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Closed-Form Solutions of Linear Ordinary Differential Equations with General Boundary Conditions

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Abstract: This paper deals with the solution of boundary value problems for ordinary differential equations with general boundary conditions. We obtain closed-form solutions in a symbolic form of problems with the general n -th order differential operator, as well as the composition of linear operators. The method is based on the theory of the extensions of linear operators in Banach spaces.

Keywords: differential equations; differential operators; non-local boundary value problems; general conditions; integral conditions; multipoint conditions; composition of operators

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1. Introduction

Differential equations model numerous phenomena and processes in sciences and engineering. Boundary value problems for elementary differential equations with classical boundary conditions have been studied exhaustively by many researchers and comprehensive material is now included in various standard texts. A more difficult and less investigated subject is the general or nonlocal boundary value problems. In many applications, the incorporation of general boundary conditions such as multipoint and integral conditions is inevitable. For example, in [1], the necessity of integral conditions in certain models of epidemics and population growth and the effects when neglecting these conditions are explained.

Ordinary differential equations with non-local boundary conditions were first studied at the beginning of the 20th century in [2–4], and later in [5]. Abstract non-local boundary value problems were considered in [6–8]. Operator methods for solving differential equations are analyzed in the books [9–11]. A description of the theory and the different directions of differential equations with non-local boundary conditions are given in the monograph [12]. An overview of non-local boundary value problems and their historical evolution can also be found in the survey papers [13–16]. Boundary value problems with integral constraints have been considered in [17–25], to mention but a few. Boundary value problems with multipoint and integral conditions have been studied in [26–32], and others. The present paper aims at providing a framework for symbolic computations for the solution of linear ordinary differential equations of order n with the most general multipoint and integral conditions, and boundary value problems for powers and products of differential operators.

In $C[a, b]$, let the general n -th order linear ordinary differential operator,

$$A = a_0(x) \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x) \frac{d}{dx} + a_n(x), \quad (1)$$

where the coefficients $a_i(x)$, $i = 0, \dots, n$, are all continuous functions on the interval $[a, b]$, $a_0(x) \neq 0$, and $D(A) = C^n[a, b]$ and $R(A) = C[a, b]$ are its domain and range, respectively.

We are concerned with the solution in the closed-form of boundary value problem for the differential equation

$$Au(x) = f(x), \quad x \in (a, b), \quad (2)$$

and the boundary conditions

$$Y(u) = \mathbf{b}, \quad (3)$$

where $Y = \text{col}(Y_1, \dots, Y_n)$ is a vector of linear bounded functionals of the general form

$$Y_i(u) = \sum_{j=0}^m \sum_{k=0}^{n-1} v_{ikj} u^{(k)}(x_j) + \sum_{k=0}^{n-1} \int_a^b h_{ik}(t) u^{(k)}(t) dt, \quad i = 1, \dots, n, \quad (4)$$

where $Y_i \in [C^{n-1}[a, b]]^*$, $i = 1, \dots, n$; the $m+1$ ordered points $a \leq x_0 < x_1 < \dots < x_m \leq b$ are fixed boundary points, $h_{ik}(x)$, $i = 1, \dots, n$, $k = 0, \dots, n-1$, are continuous functions on $[a, b]$, v_{ikj} , $i = 1, \dots, n$, $k = 0, \dots, n-1$, $j = 0, \dots, m$, are constants, and $u^{(k)}$ designates the k -th order derivative of u . The non-homogeneous term $f(x) \in C[a, b]$, while the non-homogeneous term $\mathbf{b} = \text{col}(\beta_1, \dots, \beta_n)$ is a constant vector. The function $u(x) \in C^n[a, b]$ is the sought solution. We formulate the above problem in a convenient symbolic form and establish uniqueness solvability criteria and derive the solution in closed-form. Solution formulae to some special boundary value problems for composite differential operators are also obtained. The method is based on the theory of the extensions of linear operators in Banach spaces, see, for example, [33–36], and is an extension of the work [37] by the authors.

The paper is organized as follows. In Section 2, we give some results needed for the analysis in later sections. Sections 3 and 4 contain the main findings of our investigation. In Section 5, the implementation of the technique is explained by solving two example problems. Finally, some conclusions are quoted in Section 6.

