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Criteria for the Oscillation of Solutions to Linear Second-Order Delay Differential Equation with a Damping Term

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Abstract: The aim of this work is to present new oscillation results for a class of second-order delay differential equations with damping term. The new criterion of oscillation depends on improving the asymptotic properties of the positive solutions of the studied equation by using an iterative technique. Our results extend some of the results recently published in the literature.

Keywords: second-order; delay differential equations; damping term; oscillation

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1. Introdiction

This article is concerned with studying the oscillatory behavior of the second-order delay differential equation

$$\left(r(t)\phi'(t)\right)' + p(t)\phi'(t) + q(t)\phi(\varrho(t)) = 0, \quad t \geq t_0, \tag{1}$$

where r, q, $p \in C([t_0,\infty))$, r is positive, q is nonnegative, $\varrho(t) \in C^1([t_0,\infty))$, $\varrho'(t) > 0$, $\lim_{t\to\infty} \varrho(t) = \infty$, and

$$\int_t^\infty \frac{1}{r(u)} \exp\left(-\int_{t_0}^u \left|\frac{p(s)}{r(s)}\right| \mathrm{d}s\right) \mathrm{d}u < \infty. \tag{2}$$

By a solution of (1), we mean a function $\phi \in C([t,\infty),\mathbb{R})$, $t_x \geq t_0$, which has the property $r(t)\phi'(t)$ is differentiable and satisfies (1) on $[t_x,\infty)$. We consider only those solutions ϕ of (1) which satisfy $\sup\{|\phi(t)| : t \geq t_x\} > 0$, for all $t > t_x$. Oscillatory term is related to a solution if arbitrarily large zeros on $[t_z,\infty)$ exist. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Differential and difference equations have been used, since long time ago, to describe natural phenomena and have many applications in different sciences. Therefore, it is natural to notice an increasing interest in studying the qualitative properties of these equations, for example [1–3]. Oscillation theory is a branch of the qualitative theory of functional differential equations, which is concerned with the study of the oscillatory and non-oscillatory behavior of solutions to differential equations. The growing interest and development in the oscillation theory of delay differential equations can be seen through the works [4–7].

For second-order with damping term, Saker et al. [8] established Kamenev-type and Philos-type theorems for oscillation of equation with damping term

$$\left(a(t)x'(t)\right)' + p(t)x'(\sigma(t)) + q(t)f(x(g(t))) = 0$$

The results in [8] extended and improved results in [9–11]. Graef et al. [12] studied the oscillatory behavior of equations with damping term

$$\big(r(t)z'(t)\big)' + p(t)z'(t) + q(t)x(\sigma(t)) = 0,$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{r(u)} \exp\left(-\int_{t_0}^{u} \frac{p(s)}{r(s)}\mathrm{d}s\right)\mathrm{d}u = \infty.$$

The objective of this paper is to establish new oscillation criteria for (1) in the non-canonical case. We offer a one-condition criterion which guarantees the oscillation of all solutions of the studied equation. The new criterion is also characterized by an iterative nature, meaning that it can be applied several times even if it fails in the beginning. Finally, we present an example that demonstrates the importance and applicability of the results.

2. Main Results I: Delay Equation

In this section, we assume that $\varrho(t) \leq t$. For the sake of brevity, we define

$$\mu(t) = \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)}\mathrm{d}s\right)$$

and

$$\delta(t) = \int_{t}^{\infty} \frac{1}{\mu(u)r(u)}\mathrm{d}u.$$

We provide the lemmas that help us achieve the main results.

Lemma 1. *Assume that ϕ is an eventually positive solution of (1). Then, $\big(\mu(t)r(t)\phi'(t)\big)' \leq 0$, and there exists t_1 such that one of the following cases hold:*
$(\mathbf{C}_1)$ $\phi'(t) > 0$,
$(\mathbf{C}_2)$ $\phi'(t) < 0$,
for $t \geq t_1 \geq t_0$.

