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Characterization of Wave Fronts of Ultradistributions Using Directional Short-Time Fourier Transform

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Abstract: In this paper we give a characterization of Sobolev k-directional wave front of order $p \in [1, \infty)$ of tempered ultradistributions via the directional short-time Fourier transform.

Keywords: k-directional short-time Fourier transform; ultradistributions; wave front sets

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1. Introduction

The theory of directional sensitive kind of the short-time Fourier transform, in short STFT, was initially introduced and investigated in [1,2] as a blend of Radon transform and time–frequency analysis. It allows to gain information in time and frequency of a function along a certain direction or hyperplane. Following the concept of [1], in [3], the directional STFT was extended to the space of tempered distributions. Moreover, in [4], the k-directional short-time Fourier transform, in short k-DSTFT, was introduced and the results of [5] were extended to the spaces of tempered ultradistributions of Roumieu class.

Starting form [6], wave fronts have shown to be useful concepts when analyzing the propagation of different type of singularities in the theory of partial differential equations, which led to introducing various wave front sets [7–10].

Following the recent trend on studying integral transforms on the spaces of ultradistribution [11,12], authors in [4] introduce the k-directional regular sets to analyze the regularity properties of a tempered ultradistribution of Roumieu class. Furthermore, the wave front set using the k-DSTFT (k-directional wave front) of a tempered ultradistribution of Roumieu class and the partial wave front in terms of [6] are considered, and it is shown that this partial wave front is equivalent to the k-directional wave front.

This paper is a continuation of our work presented in [4] for both Beurling and Roumieu cases. The main result is established in Theorem 2 where we give characterization of the Sobolev wave front of order $p \in [1, \infty)$ via the k-DSTFT of tempered ultradistributions. We also consider partial wave fronts in terms of [6,13], and it is shown that these notions are equivalent with the k-directional Sobolev wave front.

The main novelty of this work is the proof of Theorem 2 where we follow the idea already proved in [4] but here with another decomposition and estimates of involved integrals. On the basis of this proof we introduce a new kind of wave front and make necessary analysis of it through our main theorem, Theorem 3.

1.1. Notation

For a given multi-index $l = (l_1, \dots, l_n) \in \mathbb{N}_0^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote $x^l = x_1^{l_1} \cdots x_n^{l_n}$ and $(-i)^{|l|} D^l = \partial_x^l = \dfrac{\partial^{|l|}}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}}$, $|l| = l_1 + \cdots + l_n$. Points in $\mathbb{R}^k$ are denoted by $\tilde{x} = (x_1, \dots, x_k)$. The notation $\Omega \subseteq \mathbb{R}^n$ is used for an open set and $K \subset\subset \Omega$ for a compact set K which is contained in Ω. By $\mathcal{F}(f)(x) = \hat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i t \cdot x} dt$, $x \in \mathbb{R}^n$, we denote the Fourier transform of a function f. The inner product of f and g in L^2 is denoted by (f, g) and $\langle f, g \rangle$ means a dual paring. Thus, $(f, g) = \langle f, \overline{g} \rangle$. We also use the notation Γ_ξ for a cone neighborhood of ξ, $L_r(\xi)$ and $B_r(\xi)$ for an open and a closed ball with a center ξ and radius $r > 0$, respectively.

1.2. Ultradistribution Spaces

Let $(M_l)_{l \in \mathbb{N}}$, $M_0 = 1$ be a sequence of positive numbers which monotonically increases to infinity and satisfies the following:

$(M.1)$ $M_l^2 \leq M_{l-1} M_{l+1}$, $l \in \mathbb{N}$;
$(M.2)$ There exist constants $A, H > 1$ such that

$$M_l \leq A H^l \min_{0 \leq q \leq l} M_q M_{l-q},\ l, q \in \mathbb{N}_0;$$

$(M.3)$ There exists a constant A such that $\sum_{l=q+1}^{\infty} M_{l-1}/M_l < Aq\, M_{q+1}/M_q$, $q \in \mathbb{N}$;

Sometimes we can replace properties $(M.2)$ and $(M.3)$ by the following weaker conditions:

$(M.2')$ There exist constants $A, H > 1$ such that

$$M_{l+1} \leq A H^l M_l,\ l \in \mathbb{N}_0;$$

$(M.3')$

$$\sum_{l=1}^{\infty} M_{l-1}/M_l < \infty.$$

We will measure the decay properties of elements of Gelfand–Shilov spaces with respect to the Gevrey sequences $M_l = l!^\alpha$, $\alpha > 1$.

Let $a > 0$. Following [14], we recall the definitions of some spaces of test functions:

$$\begin{aligned}
\mathcal{E}_a^\alpha(K) &:= \{\varphi \in \mathcal{C}^\infty(\Omega) \colon \sup_{t \in K, l \in \mathbb{N}_0^n} \frac{a^{|l|}}{l!^\alpha} |D^l \varphi(t)| < \infty\}; \\
\mathcal{D}_a^\alpha(K) &:= \mathcal{E}_h^\alpha(K) \cap \{\varphi \in \mathcal{C}^\infty(\Omega) \colon \operatorname{supp} \varphi \subset K\};
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}^{(\alpha)}(K) &:= \varprojlim_{a \to \infty} \mathcal{E}_a^\alpha(K); & \mathcal{E}^{(\alpha)}(\Omega) &:= \varprojlim_{K \subset\subset \Omega} \mathcal{E}^{(\alpha)}(K); \\
\mathcal{D}^{(\alpha)}(K) &:= \varprojlim_{a \to \infty} \mathcal{D}_a^\alpha(K); & \mathcal{D}^{(\alpha)}(\Omega) &:= \varinjlim_{K \subset\subset \Omega} \mathcal{D}^{(\alpha)}(K). \\
\mathcal{E}^{\{\alpha\}}(K) &:= \varinjlim_{a \to 0} \mathcal{E}_a^\alpha(K); & \mathcal{E}^{\{\alpha\}}(\Omega) &:= \varprojlim_{K \subset\subset \Omega} \mathcal{E}^{\{\alpha\}}(K); \\
\mathcal{D}^{\{\alpha\}}(K) &:= \varinjlim_{a \to 0} \mathcal{D}_a^{\{\alpha\}}(K); & \mathcal{D}^{\{\alpha\}}(\Omega) &:= \varinjlim_{K \subset\subset \Omega} \mathcal{D}^{\{\alpha\}}(K).
\end{aligned}$$

The elements of the space $\mathcal{D}^{(\alpha)}(\Omega)$ (resp. $\mathcal{D}^{\{\alpha\}}(\Omega)$) are called ultradifferentiable functions with compact support of Beurling class (resp. Roumieu class). Their strong duals are spaces of ultradistributions $\mathcal{D}'^{(\alpha)}(\Omega)$ (resp. $\mathcal{D}'^{\{\alpha\}}(\Omega)$). $\mathcal{E}'^{(\alpha)}(\Omega)$ (resp. $\mathcal{E}'^{\{\alpha\}}(\Omega)$) is a subspace of $\mathcal{D}'^{(\alpha)}(\Omega)$ (resp. $\mathcal{D}'^{\{\alpha\}}(\Omega)$) that consists of all compactly supported ultradistributions.