2. Preliminaries

Let X, Y be Banach spaces and $P : X \rightarrow Y$ a linear operator. The operator P is *injective* or *one-to-one* if for every $u_1, u_2 \in D(P)$, $u_1 \neq u_2$ implies $Pu_1 \neq Pu_2$. The operator P is *surjective* or *onto* Y if $R(P) = Y$. If P is both injective and onto, then there exists the inverse operator $P^{-1} : Y \rightarrow X$ defined by $P^{-1}f = u$ if and only if $Pu = f$ for each $f \in Y$; in this case, $R(P^{-1}) = D(P)$.

The operator P is called *closed* if for every sequence u_m in $D(P)$ converging to u_0 with $Pu_m \rightarrow y_0$, $y_0 \in Y$, it follows that $u_0 \in D(P)$ and $Pu_0 = y_0$. A closed operator P is called *maximal* if $R(P) = Y$ and $\ker P \neq \{0\}$.

The operator P is *correct* if it is both injective and onto, and the inverse operator P^{-1} is bounded on Y . The problem $Pu = f$ is correct if the operator P is correct.

An operator $P_r : X \rightarrow Y$ is a *restriction* of P , or P is an *extension* of P_r , if $D(P_r) \subset D(P)$ and $P_ru = Pu$ for all $u \in D(P_r)$.

Let $\Psi = \text{col}(\Psi_1, \dots, \Psi_n)$ be a column vector of functionals $\Psi_i \in X^*$, $i = 1, \dots, n$, and $\mathbf{v} = (v_1, \dots, v_n)$ a row vector of elements $v_i \in X$, $i = 1, \dots, n$. By $\Psi(\mathbf{v})$, we symbolize the $n \times n$ matrix

$$\Psi(\mathbf{v}) = \begin{pmatrix} \Psi_1(v_1) & \cdots & \Psi_1(v_n) \\ \vdots & \ddots & \vdots \\ \Psi_n(v_1) & \cdots & \Psi_n(v_n) \end{pmatrix},$$

whose $\Psi_i(v_j)$ element is the value of the functional Ψ_i on the element v_j . It is easy to show that

$$\Psi(\mathbf{vN}) = \Psi(\mathbf{v})\mathbf{N}, \quad (5)$$

where $\mathbf{N}$ is a $n \times m$ constant matrix.

Proposition 1. Let X, Y be real Banach spaces, $A : X \xrightarrow{on} Y$ a linear operator, $\mathbf{z} = (z_1, \dots, z_n)$ a basis of $\ker A$, and $\hat{A}$ the restriction of A defined by

$$\hat{A} \subset A, \quad D(\hat{A}) = \{u : u \in D(A), \Phi(u) = \mathbf{0}\},$$

where the components of the vector $\Phi = \text{col}(\Phi_1, \dots, \Phi_n)$ are linear bounded functionals on X . Then:

(i) The operator $\hat{A}$ is injective if and only if

$$\det \Phi(\mathbf{z}) = \det \begin{pmatrix} \Phi_1(z_1) & \dots & \Phi_1(z_n) \\ \vdots & \ddots & \vdots \\ \Phi_n(z_1) & \dots & \Phi_n(z_n) \end{pmatrix} \neq 0.$$

(ii) If additionally to (i), the operator $\hat{A}^{-1}$ is bounded on the whole Y , then the operator $\hat{A}$ is correct.

Proof. (i) Let $\det \Phi(\mathbf{z}) \neq 0$. Take $u_0 \in \ker \hat{A}$, then $\hat{A}u_0 = 0$, $\Phi(u_0) = \mathbf{0}$ and

$$u_0 = \mathbf{z}\mathbf{a} = a_1z_1 + \dots + a_nz_n,$$

where $\mathbf{a} = \text{col}(a_1, \dots, a_n) \in \mathbb{R}_n$. Acting by the vector Φ on u_0 , we get

$$\Phi(u_0) = \Phi(\mathbf{z}\mathbf{a}) = \Phi(\mathbf{z})\mathbf{a} = \mathbf{0}.$$

Since $\det \Phi(\mathbf{z}) \neq 0$ by hypothesis, it is implied that $\mathbf{a} = \mathbf{0}$ and so $u_0 = 0$. That is $\ker \hat{A} = \{0\}$, and therefore $\hat{A}$ is injective. Conversely, let $\det \Phi(\mathbf{z}) = 0$. Then, there exists a nonzero vector $\mathbf{c} = \text{col}(c_1, \dots, c_n) \in \mathbb{R}_n$ such that $\Phi(\mathbf{z})\mathbf{c} = \mathbf{0}$. Consider the element $u_0 = \mathbf{z}\mathbf{c}$. Note that $u_0 \neq 0$, since the components of $\mathbf{z}$ are linearly independent, and that $\Phi(u_0) = \Phi(\mathbf{z})\mathbf{c} = \mathbf{0}$. This means that $u_0 \in D(\hat{A})$. Furthermore, $\hat{A}u_0 = Au_0 = (A\mathbf{z})\mathbf{c} = 0$. From the above, it follows that $u_0 \in \ker \hat{A}$. Hence, $\hat{A}$ is not injective.