Proof. Suppose that there exists $t \geq t_0$ such that $\phi(t)$ and $\phi(\varrho(t))$ are positive functions. As a direct conclusion from (1) and the definition of $\mu(t)$, we get

$$\big(\mu(t)r(t)\phi'(t)\big)' = -\mu(t)q(t)\phi(\varrho(t)),$$

which means that $\mu(t)r(t)\phi'(t)$ is a nonincreasing function and has one sign eventually. This means that $\phi'(t)$ is of one sign eventually, since $\mu(t)r(t)$ is positive. This ends the proof. □

Lemma 2. *Assume that ϕ is a positive solution of (1) and there exists a $\beta_0 \in (0,1)$ such that*

$$\mu^2(t)r(t)q(t)\delta^2(t) \geq \beta_0. \tag{3}$$

Then, ϕ satisfies $(\mathbf{C}_2)$,
(A_1) $\lim_{t\to\infty} \phi(t) = 0$,
(A_2) $\phi(t) + \mu(t)r(t)\phi'(t)\delta(t) \geq 0$,
(A_3) ϕ/δ is non-decreasing,
(A_4) ϕ/δ^{β_0} is non-increasing,
(A_5) $\lim_{t\to\infty} \phi(t)/\delta^{\beta_0}(t) = 0$, and
(A_6) $\phi/\delta^{1-\beta_0}$ is increasing.

Proof. Assume on the contrary that ϕ is an eventually positive solution of (1) satisfying case $(\mathbf{C}_1)$ for $t \geq t_1 \geq t_0$. As a direct conclusion from (1) and the definition of $\mu(t)$, we get

$$\big(\mu(t)r(t)\phi'(t)\big)' + \mu(t)q(t)\phi(\varrho(t)) = 0. \tag{4}$$

Integrating (4) from t_1 to t, we get

$$\mu(t)r(t)\phi'(t) - \mu(t_1)r(t_1)\phi'(t_1) = -\int_{t_1}^{\infty} \mu(s)q(s)\phi(\varrho(s))\mathrm{d}s. \tag{5}$$

Since ϕ is positive and increasing, there exists k which is a positive constant such that $\phi(t) \geq k$ and $\phi(\varrho(t)) \geq k$ eventually. Therefore, we obtain

$$\begin{aligned} \mu(t)r(t)\phi'(t) - \mu(t_1)r(t_1)\phi'(t_1) &\leq -k\int_{t_1}^{t} \mu(s)q(s)\mathrm{d}s \\ &\leq -k\beta_0 \int_{t_1}^{t} \frac{1}{\mu(s)r(s)\delta^2(s)}\mathrm{d}s = -k\beta_0 \int_{t_1}^{t} \frac{-\delta'(s)}{\delta^2(s)}\mathrm{d}s, \end{aligned}$$

and so

$$-\mu(t_1)r(t_1)\phi'(t_1) \leq k\beta_0\left(\frac{1}{\delta(t_1)} - \frac{1}{\delta(t)}\right) \to -\infty \text{ as } t \to \infty,$$

which is a contradiction, and hence ϕ satisfies $(\mathbf{C}_2)$.

(A_1): Consequently, $\lim_{t\to\infty} \phi(t) = \lambda \geq 0$. We claim that $\lambda = 0$. If not, then there exists a $t_1 \geq t_0$ such that $\phi(t) \geq \lambda > 0$ for $t \geq t_1$. Thus, from (5), we have

$$\mu(t)r(t)\phi'(t) \leq -\lambda \int_{t_1}^{t} \mu(s)q(s)\mathrm{d}s.$$

Then,

$$\begin{aligned} \phi'(t) &\leq -\frac{\lambda\beta_0}{\mu(t)r(t)} \int_{t_1}^{t} \frac{1}{\mu(s)r(s)\delta^2(s)}\mathrm{d}s \\ &= \frac{\lambda\beta_0}{\mu(t)r(t)}\left(\frac{1}{\delta(t_1)} - \frac{1}{\delta(t)}\right) \end{aligned}$$

By integrating once again from t_1 to ∞, provides

$$\begin{aligned} -\phi(t_1) &\leq \frac{\lambda\beta_0}{\delta(t_1)}\delta(t_1) - \lambda\beta_0 \lim_{h\to\infty} \int_{t_1}^{h} \frac{1}{\mu(v)r(v)\delta(v)}\mathrm{d}v \\ &\leq \lambda\beta_0 - \lambda\beta_0 \lim_{h\to\infty} \left(\ln \frac{\delta(t_1)}{\delta(t)}\right) \to -\infty \end{aligned}$$

which is a contradiction, and thus $\lambda = 0$.