Following [11], we introduce the test spaces for spaces of Beurling and Roumieu tempered ultradistributions as a special case of ultradistributional spaces.

Let $\alpha, \beta > 0$, and $\alpha + \beta > 1$ (resp. $\alpha + \beta \geq 1$). If $\alpha + \beta = 1$, then we presume also $\alpha, \beta \neq 0$.

Let $a > 0$. We denoted by $(\mathcal{S}_a)^{\alpha}_{\beta}(\mathbb{R}^n)$ the Banach space of all smooth functions φ on $\mathbb{R}^n$ for which

$$\sigma_a^{\alpha,\beta}(\varphi) = \sup_{t \in \mathbb{R}^n, l, q \in \mathbb{N}_0^n} \frac{a^{|l|+|q|}}{l!^{\beta} q!^{\alpha}} |t^l \varphi^{(q)}(t)| < \infty. \tag{1}$$

The space $\Sigma^{\alpha}_{\beta}(\mathbb{R}^n)$ (resp. $\mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n)$) is defined as a projective (resp. an inductive) limit of the space $(\mathcal{S}_a)^{\alpha}_{\beta}(\mathbb{R}^n)$:

$$\Sigma^{\alpha}_{\beta}(\mathbb{R}^n) = \varprojlim_{a \to \infty} (\mathcal{S}_a)^{\alpha}_{\beta}(\mathbb{R}^n) \quad (\text{resp. } \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n) = \varinjlim_{a \to 0} (\mathcal{S}_a)^{\alpha}_{\beta}(\mathbb{R}^n)),$$

and its strong dual $\Sigma'^{\beta}_{\alpha}(\mathbb{R}^n)$ (resp. $\mathcal{S}'^{\beta}_{\alpha}(\mathbb{R}^n)$) is called the space of ultradistibutions of Beurling type (resp. Roumieu type). These spaces ($\alpha + \beta > 1$, resp. $\alpha + \beta \geq 1$) are closed under translation, dilation, multiplication, differentiation, and under the action of specified infinite order differential operators (see Section 1.2.1). The Roumieu type spaces are the well-known spaces of Gelfand–Shilov.

When $\alpha = \beta$, we use $\mathcal{S}^{(\alpha)}(\mathbb{R}^n)$ (resp. $\mathcal{S}^{\{\alpha\}}(\mathbb{R}^n)$) instead of $\Sigma^{\alpha}_{\alpha}(\mathbb{R}^n)$ (resp. $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^n)$).

1.2.1. Ultradifferential Operators

It is said that $P(\xi) = \sum\limits_{l \in \mathbb{N}_0^n} a_l \xi^l$, $\xi \in \mathbb{R}^n$, is an *ultrapolynomial of Beurling class* (*of Roumieu class*), if the coefficients a_l satisfy:

$$(\exists a > 0,\ \exists C_a > 0)\ \ (\text{resp. } \forall a > 0,\ \exists C_a > 0)\ (\forall l \in \mathbb{N}_0^n)\ \ |a_l| \leq C_a a^{|l|}/M_l.$$

The corresponding operator $P(D) = \sum_{l \in \mathbb{N}_0^n} a_l D^l$ is an *ultradifferential operator* of Beurling class (resp. Roumieu class). When $M_l = l!^{\alpha}$ it is called ultradifferential operator of class (α) (resp. class $\{\alpha\}$). As M_l satisfies $(M.2)$, they act continuously on $\mathcal{E}^{(\alpha)}$ and $\mathcal{D}^{(\alpha)}$ (resp. $\mathcal{E}^{\{\alpha\}}$ and $\mathcal{D}^{\{\alpha\}}$), and the corresponding spaces of ultradistributions.

The following representation theorem holds [11]:

For any $f \in \Sigma'^{\alpha}_{\beta}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^n)$) there exist $P_1(D)$-ultradifferential operator of class (α) (resp. class $\{\alpha\}$), an ultrapolynomial $P_2(\xi)$ of class (β) (resp. class $\{\beta\}$) and an $F \in L^2(\mathbb{R}^n)$ such that

$$f(\xi) = P_1(D)(P_2(\xi)F(\xi)). \tag{2}$$

We will deal only with elliptic operators for which the function $P(\xi)$ satisfies [14] (Proposition 4.5): there exist $a > 0$ and $C_a > 0$ (resp. for every $a > 0$ there exists $C_a > 0$) such that

$$C_a^{-1} e^{a|\xi|^{1/\alpha}} \leq |P(\xi)| \leq C_a e^{a|\xi|^{1/\alpha}}, \quad \forall \xi \in \mathbb{R}^n. \tag{3}$$

In the quasi-analytic case (when $(M.3)'$ does not hold) we have [4]: Let $r \geq 1$ there is $C > 0$ such that for all $\xi \in \mathbb{R}^n$, $l \in \mathbb{N}_0^n$

$$|D^l_{\xi} \frac{1}{P(\xi)}| \leq C \frac{l!}{r^{|l|}|P(\xi)|}. \tag{4}$$

1.3. The k-DSTFT and the k-Directional Synthesis Operator

We recall some definitions and assertions from [4], where only the Roumieu case was considered. Here we state also the Beurling case, since the results of [4] also hold for the Beurling-type spaces.

Let $\mathbf{u}^k = (u_1, \dots, u_k)$, where u_i, $i = 1, \dots, k$, are independent vectors of $\mathbb{S}^{n-1}$. Let $\tilde{y} = (y_1, \dots, y_k) \in \mathbb{R}^k$ and $g \in \Sigma^\alpha_\beta(\mathbb{R}^k) \setminus \{0\}$ (resp. $g \in \mathcal{S}^\alpha_\beta(\mathbb{R}^k) \setminus \{0\}$). The k-directional short-time Fourier transform of $f \in L^2(\mathbb{R}^n)$ is defined by [4]

$$DS_{g,\mathbf{u}^k} f(\tilde{y}, \xi) = \int_{\mathbb{R}^n} f(t) \overline{g_{\mathbf{u}^k,\tilde{y},\xi}(t)} dt, \quad \xi \in \mathbb{R}^n, \tag{5}$$

and the k-directional synthesis operator of $F \in L^2(\mathbb{R}^{k+n})$ is defined by [4]

$$DS^*_{g,\mathbf{u}^k} F(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(\tilde{y}, \xi) g_{\mathbf{u}^k,\tilde{y},\xi}(t) d\tilde{y} d\xi, \quad t \in \mathbb{R}^n, \tag{6}$$