(ii) Since $\hat{A}$ is injective and the operator $\hat{A}^{-1}$ is bounded on the whole Y by hypothesis, it follows that the operator $\hat{A}$ is correct. $\square$

Proposition 2. Let X, Y be real Banach spaces, $A : X \xrightarrow{on} Y$ a linear operator, and $\mathbf{z} = (z_1, \dots, z_n)$ a basis of $\ker A$. If there exists a correct restriction $\hat{A}$ of A defined by

$$\hat{A} \subset A, \quad D(\hat{A}) = \{u : u \in D(A), \Phi(u) = \mathbf{0}\},$$

where the components of the vector $\Phi = \text{col}(\Phi_1, \dots, \Phi_n)$ are linear bounded functionals on X , then A is closed and so A is a maximal operator.

Proof. Let $u_m \in D(A)$, $u_m \rightarrow u$ and $Au_m \rightarrow y$, $m \rightarrow \infty$. Denote $Au_m = y_m$. Since A is a linear operator and $\hat{A}^{-1}y_m$ is a particular solution of $Au_m = y_m$, then every solution u_m to this equation can be represented as

$$u_m = \hat{A}^{-1}y_m + \mathbf{z}\mathbf{a}_m = \hat{A}^{-1}y_m + a_{m1}z_1 + \dots + a_{mn}z_n, \quad (6)$$

where $\mathbf{a}_m = \text{col}(a_{m1}, \dots, a_{mn}) \in \mathbb{R}_n$. Acting by the vector Φ on (6), we obtain

$$\Phi(u_m) = \Phi(\mathbf{z})\mathbf{a}_m.$$

Since, by hypothesis, the operator $\hat{A}$ is correct, it follows from Proposition 1 that $\det \Phi(\mathbf{z}) \neq 0$ and hence

$$\mathbf{a}_m = \Phi^{-1}(\mathbf{z})\Phi(u_m). \quad (7)$$

Substitution of (7) into (6) yields

$$u_m = \hat{A}^{-1}y_m + \mathbf{z}\Phi^{-1}(\mathbf{z})\Phi(u_m).$$

Since $\hat{A}^{-1}, \Phi_1, \dots, \Phi_n$ are bounded and $u_m \rightarrow u, y_m \rightarrow y$ for $m \rightarrow \infty$, it follows that

$$u = \hat{A}^{-1}y + \mathbf{z}\Phi^{-1}(\mathbf{z})\Phi(u).$$

Further, taking into account that $D(A) = D(\hat{A}) \oplus \ker A$ [34], we conclude that $u \in D(A)$ and $Au = y$. So the operator A is closed and hence A is a maximal operator. $\square$

3. General Boundary Conditions

In this Section, we study boundary value problems for ordinary differential equations with general homogeneous and nonhomogeneous boundary conditions.

Let now $X = Y = C[a, b]$ and $X^n = C^n[a, b]$. Let $A : X \rightarrow X$ be the n -th order linear operator in (1), $\hat{A} : X \rightarrow X$ a correct restriction of A defined by

$$\begin{aligned} \hat{A}u(x) &= Au(x), \quad x \in (a, b) \\ D(\hat{A}) &= \{u : u \in D(A), \Phi(u) = \mathbf{0}\}, \end{aligned} \quad (8)$$

where $\Phi = \text{col}(\Phi_1, \dots, \Phi_n)$ is a vector of n linear bounded functionals $\Phi_i \in [X^{n-1}]^*$, $i = 1, \dots, n$, and $\hat{A}^{-1}$ the inverse of $\hat{A}$.