(A_2): Using that $\mu(t)r(t)\phi'(t)$ is non-increasing, we have

$$\begin{aligned} \phi(t) &\geq -\int_{t}^{\infty} \frac{1}{\mu(s)r(s)} \mu(s)r(s)\phi'(s)\mathrm{d}s \\ &\geq -\mu(t)r(t)\phi'(t) \int_{t}^{\infty} \frac{1}{\mu(s)r(s)}\mathrm{d}s \\ &\geq -\mu(t)r(t)\phi'(t)\delta(t). \end{aligned}$$

(A_3): Using the product rule to differentiate ϕ/δ and using (A_2) in the numerator it follows that ϕ/δ is non-increasing.

(A_4): An integration of (4) from t_1 to t yields

$$\begin{aligned} -\mu(t)r(t)\phi'(t) &= -\mu(t_1)r(t_1)\phi'(t_1) + \int_{t1}^{t} \mu(s)q(s)\phi(\varrho(s))\mathrm{d}s \\ &\geq -\mu(t_1)r(t_1)\phi'(t_1) + \phi(t)\int_{t1}^{t} \mu(s)q(s)\mathrm{d}s. \end{aligned}$$

From this inequality, (3), (A_1) and $\delta' = -1/\mu r$, we have

$$\begin{aligned} -\mu(t)r(t)\phi'(t) &\geq -\mu(t_1)r(t_1)\phi'(t_1) + \beta_0\phi(t)\int_{t1}^{t}\frac{\mu(s)}{\mu^2(s)r(s)\delta^2(s)}\mathrm{d}s \\ &= -\mu(t_1)r(t_1)\phi'(t_1) - \beta_0\frac{\phi(t)}{\delta(t_1)} + \beta_0\frac{\phi(t)}{\delta(t)} \\ &\geq \beta_0\frac{\phi(t)}{\delta(t)}, \end{aligned} \tag{6}$$

and then

$$\left(\frac{\phi(t)}{\delta^{\beta_0}(t)}\right)' = \frac{\delta^{\beta_0-1}(t)(\beta_0\phi(t) + \mu(t)r(t)\phi'(t)\delta(t))}{\mu(t)r(t)\delta^{2\beta_0}(t)} \leq 0.$$

(A_5) Since ϕ/δ^{β_0} is positive and decreasing, $\lim_{t\to\infty}\phi(t)/\delta^{\beta_0}(t) = \lambda_1 \geq 0$. We claim that $\lambda_1 = 0$. If not, then $\phi(t)/\delta^{\beta_0}(t) \geq \lambda_1 > 0$, eventually. Now, we define

$$z(t) = \big(\mu(t)r(t)\phi'(t)\delta(t) + \phi(t)\big)\delta^{-\beta_0}(t).$$

Then, from (A_2), $z(t) > 0$ and

$$\begin{aligned} z'(t) &= \big(\mu(t)r(t)\phi'(t)\big)'\delta^{1-\beta_0}(t) + \beta_0\phi'(t)\delta^{-\beta_0}(t) + \beta_0\frac{\phi(t)\delta^{-\beta_0-1}(t)}{\mu(t)r(t)} \\ &= -\mu(t)q(t)\phi(\varrho(t))\delta^{1-\beta_0}(t) + \beta_0\phi'(t)\delta^{-\beta_0}(t) + \beta_0\frac{\phi(t)\delta^{-\beta_0-1}(t)}{\mu(t)r(t)} \\ &\leq -\beta_0\frac{\phi(\varrho(t))\delta^{1-\beta_0}(t)}{\mu(t)r(t)} + \beta_0\phi'(t)\delta^{-\beta_0}(t) + \beta_0\frac{\phi(t)\delta^{-\beta_0-1}(t)}{\mu(t)r(t)} \\ &\leq \beta_0\phi'(t)\delta^{-\beta_0}(t). \end{aligned}$$