where $g_{\mathbf{u}^k,\tilde{y},\xi}(t) = g((u_1 \cdot t, \dots, u_k \cdot t) - (y_1, \dots, y_k)) e^{2\pi i \xi \cdot t}, t \in \mathbb{R}^n$.
It is shown in [4, Proposition 2.4] that for $f \in \mathcal{S}^\alpha_\beta(\mathbb{R}^n)$ the following reconstruction formula holds,

$$f(t) = \frac{1}{(g, \varphi)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} DS_{g,\mathbf{u}^k} f(\tilde{y}, \xi) \varphi_{\mathbf{u}^k,\tilde{y},\xi}(t) d\tilde{y} d\xi, \quad t \in \mathbb{R}^n, \tag{7}$$

where $\varphi \in \mathcal{S}^\alpha_\beta(\mathbb{R}^k)$ is the synthesis window for $g \in \mathcal{S}^\alpha_\beta(\mathbb{R}^k) \setminus \{0\}$. The same holds for $f \in \Sigma^\alpha_\beta(\mathbb{R}^n)$ when $g, \varphi \in \Sigma^\alpha_\beta(\mathbb{R}^k)$. Thus, the relation (7) takes the form

$$(DS^*_{\varphi,\mathbf{u}^k} \circ DS_{g,\mathbf{u}^k}) f = (g, \varphi) f.$$

For the sake of simplicity we transfer the STFT in direction of $\mathbf{u}^k$ into the STFT in $\mathbf{e}^k$ direction. Recall the procedure (see [4]): Let $A = [u_{i,j}]_{k \times n}$ be a matrix with rows $u^i, i = 1, \dots, k$ and I be the identity matrix of order $n - k$. Let B be an $n \times n$ matrix determined by A and I so that $Bt = s$, where $s_1 = u_{1,1} t_1 + \dots + u_{1,n} t_n, \ \dots, \ s_k = u_{k,1} t_1 + \dots + u_{k,n} t_n$, $s_{k+1} = t_{k+1}, \dots, s_n = t_n$. The matrix B is regular, so put $C = B^{-1}$ and $\mathbf{e}^k = (e_1, \dots, e_k)$, where $e_1, \cdots, e_k$ are unit vectors of the coordinate system of $\mathbb{R}^k$. If we change the variables $t = Cs$, and $\eta = C^T \xi$, then for $f \in L^2(\mathbb{R}^n), g \in \Sigma^\alpha_\beta(\mathbb{R}^k)$ (resp. $g \in \mathcal{S}^\alpha_\beta(\mathbb{R}^k)$), the equality (5) is transformed into:

$$DS_{g,\mathbf{u}^k} f(\tilde{y}, \xi) = (DS_{g,\mathbf{e}^k} h(s))(\tilde{y}, \eta) = \int_{\mathbb{R}^n} h(s) \overline{g(\tilde{s} - \tilde{y})} e^{-2\pi i s \cdot \eta} ds, \tag{8}$$

where $h(s) = \det(C) f(Cs)$ and (6) is transformed, for $F \in L^2(\mathbb{R}^{k+n})$, $g \in \Sigma^\alpha_\beta(\mathbb{R}^k)$ (resp. $g \in \mathcal{S}^\alpha_\beta(\mathbb{R}^k)$), into:

$$DS^*_{g,\mathbf{e}^k} F(s) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(\tilde{y}, \eta) g(\tilde{s} - \tilde{y}) e^{2\pi i s \cdot \eta} d\tilde{y} d\eta, \quad s \in \mathbb{R}^n. \tag{9}$$

The function $h(s) = \det(C) f(Cs)$ is in $\Sigma^\alpha_\beta(\mathbb{R}^n)$ (resp. $\mathcal{S}^\alpha_\beta(\mathbb{R}^n)$) if $f \in \Sigma^\alpha_\beta(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}^\alpha_\beta(\mathbb{R}^n)$). Additionally, if $g(s_1, \dots, s_k) = g_1(s_1) \cdots g_k(s_k) \in (\Sigma^\alpha_\beta(\mathbb{R}))^k$ (resp. $g(s_1, \dots, s_k) = g_1(s_1) \cdots g_k(s_k) \in (\mathcal{S}^\alpha_\beta(\mathbb{R}))^k$), then

$$\begin{aligned} DS_{g,u^k} f(\tilde{y}, \xi) :&= \int_{\mathbb{R}^n} f(t) \overline{g_1(u_1 \cdot t - y_1)} \cdots \overline{g_k(u_k \cdot t - y_k)} e^{-2\pi i t \cdot \xi} dt \\ &= \int_{\mathbb{R}^n} h(s) \overline{g_1(s_1 - y_1)} \cdots \overline{g_k(s_k - y_k)} e^{-2\pi i s \cdot \mu} ds, \end{aligned}$$

and it is referred to as the partial short-time Fourier transform.

We have $DS_{g,\mathbf{e}^k} : \Sigma^\alpha_\beta(\mathbb{R}^n) \times \Sigma^\alpha_\beta(\mathbb{R}^k) \to \Sigma^\alpha_\beta(\mathbb{R}^{k+n})$ (resp. $DS_{g,\mathbf{e}^k} : \mathcal{S}^\alpha_\beta(\mathbb{R}^n) \times \mathcal{S}^\alpha_\beta(\mathbb{R}^k) \to \mathcal{S}^\alpha_\beta(\mathbb{R}^{k+n})$) is a continuous bilinear mapping. This is proved in Theorem 2.3 in [4] in the Roumieu case. With Theorem 2.5 and Corollary 2.7 in [4] in Roumieu case, and similarly in the Beurling case, it follows that $DS^*_{g,\mathbf{e}^k} : \Sigma^\alpha_\beta(\mathbb{R}^{k+n}) \times \Sigma^\alpha_0(\mathbb{R}^k) \to \Sigma^\alpha_\beta(\mathbb{R}^n)$

(resp. $DS^{*}_{g,\mathbf{e}^k} : \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^{k+n}) \times \mathcal{S}^{\alpha}_{0}(\mathbb{R}^{k}) \to \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^{n})$) is also continuous. This allows us to extend the definitions of the k-DSTFT and its synthesis operator to their duals (see [4] (Proposition 2.10)).