For example, the operator

$$\begin{aligned} \hat{A}u(x) &= Au(x), \quad x \in (a, b) \\ D(\hat{A}) &= \{u : u \in X^n, u(x_0) = u'(x_0) = \dots = u^{(n-1)}(x_0) = 0\}, \end{aligned}$$

where $\Phi_i(u) = u^{(i-1)}(x_0) = 0$, $i = 1, \dots, n$, with x_0 being a fixed point in $[a, b]$, known as Cauchy boundary conditions, is correct. In the particular case where $A = \frac{d^n}{dx^n}$, the inverse $\hat{A}$ and the unique solution of the correct problem $\hat{A}u = f$, for any $f \in X$, is given explicitly by

$$u = \hat{A}^{-1}f = \frac{1}{(n-1)!} \int_{x_0}^x (x-t)^{n-1} f(t) dt, \quad \forall f \in X.$$

From the above and Proposition 2, it is concluded that the n -th order linear operator A in (1) is closed and maximal.

3.1. Homogeneous Boundary Conditions

First, we consider the boundary value problem with homogeneous boundary conditions, namely

$$\begin{aligned} A_0u &= Au = f, \quad x \in (a, b), \\ D(A_0) &= \{u : u \in D(A), Y(u) = \mathbf{0}\}, \end{aligned} \quad (9)$$

where the linear operator $A_0 : X \rightarrow X$ is a restriction of the n -th order linear operator A in (1), the components of the vector $Y = \text{col}(Y_1, \dots, Y_n) \in [X_n^{n-1}]^*$ are as in (3), (4), and $f \in X$.

Lemma 1. *The linear operator A_0 is a closed operator.*

Proof. Let $u_r, r = 1, 2, \dots$, be a sequence in $D(A_0)$, $u_r \rightarrow u_0$ and $A_0u_r \rightarrow f$. Then, $Au_r \rightarrow f$ and since A is a closed operator, we get that $u_0 \in D(A)$ and $Au_0 = f$. Moreover, since $Y_1, \dots, Y_n$ are bounded functionals on X , we get

$$0 = Y_i(u_r) \rightarrow Y_i(u_0) = 0, \quad i = 1, \dots, n, \quad r \rightarrow \infty.$$

This is that $u_0 \in D(A_0)$ and so $Au_0 = A_0u_0 = f$. Hence, A_0 is a closed operator. $\square$

Theorem 1. Let A_0 be the linear operator defined by (9), $\mathbf{z} = (z_1, \dots, z_n)$ a basis of $\ker A$, and $\hat{A}^{-1}$ the inverse of the correct operator $\hat{A}$ in (8). Then:

- (i) The operator A_0 is injective if and only if $\det Y(\mathbf{z}) \neq 0$.
- (ii) In addition, under (i), the operator A_0 is correct and the unique solution to the boundary value problem (9), for every $f \in X$, is given by

$$u = A_0^{-1}f = \hat{A}^{-1}f - \mathbf{z}Y^{-1}(\mathbf{z})Y(\hat{A}^{-1}f). \quad (10)$$

Proof. (i) Suppose $\det Y(\mathbf{z}) \neq 0$. Let $u \in \ker A_0$. Then, $A_0u = Au = 0$ and $u = \mathbf{z}\mathbf{c}$, where $\mathbf{c} = \text{col}(c_1, \dots, c_n)$ is a vector of arbitrary constants. Additionally, $Y(u) = Y(\mathbf{z}\mathbf{c}) = Y(\mathbf{z})\mathbf{c} = \mathbf{0}$, which implies that $\mathbf{c} = \mathbf{0}$. That is $u = 0$ and consequently $\ker A_0 = \{0\}$. This proves that A_0 is injective. Conversely, let $\det Y(\mathbf{z}) = 0$. Then, there exists a nonzero vector of constants $\mathbf{c} = \text{col}(c_1, \dots, c_n)$ such that $Y(\mathbf{z})\mathbf{c} = \mathbf{0}$. Let the element $u_0 = \mathbf{z}\mathbf{c}$ and notice that $u_0 \neq 0$ since the components of $\mathbf{z}$ are linearly independent. Then, $A_0u_0 = Au_0 = A(\mathbf{z}\mathbf{c}) = (A\mathbf{z})\mathbf{c} = 0$ and $Y(u_0) = Y(\mathbf{z}\mathbf{c}) = Y(\mathbf{z})\mathbf{c} = \mathbf{0}$. That is $u_0 = \mathbf{z}\mathbf{c} \in \ker A_0$ and as a consequence A_0 is not injective.