Using (6) and the fact that $\phi(t) \geq \lambda\delta^{\beta_0}(t)$, we get

$$z'(t) \leq \frac{-\beta_0^2\lambda}{\mu(t)r(t)\delta(t)} < 0.$$

Integrating the last inequality from t_1 to t, we obtain

$$z(t_1) \geq -\beta_0^2\lambda\left(\ln\frac{\delta(t_1)}{\delta(t)}\right) \to \infty \text{ as } t \to \infty,$$

which is a contradiction and we conclude that $\lim_{t\to\infty}\phi(t)/\delta^{\beta_0}(t) = 0$.

(A_6): We can rewrite (4) as

$$\big(\mu(t)r(t)\phi'(t)\delta(t) + \phi(t)\big)' + \mu(t)\delta(t)q(t)\phi(\varrho(t)) = 0. \tag{7}$$

An integration of (7) from t to ∞, and using (A_3), yields

$$\begin{aligned} \mu(t)r(t)\phi'(t)\delta(t) + \phi(t) &\geq \int_t^\infty \mu(s)\delta(t)q(s)\phi(\varrho(s))\mathrm{d}s \\ &\geq \int_t^\infty \mu(s)\delta(t)q(s)\phi(s)\mathrm{d}s \\ &\geq \frac{\phi(t)}{\delta(t)}\int_t^\infty \mu(s)q(s)\delta^2(t)\mathrm{d}s \\ &\geq \beta_0\phi(t). \end{aligned} \tag{8}$$

The last inequality implies that

$$\left(\frac{\phi(t)}{\delta^{1-\beta_0}(t)}\right)' = \frac{\delta^{-\beta_0}(t)[\mu(t)r(t)\phi'(t)\delta(t) + \phi(t)(1-\beta_0)]}{\mu(t)r(t)\delta^{2-2\beta_0}(t)} \geq 0.$$

This ends the proof. □

Lemma 3. *Assume that there exists a* $\beta_0 \in (0,1)$ *such that (3) holds and* $\delta(\varrho(t))/\delta(t) \geq \kappa > 0$. *If* ϕ *is a positive solution of (1) then*
(A_7) $\phi(t)/\delta^{\beta_1}(t)$ *is decreasing,*
(A_8) $\lim_{t\to\infty} \phi(t)/\delta^{\beta_1}(t) = 0$, *and*
(A_9) $\phi(t)/\delta^{1-\beta_1}(t)$ *is increasing,*
where

$$\beta_1 = \frac{\kappa^{\beta_0}\beta_0}{1-\beta_0}. \tag{9}$$

Proof. Assume that ϕ is an eventually positive solution of (1). From Lemma 2, we have ϕ satisfies $(\mathbf{C}_2)$ and $(A_1)-(A_6)$ hold. Integrating (4) from t_1 to t and using the fact that ϕ/δ^{β_0} is decreasing, we get

$$\begin{aligned} -\mu(t)r(t)\phi'(t) &\geq -\mu(t_1)r(t_1)\phi'(t_1) + \int_{t_1}^{t} \mu(s)q(s)\frac{\phi(s)\delta^{\beta_0}(\varrho(s))}{\delta^{\beta_0}(s)}\mathrm{d}s \\ &\geq -\mu(t_1)r(t_1)\phi'(t_1) + \frac{\phi(t)}{\delta^{\beta_0}(t)}\int_{t_1}^{t} \mu(s)q(s)\delta^{\beta_0}(\varrho(s))\mathrm{d}s, \end{aligned} \tag{10}$$

which in view of (3) implies

$$-\mu(t)r(t)\phi'(t) \geq -\mu(t_1)r(t_1)\phi'(t_1) + \frac{\kappa^{\beta_0}\beta_0\phi(t)}{\delta^{\beta_0}(t)}\int_{t_1}^{t} \frac{1}{\mu(s)r(s)}\delta^{\beta_0-2}(s)\mathrm{d}s,$$

and

$$-\mu(t)r(t)\phi'(t) \geq -\mu(t_1)r(t_1)\phi'(t_1) + \beta_1\frac{\phi(t)}{\delta(t)} - \beta_1\delta^{\beta_0-1}(t_1)\frac{\phi(t)}{\delta^{\beta_0}(t)}.$$

Since $\phi(t)/\delta^{\beta_0}(t) \to 0$ as $t \to \infty$, we obtain

$$-\mu(t)r(t)\phi'(t) \geq \beta_1\frac{\phi(t)}{\delta(t)}, \tag{11}$$

and then $\phi(t)/\delta^{\beta_1}(t)$ is decreasing.