The relation of the k-DSTFTs with respect to different windows is presented with the following assertion. It is given in [4] (Theorem 2.11) for the Roumieu case:

Theorem 1. *Let $\mathbf{u}^k = (u_1, \ldots, u_k)$, where u_i, $i = 1, \ldots, k$ are independent vectors of $\mathbb{S}^{n-1}$. Let $\varphi, g, \gamma_1 \in \mathcal{S}^{(\alpha)}(\mathbb{R}^k)$ (resp. $\varphi, g, \gamma_1 \in \mathcal{S}^{\{\alpha\}}(\mathbb{R}^k)$) where γ_1 is the synthesis window for g and $\gamma_0 \in \mathcal{S}^{(\alpha)}(\mathbb{R}^{n-k})$ (resp. $\gamma_0 \in \mathcal{S}^{\{\alpha\}}(\mathbb{R}^{n-k})$) so that $\int_{\mathbb{R}^{n-k}} \gamma_0(t_{n-k+1}, \ldots, t_n) dt_{n-k+1} \cdots dt_n \neq 0$. Put*

$$\gamma(t_1, \ldots, t_n) = \gamma_1(t_1, \ldots, t_k)\gamma_0(t_{n-k+1}, \ldots, t_n). \tag{10}$$

Let $f \in \Sigma'^{\alpha}_{\beta}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^n)$), then

$$DS_{\varphi,\mathbf{u}^k} f(\tilde{x}, \eta) = (DS_{g,\mathbf{u}^k} f(\tilde{s}, \zeta)) * (DS_{\varphi,\mathbf{u}^k} \gamma(\tilde{s}, \zeta))(\tilde{x}, \eta), \tag{11}$$

$\tilde{x}, \tilde{s} \in \mathbb{R}^k$, $\eta, \zeta \in \mathbb{R}^n$.

2. The Main Results

The STFT in the direction of $\mathbf{u}^k$ can be used in the detection of singularities determined by the hyperplanes orthogonal to vectors $u_1, \ldots, u_k$. For this purpose, we introduce k-directional regular sets and wave front sets for the Beurling (resp. Roumieu)-tempered ultradistributions using the STFT in the direction of $\mathbf{u}^k$. To simplify our exposition we transfer the STFT in direction of $\mathbf{u}^k$ into the STFT in $\mathbf{e}^k$ direction by the use of (8).

As in [5], if $k = 1$, we consider direction $\mathbf{e}^1 = e_1$ while for $1 < k \leq n$, we consider direction $\mathbf{e}^k = (e_1, \ldots, e_k)$. Let $k = 1$ and $y_0 = y_{0,1} \in \mathbb{R}$, and let $\Pi_{e_1,y_0,\varepsilon} = \Pi_{y_0,\varepsilon} := \{t \in \mathbb{R}^n : |t_1 - y_0| < \varepsilon\}$. It is a part of $\mathbb{R}^n$ between two hyperplanes orthogonal to e_1, that is,

$$\Pi_{y_0,\varepsilon} = \bigcup_{y \in (y_0-\varepsilon, y_0+\varepsilon)} P_y, \quad (y_0 = (y_0, 0, \ldots, 0), y = (y, 0, \ldots, 0)),$$

and P_y denotes the hyperplane orthogonal to e_1 passing through y. Let

$$\Pi_{\mathbf{e}^k,\tilde{y},\varepsilon} = \Pi_{e_1,y_1,\varepsilon} \cap \ldots \cap \Pi_{e_k,y_k,\varepsilon}, \quad \Pi_{\mathbf{e}^k,\tilde{y}} = \Pi_{e_1,y_1} \cap \ldots \cap \Pi_{e_k,y_k}.$$

The set $\Pi_{\mathbf{e}^k,\tilde{y},\varepsilon}$ is a parallelepiped in $\mathbb{R}^k$. In $\mathbb{R}^n$ this parallelepiped is determined by $2k$ finite edges while the other edges are infinite. The set $\Pi_{\mathbf{e}^k,\tilde{y}}$ equals $\mathbb{R}^{n-k}$ translated by vectors $\vec{y}_1, \ldots, \vec{y}_k$. We call it $n-k$-dimensional element of $\mathbb{R}^n$ and it is denoted by $P_{\mathbf{e}^k,\tilde{y}} \in \mathbb{R}^{n-k}$. When $k = n$, we have the point $y = (y_1, \ldots, y_n)$.

Definition 1. *Let $f \in \mathcal{S}'^{(\alpha)}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'^{\{\alpha\}}(\mathbb{R}^n)$), $\alpha > 1$ and $p \in [1, \infty)$. It is said that f is (α)-p-k-directionally microlocally regular (in short, (α)-p-k-d.m.r.) (resp. $\{\alpha\}$-p-k-directionally microlocally regular (in short, $\{\alpha\}$-p-k-d.m.r.)) at $(P_{\mathbf{e}^k,\tilde{y}_0}, \xi_0) \in \mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$, that is, at every point of the form $(\tilde{y}_0, \xi_0)$ if there exist $g \in \mathcal{D}^{(\alpha)}(\mathbb{R}^k)$ (resp. $g \in \mathcal{D}^{\{\alpha\}}(\mathbb{R}^k)$), $g(\tilde{0}) \neq 0$, a product of open balls $L_r(\tilde{y}_0) = L_r(y_{0,1}) \times \ldots \times L_r(y_{0,k}) \in \mathbb{R}^k$, a cone Γ_{ξ_0} and for each $N \in \mathbb{N}$ (resp. for some $N \in \mathbb{N}$) there exists $C_N > 0$ such that*

$$\begin{aligned} &\sup_{\tilde{y} \in L_r(\tilde{y}_0)} ||DS_{g,\mathbf{e}^k} f(\tilde{y}, \xi) e^{N|\xi|^{1/\alpha}}||_{L^p(\Gamma_{\xi_0})} \\ &= \sup_{\tilde{y} \in L_r(\tilde{y}_0)} \left(\int_{\Gamma_{\xi_0}} |\mathcal{F}(f(t)\overline{g(\tilde{t} - \tilde{y})})(\xi)|^p e^{pN|\xi|^{1/\alpha}} d\xi \right)^{1/p} \leq C_N. \end{aligned} \tag{12}$$

If $k = n$, Definition 1 gives the classical Hörmander's regularity [6].

Remark 1. *(a) If f is (α)-p-k-d.m.r. (resp. $\{\alpha\}$-p-k-d.m.r.) at $(P_{\mathbf{e}^k,\tilde{y}_0}, \xi_0)$, then there exist an open ball $L_r(\tilde{y}_0)$ and an open cone $\Gamma \subset \Gamma_{\xi_0}$ so that f is (α)-p-k-d.m.r. (resp. $\{\alpha\}$-p-k-d.m.r.) at*

$(P_{\mathbf{e}^k,\tilde{z}_0}, \theta_0)$ for any $\tilde{z}_0 \in L_r(\tilde{y}_0)$ and $\theta_0 \in \Gamma$. This implies that the union of all (α)-p-k-d.m.r. (resp. $\{\alpha\}$-p-k-d.m.r) points $(P_{\mathbf{e}^k,\tilde{z}_0}, \theta_0)$, $(\tilde{z}_0, \theta_0) \in L_r(\tilde{y}_0) \times \Gamma$ is an open set of $\mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$.