(ii) Let $\det Y(\mathbf{z}) \neq 0$. Then, from statement (i) follows that the operator A_0 is injective and hence there exists the inverse operator $A_0^{-1} : R(A_0) \subseteq X \rightarrow X$. Since by Lemma 1 the operator A_0 is closed, it is implied that A_0^{-1} is a closed operator. Furthermore, $\hat{A}$ is a correct restriction of the linear operator A and therefore the general solution of the problem $Au = f$, for every $f \in X$, may be written as follows

$$u = \hat{A}^{-1}f + \mathbf{z}\mathbf{c}, \quad (11)$$

where $\mathbf{c} = \text{col}(c_1, \dots, c_n)$ is a vector of arbitrary constants. By requiring u to satisfy the boundary conditions in (9), we have

$$Y(u) = Y(\hat{A}^{-1}f) + Y(\mathbf{z})\mathbf{c} = \mathbf{0}, \quad \text{and hence} \quad \mathbf{c} = -Y^{-1}(\mathbf{z})Y(\hat{A}^{-1}f).$$

Substitution of $\mathbf{c}$ into (11) yields (10), which is the unique solution of the boundary value problem $A_0u = f$. In addition, it follows that $R(A_0) = X$ and since A_0^{-1} is a closed operator with $D(A_0^{-1}) = R(A_0) = X$, it is implied that A_0^{-1} is bounded on X . This proves that the operator A_0 is correct. $\square$

3.2. Non-Homogeneous Boundary Conditions

Next, we consider the complete non-homogeneous boundary value problem

$$\begin{aligned} A_1u &= Au = f, \quad x \in (a, b) \\ D(A_1) &= \{u : u \in D(A), Y(u) = \mathbf{b}\}, \end{aligned} \quad (12)$$

where $A_1 : X \rightarrow X$ is a restriction of the n -th order linear operator A in (1), the components of the vector $Y = \text{col}(Y_1, \dots, Y_n) \in [X_n^{n-1}]^*$ are as in (3), (4), $\mathbf{b} = \text{col}(\beta_1, \dots, \beta_n) \in \mathbb{R}_n$, and $f \in X$. It is noted that the operator A_1 is not linear, since its domain is a nonlinear set.

We state and prove the next theorem for the existence and construction of the unique solution of the boundary value problem (12).

Theorem 2. Let A_1 be the operator defined by (12), $\mathbf{z} = (z_1, \dots, z_n)$ a basis of $\ker A$, $\hat{A}^{-1}$ the inverse of the correct operator $\hat{A}$ in (8), and A_0 the operator defined in (9). Then, the operator A_1 is injective if and only if A_0 is injective. In this case, for every $f \in X$ and $\mathbf{b} \in \mathbb{R}_n$, the unique solution of (12) is given by

$$u = A_1^{-1}f = \hat{A}^{-1}f + \mathbf{z}Y^{-1}(\mathbf{z})[\mathbf{b} - Y(\hat{A}^{-1}f)]. \quad (13)$$

Proof. Suppose A_0 is injective. Then, $\ker A_0 = \{0\}$ and $\det Y(\mathbf{z}) \neq 0$ by Theorem 1. Let $u_1, u_2 \in D(A_1)$ and $A_1 u_1 = A_1 u_2 = f$. That is,

$$A_1 u_i = A u_i = f, \quad Y(u_i) = \mathbf{b}, \quad i = 1, 2,$$

from which we get

$$A_1 u_i = A u_i = A(u_i - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b}) = f, \quad Y(u_i - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b}) = \mathbf{0}, \quad i = 1, 2, \quad (14)$$

by taking into account that $A\mathbf{z} = \mathbf{0}$ and (5). From (14), it is implied that $u_i - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b} \in D(A_0)$ and

$$A_1 u_i = A_0(u_i - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b}) = f, \quad i = 1, 2.$$

From Theorem 1, we have

$$u_i - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b} = A_0^{-1}f \quad \text{or} \quad u_i = A_0^{-1}f + \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b}, \quad i = 1, 2.$$

Since A_0 is injective, it is concluded that $u_1 = u_2$ and therefore A_1 is an injective operator. Conversely, suppose A_1 is injective. Let $u \in \ker A_1$, which means

$$A_1 u = A u = 0, \quad Y(u) = \mathbf{b}, \quad \mathbf{b} \in \mathbb{R}_n. \quad (15)$$

It follows that $u = \mathbf{z}\mathbf{c}$, where $\mathbf{c}$ is a vector of constants, and

$$Y(u) = Y(\mathbf{z}\mathbf{c}) = Y(\mathbf{z})\mathbf{c} = \mathbf{b}. \quad (16)$$

Since A_1 is injective, the system (16) has only one solution, that is $\det Y(\mathbf{z}) \neq 0$, and hence by Theorem 1 A_0 is injective.