Next, proceeding as in the proof of (A_5) and (A_6) in Lemma 2, we can prove (A_8) and (A_9). □

Theorem 1. *Assume that there exists a* $\beta_0 \in (0,1)$ *such that (3) holds and* $\delta(\varrho(t))/\delta(t) \geq \kappa > 0$. *Let there exists* $n \in \mathbb{N}$ *such that* $\beta_j < 1$ *for* $j = 1, \ldots, n-1$,

$$\beta_k = \beta_0\frac{\kappa^{\beta_{k-1}}}{1-\beta_{k-1}},\ k = 0, 1, \ldots n, \tag{12}$$

and

$$\beta_n > \frac{1}{2} \tag{13}$$

Then, under assumption (3), every solution of (1) is oscillatory.

Proof. Assume the contrary that ϕ is an eventually positive solution of (1). From Lemmas 2 and 3, we have $\phi \in (\mathbf{C}_2)$ and $(A_1)-(A_9)$ hold. Thus, by using induction, we can prove that

$$\frac{\phi(t)}{\delta^{\beta_k}(t)} \text{is decreasing and } \frac{\phi(t)}{\delta^{1-\beta_k}(t)} \text{ is increasing for all } k = 0, 1, ...n.$$

Hence, we obtain

$$\beta_n\phi(t) + \mu(t)r(t)\phi'(t)\delta(t) \leq 0$$

and

$$\mu(t)r(t)\phi'(t)\delta(t) + \phi(t)(1-\beta_n) \geq 0.$$

This implies

$$(2\beta_n - 1)\phi(t) \leq 0.$$

Then $\beta_0 \leq \frac{1}{2}$, which is a contradiction. Therefore, the proof is complete. □

Theorem 2. *Assume that there exists a $\beta_0 \in (0,1)$ such that (3) holds and $\delta(\varrho(t))/\delta(t) \geq \kappa > 0$. If there exists $n \in \mathbb{N}$ such that*

$$\liminf_{t\to\infty} \int_{\varrho(t)}^{t} \mu(s)q(s)\delta(s)\mathrm{d}s > \frac{1-\beta_n}{\mathrm{e}}, \tag{14}$$

where β_n is defined as in Theorem 1, then every solution of (1) is oscillatory.

Proof. Assume the contrary that ϕ is an eventually positive solution of (1). We define the function

$$w(t) = \mu(t)r(t)\phi'(t)\delta(t) + \phi(t).$$

It follows from Lemma 2 that $w(t) > 0$ and; moreover

$$w'(t) = \left(\mu(t)r(t)\phi'(t)\right)'\delta(t) = -\mu(t)q(t)\phi(\varrho(t))\delta(t). \tag{15}$$

Since ϕ/δ^{β_n} is decreasing, then $\mu(t)r(t)\phi'(t)\delta(t) + \beta_n\phi(t) \leq 0$ and so

$$w(t) \leq (1-\beta_n)\phi(t).$$

Setting the last inequality into (15), we see that w is a positive solution of

$$w'(t) + \frac{\mu(t)r(t)\delta(t)}{(1-\beta_n)}w(\varrho(t)) \leq 0. \tag{16}$$

By Theorem 2.1.1 in [13], (14) guarantees that (16) has no positive solution, a contradiction. This ends the proof. □