(b) Denote by Pr_k the projection of $\mathbb{R}^n$ onto $\mathbb{R}^k$. Then, the (α)-p-k-d.m.r. (resp. $\{\alpha\}$-p-k-d.m.r.) point $(P_{\mathbf{e}^k,\tilde{y}_0}, \xi_0)$, considered in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ with respect to the first k variables, equals $(Pr_k^{-1} \times I_\xi)(P_{\mathbf{e}^k,\tilde{y}_0}, \xi_0)$ (I_ξ is the identity matrix on $\mathbb{R}^n$).

A p-Sobolev k-directional wave front of Beurling (resp. Roumieu) type is defined as the complement in $\mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$ of all (α)-p-k-d.m.r. (resp. $\{\alpha\}$-p-k-d.m.r.) points $(P_{\mathbf{e}^k,\tilde{y}_0}, \xi_0)$, and we denoted as $\Sigma WF_{\mathbf{e}^k}(f)$ (resp. $SWF_{\mathbf{e}^k}(f)$).

2.1. Independence with Respect to a Window Function

One of the main results in [4] (Theorem 3.4) shows that the wave front set does not depend on the used window. Here, we prove the same assertion for the p-Sobolev k-directional wave fronts. The idea is similar to the one in [6,13] (see [4]) but here the decomposition of the involved integrals and the use of ultradifferential operators make the proof more complex.

Theorem 2. *If (12) holds for some $g \in \mathcal{D}^{(\alpha)}(\mathbb{R}^k)$ (resp. $g \in \mathcal{D}^{\{\alpha\}}(\mathbb{R}^k)$), $g(\tilde{0}) \neq 0$, then it holds for every $h \in \mathcal{D}^{(\alpha)}(\mathbb{R}^k)$ (resp. $h \in \mathcal{D}^{\{\alpha\}}(\mathbb{R}^k)$), $h(\tilde{0}) \neq 0$ supported by a ball $B_\rho(\tilde{0})$, where $\rho \leq \rho_0$ and ρ_0 depends on r in (12).*

Proof. We will focus only on the Roumieu-type spaces. The proof in the Beurling case will follow similarly. We assume that φ, g, γ_1, belong to $\mathcal{S}_0^\alpha(\mathbb{R}^k)$ where γ_1 defined by (10) is the synthesis window for g and $\gamma_0 \in \mathcal{S}^{\{\alpha\}}(\mathbb{R}^{n-k})$. Additionally, suppose that f is a continuous function which satisfies (3), since we can use the methods of oscillatory integral and transfer the differentiation from f on other factors in integral expressions. Using [11] (Theorem 3.2.2), we obtain that $f = P_0(D)F$, where F is a continuous function which satisfies

$$\forall a > 0, \quad \exists C_a > 0, \quad \forall \xi \in \mathbb{R}^n \quad |F(\xi)| \leq C_a e^{a|\xi|^{1/\alpha}} \tag{13}$$

and $P_0(D)$ is a differential operator.

We use Theorem 1, that is, the form (11). Assume that (12) holds. The constructions of balls we repeat from [5]. The window function γ is chosen so that $\operatorname{supp} \gamma \subset B_{\rho_1}(0)$ and $\rho_1 < r - r_0$. Let $h \in \mathcal{D}^{\{\alpha\}}(\mathbb{R}^k)$ and $\operatorname{supp} h \subset B_\rho(\tilde{0})$. The aim is to find ρ_0 such that (12) holds for $DS_{h,\mathbf{e}^k} f(\tilde{x}, \eta)$, with $\tilde{x} \in B_{r_0}(\tilde{y}_0), \eta \in \Gamma_1 \subset\subset \Gamma_{\xi_0}$, for $\rho \leq \rho_0$ ($\Gamma_1 \subset\subset \Gamma_{\xi_0}$ implies that $\Gamma_1 \cap \mathbb{S}^{n-1}$ is a compact subset of $\Gamma_{\xi_0} \cap \mathbb{S}^{n-1}$).

We choose ρ_0 such that $\rho_0 + \rho_1 < r - r_0$ and

$$\rho + \rho_1 + r_0 < r \text{ holds for } \rho \leq \rho_0. \tag{14}$$

This implies that

$$|\tilde{y} - \tilde{y}_0| < r$$

as a consequence of

$$|\tilde{q}| \leq \rho_1, \quad |\tilde{x} - \tilde{y}_0| \leq r_0 \text{ and } \quad |\tilde{q} - ((\tilde{x} - \tilde{y}_0) - (\tilde{y} - \tilde{y}_0))| \leq \rho.$$

Let $\Gamma_1 \subset\subset \Gamma_{\xi_0}$. Then, there exists $c \in (0,1)$ such that $\eta \in \Gamma_1, |\eta| > 1$ and

$$|\eta - \xi| \leq c|\eta| \Rightarrow \eta \in \Gamma_{\xi_0}; |\eta - \xi| \leq c|\eta| \Rightarrow |\eta| \leq (1-c)^{-1}|\xi|. \tag{15}$$

Let $\tilde{x} \in B_{r_0}(\tilde{y}_0), \eta \in \Gamma_1$ and $K = e^{N|\eta|^{1/\alpha} - \varepsilon|\eta|^{1/\alpha}} |DS_{h,\mathbf{e}^k} f(\tilde{x}, \eta)|$. Then, by (8) and (11)

$$\sup_{\tilde{x} \in L_{r_0}(\tilde{y}_0)} \int_{\Gamma_1} |K|^p d\eta = \sup_{\tilde{x} \in L_{r_0}(\tilde{y}_0)} \int_{\Gamma_1} e^{-\varepsilon p|\eta|^{1/\alpha}} e^{pN|\eta|^{1/\alpha}} \Big| \int_{\mathbb{R}^k} d\tilde{y}$$

$$
\Big(\int_{|\eta-\xi|\leq c|\eta|}+\int_{|\eta-\xi|\geq c|\eta|}\Big)\Big(\int_{\mathbb{R}^n_t} f(t)\overline{g(\tilde{t}-\tilde{y})}e^{-2\pi i t\cdot\xi}dt
$$

$$
\int_{\mathbb{R}^n_q}\gamma(q)\overline{h(\tilde{q}-(\tilde{x}-\tilde{y}))}e^{-2\pi i q\cdot(\eta-\xi)}dq\Big)d\xi|^p d\eta = I_1+I_2.
$$

