[1-e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]\right)+\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)]\\&\geq\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)]+\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)],\end{aligned} \tag{41}$$

*for all $\kappa,\xi>0$.*

**Proof.** From the condition (40), for any $\mu,\theta\geq 0$, we have

$$\begin{aligned}&\phi(\mu)\varphi(\mu)\chi(\mu)+\phi(\mu)\varphi(\mu)\chi(\theta)+\phi(\theta)\varphi(\theta)\chi(\mu)+\phi(\theta)\varphi(\theta)\chi(\theta)\\&\geq\phi(\mu)\varphi(\theta)\chi(\mu)+\phi(\theta)\varphi(\mu)\chi(\mu)+\phi(\theta)\varphi(\mu)\chi(\theta)+\phi(\mu)\varphi(\theta)\chi(\theta).\end{aligned} \tag{42}$$

By multiplying both sides of the inequality (42) by $\frac{1}{\kappa}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}$, which is positive, and then integrating with respect to $\mu$ from 0 to $\xi$, we get

$$\begin{aligned}&\frac{1}{\kappa}\int_0^{\xi}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\mu)\chi(\mu)d\mu+\frac{1}{\kappa}\int_0^{\xi}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\mu)\chi(\theta)d\mu\\&+\frac{1}{\kappa}\int_0^{\xi}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\theta)\varphi(\theta)\chi(\mu)d\mu+\frac{1}{\kappa}\int_0^{\xi}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\theta)\varphi(\theta)\chi(\theta)d\mu\\&\geq\frac{1}{\kappa}\int_0^{\xi}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\theta)\chi(\mu)d\mu+\frac{1}{\kappa}\int_0^{\xi}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\theta)\varphi(\mu)\chi(\mu)d\mu\\&+\frac{1}{\kappa}\int_0^{\xi}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\theta)\varphi(\mu)\chi(\theta)d\mu+\frac{1}{\kappa}\int_0^{\xi}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\theta)\chi(\theta)d\mu,\end{aligned} \tag{43}$$

which implies that

$$\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi\chi(\xi)] + \chi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)] + \phi(\theta)\varphi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)] \\
&+ \phi(\theta)\varphi(\theta)\chi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}d\mu \geq \varphi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)] \\
&+ \phi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)] + \phi(\theta)\chi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)] + \varphi(\theta)\chi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)].
\end{aligned} \tag{44}$$

Again, multiplying inequality (44) by $\frac{1}{\kappa}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}$, which is positive, and integrating with respect to $\theta$ from 0 to $\xi$, we have

$$\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi\chi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}d\theta + \mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\chi(\theta)d\theta \\
&+ \mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\rho)}\phi(\theta)\varphi(\theta)d\theta \\
&+ \left(\frac{1}{1-\kappa}\left[1-e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]\right)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\phi(\theta)\varphi(\theta)\chi(\theta)d\theta \\
&\geq \frac{\mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)]}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\varphi(\theta)d\theta + \mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)]\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\phi(\theta)d\theta \\
&+ \frac{\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)]}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\phi(\theta)\chi(\theta)d\theta + \frac{\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\theta)}\varphi(\theta)\chi(\theta)d\theta.
\end{aligned} \tag{45}$$

Hence,

$$\begin{aligned}
&\left(\frac{1}{1-\kappa}\left[1-e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]\right)\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi\chi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)] \\
&+ \mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)] + \left(\frac{1}{1-\kappa}\left[1-e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]\right)\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi\chi(\xi)] \\
&\geq \mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)] \\
&+ \mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)].
\end{aligned} \tag{46}$$

This completes the proof of the inequality (41). □

**Theorem 7.** *Let $\phi$, $\varphi$ and $\chi$ be three positive functions on $[0,\infty)$ such that*

$$(\phi(\mu)-\phi(\theta))(\varphi(\mu)+\varphi(\theta))(\chi(\mu)+\chi(\theta)) \geq 0, \tag{47}$$

*for all $\mu,\theta \geq 0$. Then*

$$\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)] \\
&\geq \mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)],
\end{aligned} \tag{48}$$

*for all $\kappa,\xi > 0$.*

**Proof.** From the condition (40), for any $\mu,\theta \geq 0$, we have

$$\begin{aligned}
&\phi(\mu)\varphi(\mu)\chi(\mu) + \phi(\mu)\varphi(\mu)\chi(\theta) + \phi(\mu)\varphi(\theta)\chi(\mu) + \phi(\mu)\varphi(\theta)\chi(\theta) \\
&\geq \phi(\theta)\varphi(\mu)\chi(\mu) + \phi(\theta)\varphi(\mu)\chi(\theta) + \phi(\theta)\varphi(\theta)\chi(\mu) + \phi(\theta)\varphi(\theta)\chi(\theta).
\end{aligned} \tag{49}$$

By multiplying both sides of the inequality (49) by $\frac{1}{\kappa}e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}$, which is positive, and then integrating with respect to $\mu$ from 0 to $\xi$, we get

$$
\begin{aligned}
&\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\mu)\chi(\mu)d\mu + \chi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\mu)d\mu \\
&+ \varphi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\chi(\mu)d\mu + \varphi(\theta)\chi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)d\mu \\
&\geq \phi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\varphi(\mu)\chi(\mu)d\mu + \chi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\phi(\mu)\varphi(\mu)d\mu \\
&+ \phi(\theta)\varphi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}\chi(\mu)d\mu + \phi(\theta)\varphi(\theta)\chi(\theta)\frac{1}{\kappa}\int_0^{\xi} e^{-\left(\frac{1-\kappa}{\kappa}\right)(\xi-\mu)}d\mu,
\end{aligned} \tag{50}
$$

which implies that

$$
\begin{aligned}
&\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi\chi(\xi)] + \chi(\theta)\,\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)] + \varphi(\theta)\mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)] + \varphi(\theta)\chi(\theta)\,\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)] \\
&\geq \chi(\theta)\,\mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)] + \chi(\theta)\,\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)] + \phi(\theta)\varphi(\theta)\,\mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)] \\
&+ \phi(\theta)\varphi(\theta)\chi(\theta)\left(\frac{1}{1-\kappa}\left[1-e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]\right).
\end{aligned} \tag{51}
$$

With the same argument as in inequality (45), we obtain

$$
\begin{aligned}
&\left(\frac{1}{1-\kappa}\left[1-e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]\right)\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi\chi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)] \\
&+ \mathcal{I}^{\kappa}_{0,\xi}[\phi\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\varphi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)] \\
&\geq \mathcal{I}^{\kappa}_{0,\xi}[\varphi\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi(\xi)] + \mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)] \\
&+ \mathcal{I}^{\kappa}_{0,\xi}[\chi(\xi)]\,\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi(\xi)] + \left(\frac{1}{1-\kappa}\left[1-e^{-\left(\frac{1-\kappa}{\kappa}\right)\xi}\right]\right)\mathcal{I}^{\kappa}_{0,\xi}[\phi\varphi\chi(\xi)].
\end{aligned} \tag{52}
$$

This completes the proof of inequality (48). □

## 5. Concluding Remarks

In this paper, we studied the novel fractional integral inequalities for the Chebyshev and extended the Chebyshev functionals by considering the Caputo–Fabrizio fractional integral operator. In addition, we studied some inequalities for three positive functions using the same operator. The inequalities investigated in this paper make some contribution to the fields of fractional calculus and Caputo–Fabrizio fractional integral operators. In the future, we hope that inequalities presented in this paper can prove the existence and uniqueness of some ordinary differential equations, as well as initial and boundary value problems involving Caputo–Fabrizio fractional operators.

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