Example 1. *Consider the second order delay differential equation*

$$\left(t^2\phi'(t)\right)' + t\phi'(t) + a\phi(0.5t) = 0, \tag{17}$$

where $a > 0$. We note that $\varrho(t) = 0.5t$, $\delta(t) = \frac{1}{2t^2}$, $\beta_0 = \frac{1}{4}a$, and $\kappa = 4$. So, condition (14) reduces to

$$\frac{a}{2}\ln 2 > \frac{1-\beta_n}{\mathrm{e}} \tag{18}$$

with β_n iterative defined by (12). A simple computation reveals that for a =0.66 desired sequence

$$\begin{aligned} \beta_1 &= 0.248391864, \\ \beta_2 &= 0.309769942, \\ \beta_3 &= 0.367274043, \end{aligned}$$

and (18) holds for $n = 3$, that is for $a = 0.66$ (17) is oscillatory.

Example 2. *Consider the second order delay differential equation*

$$\left(t^2\phi'(t)\right)' + p_0t\phi'(t) + q_0\phi(\lambda t) = 0, \tag{19}$$

where $q_0 > 0$ *and* $\lambda \in (0,1)$. *We note that* $\mu(t) = t^{p_0}$, $\delta(t) = \frac{1}{(p_0+1)t^{p_0+1}}$ *and* $\beta_0 = \frac{q_0}{(p_0+1)^2}$. *So, condition (14) reduces to*

$$\frac{1}{p_0+1}\ln\frac{1}{\lambda} > \frac{1-\beta_n}{\mathrm{e}} \tag{20}$$

with β_n *iterative defined by (12). If* $p_0 = 1$, *then we get* $\beta_0 = \frac{q_0}{4}$ *and condition (20) reduces to*

$$\ln\frac{1}{\lambda} > \frac{4-q_0}{2q_0\mathrm{e}}. \tag{21}$$

See Figure 1. When $\lambda = 0.5$, *we get* $\beta_0 = \frac{q_0}{(p_0+1)^2}$ *and condition (20) reduces to*

$$q_0 \ln 2 > \frac{(p_0+1)^2 - q_0}{(p_0+1)\mathrm{e}}. \tag{22}$$

See Figure 2. At $q_0 = 0.6$, *we get* $\beta_0 = \frac{0.6}{(p_0+1)^2}$ *and condition (20) becomes*

$$0.6\ln\frac{1}{\lambda} > \frac{(p_0+1)^2 - 0.6}{(p_0+1)\mathrm{e}}. \tag{23}$$

Remark 1. *If* $p(t) = 0$, *then then our results are reduced to the results of Baculikova in [14]. However, Baculikova's results require the extra condition*

$$\int_{t_0}^{\infty} q(s)\left(\int_s^{\infty} r^{-1}(u)\mathrm{d}u\right)\mathrm{d}s = \infty,$$

which we do not use here, instead we use (3).