We continue to estimate I_1

$$
I_1 \leq C\sup_{\tilde{x}\in L_{r_0}(\tilde{y}_0)}\int_{\Gamma_1} e^{-\varepsilon p|\eta|^{1/\alpha}}\Big|\int_{\mathbb{R}^k} d\tilde{y}\Big(\int_{\Gamma_{\xi_0}} e^{pN|\eta-\xi|^{1/\alpha}}\Big(\int_{\mathbb{R}^n_t}|f(t)\overline{g(\tilde{t}-\tilde{y})}|dt\Big)^p
$$

$$
e^{pN|\xi|^{1/\alpha}}\Big|\int_{\mathbb{R}^n_q} P(D_q)(\gamma(q)\overline{h(\tilde{q}-(\tilde{x}-\tilde{y}))})\frac{e^{-2\pi i q\cdot(\eta-\xi)}}{P(2\pi(\eta-\xi))}dq|^p d\xi\Big)\Big|d\eta
$$

$$
\leq C\sup_{\tilde{x}\in L_{r_0}(\tilde{y}_0)}\int_{\Gamma_1} e^{-\varepsilon p|\eta|^{1/\alpha}}\Big|\int_{\mathbb{R}^k} d\tilde{y}\Big(\int_{\Gamma_{\xi_0}} e^{pN|\xi|^{1/\alpha}}\Big(\int_{\mathbb{R}^n_t}|f(t)\overline{g(\tilde{t}-\tilde{y})}|dt\Big)^p d\xi\Big)
$$

$$
\sup_{\xi\in\mathbb{R}^n}\frac{e^{pN|\eta-\xi|^{1/\alpha}}}{P(2\pi(\eta-\xi))}\Big|\int_{\mathbb{R}^n_q} P(D_q)(\gamma(q)\overline{h(\tilde{q}-(\tilde{x}-\tilde{y}))})dq|^p\Big|d\eta
$$

By the assumptions that g, h, γ are with compact support, integrals over $\mathbb{R}^k$ and $\mathbb{R}^n_q$ are finite while the integral over Γ_{ξ_0} is finite because of the assumption (1).

Now we consider I_2.

$$
I_2 = \sup_{\tilde{x}\in L_{r_0}(\tilde{y}_0)}\int_{\Gamma_1} e^{-\varepsilon p|\eta|^{1/\alpha}}e^{pN|\eta|^{1/\alpha}}\Big|\int_{\mathbb{R}^k} d\tilde{y}\int_{|\eta-\xi|\geq c|\eta|} DS_{g,\mathbf{e}^k}f(\tilde{y},\eta-\xi)
$$

$$
DS_{h,\mathbf{e}^k}\gamma(\tilde{x}-\tilde{y},\xi)d\xi|^p d\eta.
$$

Let $\Omega=\{\xi : |\eta-\xi|\geq c|\eta|\}$. By $\kappa^0_d, 0<d<1$, we denote the characteristic function of $\Omega_d=\bigcup_{\xi\in\Omega} L_d(\xi)$, where Ω_d is an open d-neighborhood of Ω. Then, put

$$
\kappa_d=\kappa^0_d*\varphi_d,
$$

where $\varphi_d=\frac{1}{d^n}\varphi(\cdot/d)$, $\varphi\in\mathcal{D}^{\{\alpha\}}(\mathbb{R}^n)$ is non-negative, supported in the ball $B_1(0)$ and equals 1 on $B_{1/2}(0)$. By the construction we have that κ_d equals one on Ω, it is supported in Ω_{2d}, and all the derivatives of κ_d are bounded. We note that

$$
\Big|\int_\Omega\ldots d\xi\Big|\leq\Big|\int_{\Omega_{2d}}\kappa_d(\xi)\ldots d\xi\Big|+\Big|\int_{\Omega_{2d}\cap\{\xi:|\eta-\xi|\leq c\eta\}}\kappa_d(\xi)\ldots d\xi\Big|.
$$

Then,

$$
I_2\leq\sup_{\tilde{x}\in L_{r_0}(\tilde{y}_0)}\int_{\Gamma_1} e^{-\varepsilon p|\eta|^{1/\alpha}}\Big(\int_{\mathbb{R}^k}\Big|\int_{\mathbb{R}^n}\kappa_d e^{N|\eta|^{1/\alpha}}DS_{g,\mathbf{e}^k}f(\tilde{y},\eta-\xi)
$$

$$
DS_{h,\mathbf{e}^k}\gamma(\tilde{x}-\tilde{y},\xi)d\xi|^p d\tilde{y}\Big)d\eta
$$

$$
+\sup_{\tilde{x}\in L_{r_0}(\tilde{y}_0)}\int_{\Gamma_1} e^{-\varepsilon p|\eta|^{1/\alpha}}\Big(\int_{\mathbb{R}^k}\Big|\int_{\Omega_{2d}\cap\{\xi:|\eta-\xi|\leq c|\eta|\}}\kappa_d e^{N|\eta|^{1/\alpha}}DS_{g,\mathbf{e}^k}f(\tilde{y},\eta-\xi)
$$

$$
DS_{h,\mathbf{e}^k}\gamma(\tilde{x}-\tilde{y},\xi)d\xi|^p d\tilde{y}\Big)d\eta = I_{2,1}+I_{2,2}.
$$

We first estimate $I_{2,1}$.

$$
I_{2,1}\leq C\sup_{\tilde{x}\in L_{r_0}(\tilde{y}_0)}\int_{\Gamma_1} e^{-\varepsilon p|\eta|^{1/\alpha}}\Big|\int_{\mathbb{R}^k} d\tilde{y}\Big(\int_{\mathbb{R}^d_\xi} e^{pN|\eta-\xi|^{1/\alpha}}\Big(\int_{\mathbb{R}^n_t}\Big|\frac{f(t)\overline{g(\tilde{t}-\tilde{y})}}{P(2\pi t)}\Big|dt\Big)^p\cdot
$$

$$
e^{pN|\xi|^{1/\alpha}}\Big|\int_{\mathbb{R}^n_q} P(D_q)(\gamma(q)\overline{h(\tilde{q}-(\tilde{x}-\tilde{y}))})\cdot
$$

$$\frac{P(D_\xi)(\kappa_d(\eta-\xi)e^{-2\pi i q\cdot(\eta-\xi)})}{P(2\pi(\eta-\xi))}dq|^p d\xi)|d\eta$$

$$\leq C \sup_{\tilde{x}\in L_{r_0}(\tilde{y}_0)} \int_{\Gamma_1} e^{-\varepsilon p|\eta|^{1/\alpha}} |\int_{\mathbb{R}^k} d\tilde{y}(\int_{\mathbb{R}^d_\xi} e^{pN|\xi|^{1/\alpha}} (\int_{\mathbb{R}^n_t} |\frac{f(t)\overline{g(\tilde{t}-\tilde{y})}}{P(2\pi t)}|dt)^p d\xi)$$

$$\sup_{\xi\in\mathbb{R}^n} \frac{e^{pN|\eta-\xi|^{1/\alpha}}}{P(2\pi(\eta-\xi))} | \int_{\mathbb{R}^n_q} P(D_q)(\gamma(q)\overline{h(\tilde{q}-(\tilde{x}-\tilde{y}))})dq|^p)|d\eta$$

is finite.