Finally, under the hypothesis that A_1 is injective, for any $u \in D(A_1)$ that solves the completely nonhomogeneous problem $A_1 u = f$, we have

$$A_1 u = A u = A(u - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b}) = f, \quad Y(u - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b}) = \mathbf{0}. \quad (17)$$

This means that $u - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b} \in D(A_0)$ and

$$A_1 u = A_0(u - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b}) = f.$$

The solution to this problem follows from Theorem 1, namely

$$\begin{aligned} u - \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b} &= A_0^{-1}f \\ &= \hat{A}^{-1}f - \mathbf{z}Y^{-1}(\mathbf{z})Y(\hat{A}^{-1}f), \end{aligned}$$

from where we get

$$\begin{aligned} u &= \hat{A}^{-1}f - \mathbf{z}Y^{-1}(\mathbf{z})Y(\hat{A}^{-1}f) + \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b} \\ &= \hat{A}^{-1}f + \mathbf{z}Y^{-1}(\mathbf{z})[\mathbf{b} - Y(\hat{A}^{-1}f)]. \end{aligned}$$

□

4. Composition of Operators

In this Section, we investigate boundary value problems for special differential operators, specifically the k -th power of an operator and the product of two operators, with general homogeneous boundary conditions.

4.1. k -th Power of an Operator

The k -th power of an operator A^k is defined as the composition of the operator with itself, repeated k times, i.e.,

$$A^k = A(A^{k-1}) = \underbrace{AA \cdots A}_k, \quad k = 2, 3, \dots$$

If $A : X \rightarrow X$ is an n -th order linear differential operator with $D(A) = X^n$ then $A^k : X \rightarrow X$ is a kn -th order linear operator with $D(A^k) = X^{kn}$.

Let the boundary value problem

$$\begin{aligned} A_0^k u &= A^k u = f, \quad x \in (a, b), \\ D(A_0^k) &= \{u : u \in D(A^k), Y(u) = \mathbf{0}, Y(Au) = \mathbf{0}, \dots, Y(A^{k-1}u) = \mathbf{0}\}, \end{aligned} \quad (18)$$

where the operator $A_0^k : X \rightarrow X$, the operators A and A_0 are defined as in (1) and (9), respectively, the components of the vector $Y = \text{col}(Y_1, \dots, Y_n)$ are as in (3), (4) where now $Y_i \in [X^{kn-1}]^*$, $i = 1, \dots, n$, and $f \in X$. We state the following theorem.

Theorem 3. Let A_0^k be the linear operator defined in (18) and $\mathbf{z} = (z_1, \dots, z_n)$ be a basis of $\ker A$. Then:

- (i) The operator A_0^k is injective if and only if $\det Y(\mathbf{z}) \neq 0$.
- (ii) Moreover, under (i), the operator A_0^k is correct and the unique solution to the boundary value problem (18), for any $f \in X$, is given by

$$u = (A_0^k)^{-1} f = A_0^{-k} f. \quad (19)$$

Proof. (i) Let $\det Y(\mathbf{z}) \neq 0$. Then, by Theorem 1, the operator A_0 is injective. Further, the operator A_0^k is injective as a composition of injective operators. Conversely, let A_0^k be injective. Then, $\ker A_0^k = \{0\}$, and from the well known relation, which holds for any linear operator A_0 ,

$$\ker A_0 \subseteq \ker A_0^2 \subseteq \dots \subseteq \ker A_0^k, \quad k \in \mathbb{N},$$

follows that $\ker A_0 = \{0\}$, i.e., A_0 is injective. Then, by Theorem 1, we have $\det Y(\mathbf{z}) \neq 0$.