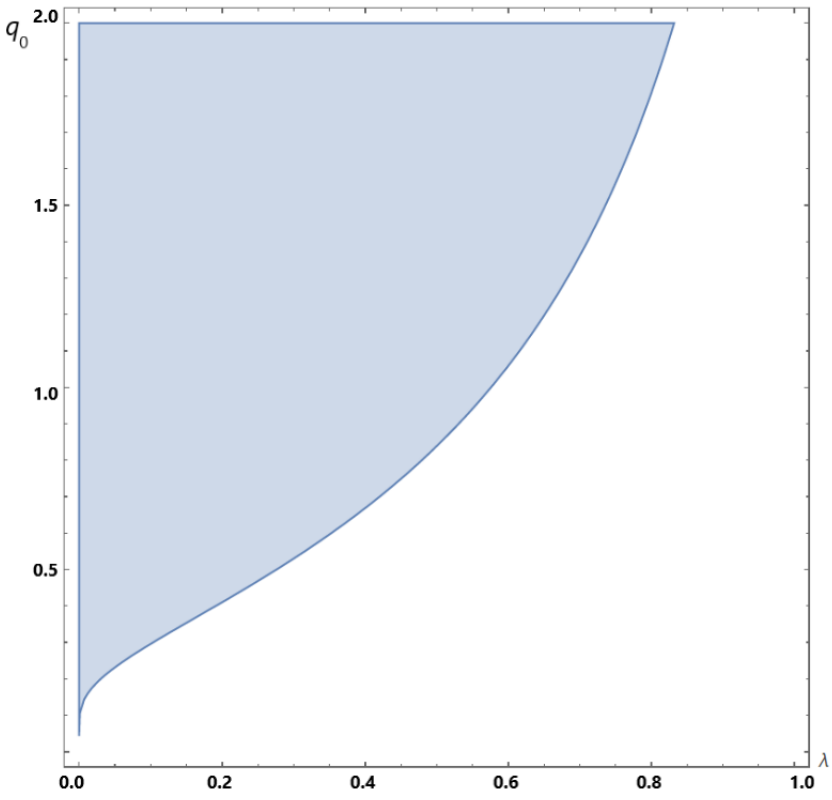


Figure 1. Regions for which Condition (21) are satisfied.

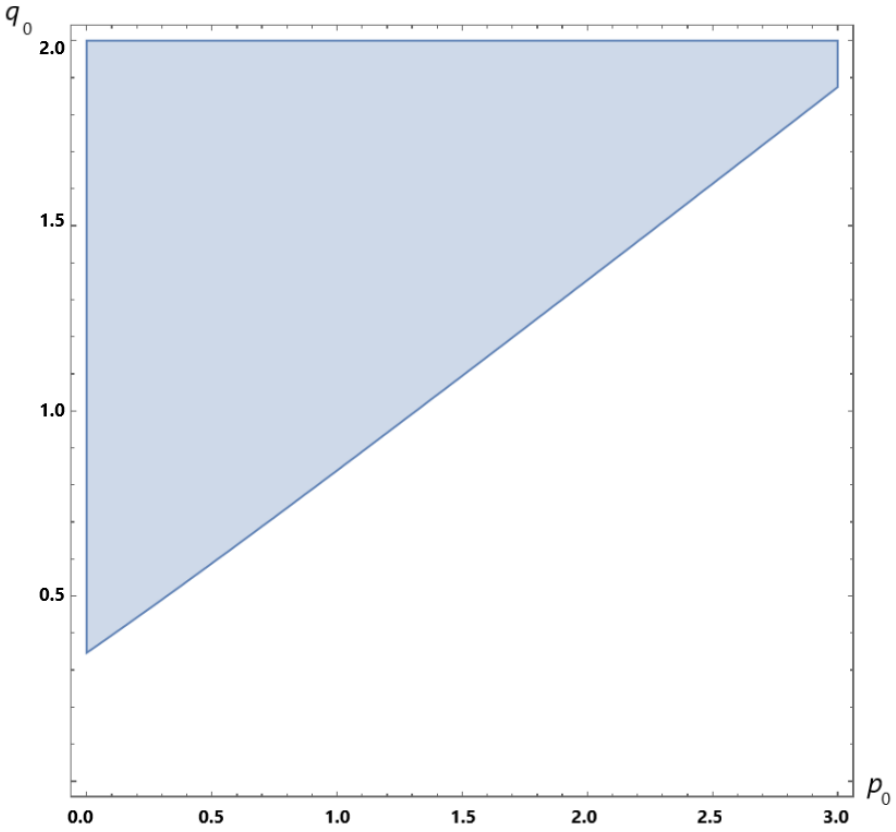


Figure 2. Regions for which Conditions (22) are satisfied.

3. Main Results II: Advanced Equation

The above method can be modified to serve also advanced differential equations, when $\varrho(t) \geq t$. We slightly modify the key constant β_0 to γ_0 as follows.

Lemma 4. *Assume that there exists a* $\gamma_0 \in (0,1)$ *such that*

$$\mu^2(t)r(t)q(t)\delta(t)\delta(\tau(t)) \geq \gamma_0 \tag{24}$$

holds and $\delta(t)/\delta(\varrho(t)) \geq \omega \geq 1$ *where* ω *is constant. If* ϕ *is a positive solution of (1) then*
(B_1) $\phi(t)/\delta^{\beta_0}(t)$ *is decreasing*
(B_2) $\lim_{t\to\infty} \phi(t)/\delta^{\beta_0}(t) = 0$, *and*
(B_3) $\phi(t)/\delta^{1-\beta_0}(t)$ *is increasing.*