Since integration goes trough a subset of $\{\xi : |\eta-\xi| \leq c|\eta|\}$, we can conclude that $I_{2,2}$ can be estimated similarly as I_1. Additional therm κ_d does not cause any problems since it belongs to $\mathcal{E}^{\{\alpha\}}$. □

2.2. Equivalent Definition

In this section we characterize the wave front sets given Definition 1 with the ones formulated in the next definition. We will use the Fourier transform as well as the cut-off function, and since we will show that these to definitions are equivalent but we need to distinguish them, we add the prefix "locally" in front of this notation in the Definition 2. We follow our ideas outlined in [10] and prove that both definitions determine the same sets. For the sake of completeness, we give all the details of the proof although it is the repetition of our proof of the theorem in [10] where we have considered distributions instead of ultradistributions.

Definition 2. *Let $f \in \mathcal{S}'^{(\alpha)}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'^{\{\alpha\}}(\mathbb{R}^n)$), $p \in [1,\infty)$ and $s \in \mathbb{R}$. The point $(\tilde{y}_0, \xi_0) \in \mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$ is locally (α)-p-k- microlocally regular, in short locally (α)-p-k- m.r., (resp. locally $\{\alpha\}$-p-k-m.r.) for f if there exists $\chi \in \mathcal{D}^{(\alpha)}(\mathbb{R}^k)$ (resp. $\chi \in \mathcal{D}^{\{\alpha\}}(\mathbb{R}^k)$) so that $\chi(\tilde{y}_0) \neq 0$ and a cone Γ_{ξ_0} such that*

$$\|e^{s|\xi|^{1/\alpha}}\mathcal{F}(\chi(\tilde{y})f(y))(\xi)\|_{L^p(\Gamma_{\xi_0})} < \infty,$$

$y = (\tilde{y}, y_{k+1}, \ldots, y_n) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$.

For the complements we use the notation $L - \Sigma WF_{\mathbf{e}^k}(f)$ (resp. $L - SWF_{\mathbf{e}^k}(f)$)

Theorem 3. *Let $f \in \mathcal{S}'^{(\alpha)}(\mathbb{R}^n)$ ($f \in \mathcal{S}'^{\{\alpha\}}(\mathbb{R}^n)$) and $p \in [1,\infty)$. The following conditions are equivalent.*

(i) *$(\tilde{y}_0, \xi_0) \notin L - \Sigma WF_{\mathbf{e}^k}(f)$ (resp. $(\tilde{y}_0, \xi_0) \notin L - SWF_{\mathbf{e}^k}(f)$).*

(ii) *There exist a compact neighborhood $\tilde{K}$ of $\tilde{y}_0$ and a cone Γ_{ξ_0} such that for every $s > 0$ (resp. for some $s > 0$) the mapping $\chi \mapsto e^{s|\cdot|^{1/\alpha}}\mathcal{F}(\chi(\tilde{y})f(y))$, $\mathcal{D}^{(\alpha)}(\tilde{K}) \to L^p(\Gamma_{\xi_0})$ (resp. $\mathcal{D}^{\{\alpha\}}(\tilde{K}) \to L^p(\Gamma_{\xi_0})$), is well-defined and continuous.*

(iii) *There exist a compact neighborhood $\tilde{K}$ of $\tilde{y}_0$, a cone Γ_{ξ_0} and $C, a > 0$ such that for all $\chi \in \mathcal{D}^{(\alpha)}(\tilde{K} - \{\tilde{y}_0\})$ and $s > 0$ (resp. for all $\chi \in \mathcal{D}^{\{\alpha\}}(\tilde{K} - \{\tilde{y}_0\})$ and some $s > 0$) there holds*

$$\sup_{\tilde{y}\in\tilde{K}} \|e^{s|\xi|^{1/\alpha}} DS_{\chi,\mathbf{e}^k} f(\tilde{y},\xi)\|_{L^p(\Gamma_{\xi_0})} \leq C \sup_{l\in\mathbb{N}_0^n} \frac{a^{|l|}}{l!^\alpha}\|D^l\chi\|_{L^\infty(\mathbb{R}^k)}.$$

Here, the set $\tilde{K} - \{\tilde{y}_0\} = \{\tilde{y} \in \mathbb{R}^k | \tilde{y} + \tilde{y}_0 \in \tilde{K}\}$.

(iv) *There exist a compact neighborhood $\tilde{K}$ of $\tilde{y}_0$, a cone Γ_{ξ_0}, such that for all $s > 0$ and corresponding $\chi \in \mathcal{D}^{(\alpha)}(\mathbb{R}^k)$ (resp. for some $s > 0$ and corresponding $\chi \in \mathcal{D}^{\{\alpha\}}(\mathbb{R}^k)$) with $\chi(\tilde{0}) \neq 0$ there holds $\sup_{\tilde{y}\in\tilde{K}} \|e^{s|\xi|^{1/\alpha}} DS_{\chi,\mathbf{e}^k} f(\tilde{y},\xi)\|_{L^p(\Gamma_{\xi_0})} < \infty$.*

Proof. $(i) \Rightarrow (ii)$ We will prove only the Roumieu case. We have that f is locally $\{\alpha\}$-p-k-m.r. at $(\tilde{y}_0, \xi_0)$, which means that there exist $\chi \in \mathcal{D}^{\{\alpha\}}(\mathbb{R}^k)$, $\chi(\tilde{y}_0) \neq 0$, and there exists a cone Γ'_{ξ_0} such that

$$C_\chi = \| e^{s|\xi|^{1/\alpha}} \mathcal{F}(\chi(\tilde{y}) f(y))(\xi) \|_{L^p(\Gamma'_{\xi_0})} < \infty.$$

There exists a compact neighborhood $\tilde{K}$ of $\tilde{y}_0$ where χ never vanishes. Moreover, there are constants C_1 and $r \geq 1$ such that

$$|\mathcal{F}(\chi(\tilde{y}) f(y))(\xi)| \leq C_1 e^{r|\xi|^{1/\alpha}}, \forall \xi \in \mathbb{R}^n.$$

Now let Γ_{ξ_0} be a cone such that $\overline{\Gamma}_{\xi_0} \subseteq \Gamma'_{\xi_0} \cup \{0\}$. One can find $0 < c < 1$ such that

$$\{\eta \in \mathbb{R}^n | \, \exists \xi \in \Gamma_{\xi_0} \text{ such that } |\xi - \eta| \leq c|\xi|\} \subseteq \Gamma'_{\xi_0}. \tag{16}$$