(ii) Let $\det Y(\mathbf{z}) \neq 0$. Then, by Theorem 1, the operator A_0 is correct. Observe that the problem (18), for any $f \in X$, by setting $Au = v_1$, $Av_1 = v_2$, $\dots$, $Av_{k-2} = v_{k-1}$, $Av_{k-1} = f$ can be decomposed into the k boundary value problems:

$$\begin{aligned} Av_{k-1} &= f, \quad Y(v_{k-1}) = \mathbf{0} \quad \text{or} \quad A_0 v_{k-1} = f, \\ Av_{k-2} &= v_{k-1}, \quad Y(v_{k-2}) = \mathbf{0} \quad \text{or} \quad A_0 v_{k-2} = v_{k-1}, \\ &\dots \\ Av_1 &= v_2, \quad Y(v_1) = \mathbf{0} \quad \text{or} \quad A_0 v_1 = v_2, \\ Au &= v_1, \quad Y(u) = \mathbf{0} \quad \text{or} \quad A_0 u = v_1. \end{aligned}$$

By applying Theorem 1 successively, we get

$$\begin{aligned} v_{k-1} &= A_0^{-1} f = \hat{A}^{-1} f - \mathbf{z} Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f), \\ v_{k-2} &= A_0^{-1} v_{k-1} = A_0^{-1} (A_0^{-1} f) = A_0^{-2} f, \\ &\dots \\ v_1 &= A_0^{-1} v_2 = A_0^{-1} (A_0^{-(k-2)} f) = A_0^{-(k-1)} f, \\ u &= A_0^{-1} v_1 = A_0^{-1} (A_0^{-(k-1)} f) = A_0^{-k} f, \end{aligned}$$

which is (19) with

$$A_0^{-k} = \underbrace{A_0^{-1} A_0^{-1} \cdots A_0^{-1}}_k.$$

Finally, since $R(A_0^{-k}) = X$ and A_0^{-k} is bounded as a composition of bounded operators, it is concluded that the operator A_0^k is correct. $\square$

For the important category of boundary value problems for $k = 2$, we state the following corollary, which follows immediately from Theorem 3.

Corollary 1. *The boundary value problem*

$$\begin{aligned} A_0^2 u &= A^2 u = f, \\ D(A_0^2) &= \{u : u \in D(A^2), Y(u) = \mathbf{0}, Y(Au) = \mathbf{0}\}, \end{aligned} \quad (20)$$

is correct if and only if $\det Y(\mathbf{z}) \neq 0$ and the unique solution, for every $f \in X$, is given by

$$\begin{aligned} u &= A_0^{-2} f \\ &= \hat{A}^{-2} f - \mathbf{z} Y^{-1}(\mathbf{z}) Y(\hat{A}^{-2} f) \\ &\quad - [\hat{A}^{-1} \mathbf{z} - \mathbf{z} Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} \mathbf{z})] Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f). \end{aligned} \quad (21)$$

4.2. Product of Two Operators

Here, we are looking at yet another special boundary value problem, which is the generalization of Corollary 1. In particular, we consider the boundary value problem

$$\begin{aligned} A_0 \tilde{A}_0 u &= A^2 u = f, \quad x \in (a, b), \\ D(A_0 \tilde{A}_0) &= \{u : u \in D(A^2), \tilde{Y}(u) = \mathbf{0}, Y(Au) = \mathbf{0}\}, \end{aligned} \quad (22)$$

where A, A_0 are defined as in (1) and (9), respectively, and $\tilde{A}_0 : X \rightarrow X$ is a restriction of A defined by

$$\begin{aligned} \tilde{A}_0 u &= Au, \quad x \in (a, b), \\ D(\tilde{A}_0) &= \{u : u \in D(A), \tilde{Y}(u) = \mathbf{0}\}, \end{aligned}$$

wherein $\tilde{Y} = \text{col}(\tilde{Y}_1, \dots, \tilde{Y}_n)$ with $\tilde{Y}_i \in [X^{n-1}]^*$ being defined by

$$\tilde{Y}_i(u) = \sum_{j=0}^m \sum_{k=0}^{n-1} \tilde{v}_{ikj} u^{(k)}(x_j) + \sum_{k=0}^{n-1} \int_a^b \tilde{h}_{ik}(t) u^{(k)}(t) dt, \quad i = 1, \dots, n, \quad (23)$$

where $\tilde{h}_{ik}(x)$, $i = 1, \dots, n$, $k = 0, \dots, n-1$, are continuous functions on $[a, b]$, $\tilde{v}_{ikj}$, $i = 1, \dots, n$, $k = 0, \dots, n-1$, $j = 0, \dots, m$, are constants.