Proof. Assume that ϕ is an eventually positive solution of (1). From Lemma 2, we have ϕ satisfies $(\mathbf{C}_2)$ and

$$\phi(\varrho(t)) \geq \frac{\delta(\varrho(t))}{\delta(t)}\phi(t)$$

Integrating (4) from t_1 to t we get

$$\begin{aligned} -\mu(t)r(t)\phi'(t) &= -\mu(t_1)r(t_1)\phi'(t_1) + \int_{t1}^{t} \mu(s)q(s)\phi(\varrho(s))\mathrm{d}s \\ &\geq -\mu(t_1)r(t_1)\phi'(t_1) + \phi(t)\int_{t1}^{t} \mu(s)q(s)\frac{\delta(\varrho(t))}{\delta(t)}\mathrm{d}s, \end{aligned}$$

from the inequality (24) we have

$$\begin{aligned} -\mu(t)r(t)\phi'(t) &\geq -\mu(t_1)r(t_1)\phi'(t_1) + \gamma_0\phi(t)\int_{t1}^{t}\frac{1}{\mu(s)r(s)\delta^2(s)}\mathrm{d}s \\ &= -\mu(t_1)r(t_1)\phi'(t_1) - \gamma_0\frac{\phi(t)}{\delta(t_1)} + \gamma_0\frac{\phi(t)}{\delta(t)} \\ &\geq \gamma_0\frac{\phi(t)}{\delta(t)} \end{aligned}$$

where we have used that $\phi(t) \to 0$ as $t \to \infty$. therefor,

$$\left(\frac{\phi(t)}{\delta^{\gamma_0}(t)}\right)' = \frac{\delta^{\gamma_0-1}(t)(\gamma_0\phi(t) + \mu(t)r(t)\phi'(t)\delta(t))}{\mu(t)r(t)\delta^{2\gamma_0}(t)} \leq 0.$$

To prove parts (B_2) and (B_3) we proceed exactly as in the proof of Lemma 2. The proof is complete. □

Theorem 3. *Assume that there exists a $\gamma_0 \in (0,1)$such that (24) holds and $\delta(t)/\delta(\varrho(t)) \geq \omega \geq 1$. If there exists $n \in \mathbb{N}$ such that $\gamma_j < 1$ for $j = 1, \ldots, n-1$,*

$$\gamma_k = \gamma_0\frac{\kappa^{\gamma_{k-1}}}{1-\gamma_{k-1}},\ k = 0, 1, \ldots n, \tag{25}$$

and

$$\gamma_n > \frac{1}{2} \tag{26}$$

then (1) is oscillatory.

Theorem 4. *Assume that there exists a $\gamma_0 \in (0,1)$ such that (24) holds and $\delta(t)/\delta(\varrho(t)) \geq \omega \geq 1$. If there exists $n \in \mathbb{N}$ such that*

$$\liminf_{t\to\infty}\int_t^{\varrho(t)}\mu(s)q(s)\delta(\varrho(s))\mathrm{d}s > \frac{1-\gamma_n}{\mathrm{e}}, \tag{27}$$

where γ_n is defined as in Theorem 3, then (1) is oscillatory.

Example 3. *Consider the second order advanced differential equation*

$$\left(t^2\phi'(t)\right)' + t\phi'(t) + a\phi(2t) = 0,\ t \geq 1 \tag{28}$$

where $a > 0$. We note that $\varrho(t) = 2t$, $\delta(t) = \frac{1}{2t^2}$, $\gamma_0 = \frac{1}{16}a$, and $\omega = 4$. So, condition (27) reduces to

$$\frac{a}{8}\ln 2 > \frac{1-\gamma_n}{\mathrm{e}} \tag{29}$$

with γ_n iterative defined by (25). A simple computation reveals that for $a = 2$ desired sequence

$$\begin{aligned} \gamma_0 &= 0.125, \\ \gamma_1 &= 0.1698867, \\ \gamma_2 &= 0.1905700, \end{aligned}$$

and (27) holds for $n = 2$ and Theorem 4 ensures oscillation of (28).

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