We take $\psi \in \mathcal{D}^{\{\alpha\}}(\tilde{K})$, then $\mathcal{F}(\psi(\tilde{y})\chi(\tilde{y}) f(y)) = \mathcal{F}(\psi(\tilde{y})) * \mathcal{F}(\chi(\tilde{y}) f(y))$. By the Minkowski integral inequality we have

$$\begin{aligned}
&\| e^{s|\xi|^{1/\alpha}} \mathcal{F}(\psi(\tilde{y})\chi(\tilde{y}) f(y))(\xi) \|_{L^p(\Gamma_{\xi_0})} \\
&\leq \int_{\mathbb{R}^n_\eta} \left(\int_{\Gamma_{\xi_0}} e^{ps|\xi|^{1/\alpha}} |\mathcal{F}(\psi(\tilde{y}))(\eta)|^p |\mathcal{F}(\chi(\tilde{y}) f(y))(\xi - \eta)|^p d\xi \right)^{1/p} d\eta \\
&\leq I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \int_{\mathbb{R}^n_\eta} |\mathcal{F}(\psi(\tilde{y}))(\eta)| \left(\int_{\substack{|\xi| \geq |\eta|/c \\ \xi \in \Gamma_{\xi_0}}} e^{ps|\xi|^{1/\alpha}} |\mathcal{F}(\chi(\tilde{y}) f(y))(\xi - \eta)|^p d\xi \right)^{1/p} d\eta,$$

$$I_2 = \int_{\mathbb{R}^n_\eta} |\mathcal{F}(\psi(\tilde{y}))(\eta)| \left(\int_{\substack{|\xi| < |\eta|/c \\ \xi \in \Gamma_{\xi_0}}} e^{ps|\xi|^{1/\alpha}} |\mathcal{F}(\chi(\tilde{y}) f(y))(\xi - \eta)|^p d\xi \right)^{1/p} d\eta.$$

Using the change of variable $\xi - \eta$ in the inner integral in I_1 we have

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n_\eta} |\mathcal{F}(\psi(\tilde{y}))(\eta)| \left(\int_{\substack{|\xi+\eta| \geq |\eta|/c \\ \xi \in \Gamma_{\xi_0} - \{\eta\}}} e^{ps|\xi+\eta|^{1/\alpha}} |\mathcal{F}(\chi(\tilde{y}) f(y))(\xi)|^p d\xi \right)^{1/p} d\eta \\
&\leq (1-c)^{-s} \int_{\mathbb{R}^n_\eta} |\mathcal{F}\psi(\eta)| \left(\int_{\Gamma'_{\xi_0}} e^{ps|\xi|^{1/\alpha}} |\mathcal{F}(\chi(\tilde{y}) f(y))(\xi)|^p d\xi \right)^{1/p} d\eta \\
&= \frac{C_\chi \| \mathcal{F}(\psi(\tilde{y})) \|_{L^1(\mathbb{R}^n)}}{(1-c)^s}.
\end{aligned}$$

For the inequality above, we have used $\{\xi \in \Gamma_{\xi_0} - \{\eta\} | \, |\xi + \eta| \geq |\eta|/c\} \subseteq \Gamma'_{\xi_0}$, which follows from (16). For I_2 we have

$$\begin{aligned}
I_2 &\leq C_1 \int_{\mathbb{R}^n_\eta} |\mathcal{F}(\psi(\tilde{y}))(\eta)| \left(\int_{\substack{|\xi| < |\eta|/c \\ \xi \in \Gamma_{\xi_0}}} e^{ps|\xi|^{1/\alpha} + pr|\xi - \eta|^{1/\alpha}} d\xi \right)^{1/p} d\eta \\
&\leq C_1 (1 + c^{-1})^r c^{-s-n-1} \| e^{(-n-1)|\cdot|^{1/\alpha}} \|_{L^p(\mathbb{R}^n)} \cdot \\
&\quad \cdot \| e^{(r+s+n+1)|\cdot|^{1/\alpha}} \mathcal{F}(\psi(\tilde{y})) \|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

From the estimates of I_1 and I_2, we conclude that there exists $C_\chi > 0$ such that

$$\|e^{s|\cdot|^{1/\alpha}}\mathcal{F}(\psi(\tilde{y})\chi(\tilde{y})f(y))\|_{L^p(\Gamma_{\xi_0})} \le C_\chi \|e^{(s+r+n+1)|\cdot|^{1/\alpha}}\mathcal{F}(\psi(\tilde{y}))\|_{L^1(\mathbb{R}^n)},$$

$\forall\psi \in \mathcal{D}^{\{\alpha\}}(\tilde{K})$. Now, the claim in (ii) can be deduced since for $\psi \in \mathcal{D}^{\{\alpha\}}(\tilde{K})$, we have $\psi f = (\psi/\chi)\chi f$ with $\psi/\chi \in \mathcal{D}^{\{\alpha\}}(\tilde{K})$.

$(ii) \Rightarrow (iii)$ Let $\tilde{K}_1$ be a compact neighborhood of $\tilde{y}_0$, and we choose a cone Γ_{ξ_0} such that the mapping

$$\chi \mapsto e^{s|\cdot|^{1/\alpha}}\mathcal{F}(\chi(\tilde{y})f(y)), \quad \mathcal{D}^{\{\alpha\}}(\tilde{K}_1) \to L^p(\Gamma_{\xi_0}),$$

is well-defined and continuous. Under the assumption that $\tilde{K}_1 = B_r(\tilde{y}_0)$, for some $r > 0$, there exist $C > 0$ and $a > 0$ such that

$$\|e^{s|\xi|^{1/\alpha}}\mathcal{F}(\chi(\tilde{y})f(y))(\xi)\|_{L^p(\Gamma_{\xi_0})} \le C \sup_{l\in\mathbb{N}_0^n} \frac{a^{|l|}}{l!^\alpha}\|D^l\chi\|_{L^\infty(\tilde{K}_1)},$$

$\forall\chi \in \mathcal{D}^{\{\alpha\}}(\tilde{K}_1)$.

Let $\tilde{K} = B_{r/2}(\tilde{y}_0)$. For $\chi \in \mathcal{D}^{\{\alpha\}}(\tilde{K} - \{\tilde{y}_0\})$ and $\tilde{y} \in \tilde{K}$, the function $\tilde{t} \mapsto \overline{\chi(\tilde{t} - \tilde{y})}$ belongs to $\mathcal{D}^{\{\alpha\}}(\tilde{K}_1)$ and, as $\mathcal{F}(\chi(\tilde{\cdot} - \tilde{y})f(y))(\xi) = DS_{\chi,\mathbf{e}^k} f(\tilde{y},\xi)$, we have

$$\begin{aligned}\sup_{\tilde{y}\in K}\|e^{s|\xi|^{1/\alpha}}DS_{\chi,\mathbf{e}^k} f(\tilde{y},\xi)\|_{L^p(\Gamma_{\xi_0})} &\le C\sup_{\tilde{y}\in K}\sup_{l\in\mathbb{N}_0^n}\frac{a^{|l|}}{l!^\alpha}\|D^l\chi(\tilde{\cdot}-\tilde{y})\|_{L^\infty(\tilde{K}_1)} \\ &= C\sup_{l\in\mathbb{N}_0^n}\frac{a^{|l|}}{l!^\alpha}\|D^l\chi\|_{L^\infty(\mathbb{R}^k)}.\end{aligned}$$

The implication $(iii) \Rightarrow (iv)$ is trivial. If we let $\tilde{y} = \tilde{y}_0$, the implication $(iv) \Rightarrow (i)$ follows immediately. □

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