Theorem 4. *Let A_0 and $\tilde{A}_0$ be the linear operators defined in (22), $\mathbf{z} = (z_1, \dots, z_n)$ a basis of $\ker A$, and $\hat{A}^{-1}$ the inverse of the correct operator $\hat{A}$ in (8). Then:*

(i) *The operator $A_0 \tilde{A}_0$ is injective if and only if*

$$\det Y(\mathbf{z}) \neq 0, \quad \det \tilde{Y}(\mathbf{z}) \neq 0. \quad (24)$$

(ii) *Furthermore, under (24), the operator $A_0 \tilde{A}_0$ is correct and the unique solution to the boundary value problem (22), for any $f \in X$, is given by*

$$\begin{aligned} u &= \hat{A}^{-2} f - \mathbf{z} \tilde{Y}^{-1}(\mathbf{z}) \tilde{Y}(\hat{A}^{-2} f) \\ &\quad - [\hat{A}^{-1} \mathbf{z} - \mathbf{z} \tilde{Y}^{-1}(\mathbf{z}) \tilde{Y}(\hat{A}^{-1} \mathbf{z})] Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f). \end{aligned} \quad (25)$$

Proof. (i)–(ii) By setting $\tilde{A}_0 u = v$, the problem (22) may be decomposed into the following two boundary value problems:

$$Av = f, \quad Y(v) = \mathbf{0} \quad \text{or} \quad A_0 v = f, \quad (26)$$

$$Au = v, \quad \tilde{Y}(u) = \mathbf{0} \quad \text{or} \quad \tilde{A}_0 u = v. \quad (27)$$

By Theorem 1, the boundary value problem (26) is correct if and only if $\det Y(\mathbf{z}) \neq 0$ and its unique solution is given by

$$v = A_0^{-1} f = \hat{A}^{-1} f - \mathbf{z} Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f). \quad (28)$$

Similarly, the boundary value problem (27) is correct if and only if $\det \tilde{Y}(\mathbf{z}) \neq 0$ and its solution is

$$u = \tilde{A}_0^{-1} v = \hat{A}^{-1} v - \mathbf{z} \tilde{Y}^{-1}(\mathbf{z}) \tilde{Y}(\hat{A}^{-1} v). \quad (29)$$

Substitution of (28) into (29) yields

$$\begin{aligned} u &= \hat{A}^{-1} \left[\hat{A}^{-1} f - \mathbf{z} Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f) \right] \\ &\quad - \mathbf{z} \tilde{Y}^{-1}(\mathbf{z}) \tilde{Y} \left(\hat{A}^{-1} \left[\hat{A}^{-1} f - \mathbf{z} Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f) \right] \right) \\ &= \hat{A}^{-2} f - \hat{A}^{-1} \mathbf{z} Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f) \\ &\quad - \mathbf{z} \tilde{Y}^{-1}(\mathbf{z}) \tilde{Y}(\hat{A}^{-2} f) + \mathbf{z} \tilde{Y}^{-1}(\mathbf{z}) \tilde{Y}(\hat{A}^{-1} \mathbf{z}) Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f) \\ &= \hat{A}^{-2} f - \mathbf{z} \tilde{Y}^{-1}(\mathbf{z}) \tilde{Y}(\hat{A}^{-2} f) \\ &\quad - \left[\hat{A}^{-1} \mathbf{z} - \mathbf{z} \tilde{Y}^{-1}(\mathbf{z}) \tilde{Y}(\hat{A}^{-1} \mathbf{z}) \right] Y^{-1}(\mathbf{z}) Y(\hat{A}^{-1} f), \end{aligned}$$

which is (25). Thus, the operator $A_0 \tilde{A}_0$ is correct if and only if $\det Y(\mathbf{z}) \neq 0$ and $\det \tilde{Y}(\mathbf{z}) \neq 0$, and the unique solution of $A_0 \tilde{A}_0 u = f$ is given explicitly by (25). $\square$

5. Examples

To explain the implementation of the results presented in the previous section and to show the efficiency of the proposed solution routine, we solve two example problems.

Example 1. Consider the differential Equation [29]

$$u''(x) + u(x) = 0, \quad 0 < x < \frac{\pi}{2},$$

with the constraints

$$\int_0^{\pi/2} u(x) dx = 1, \quad \int_0^{\pi/2} u(x) \sin x dx = 1.$$

Comparing with (12), it is natural to take

$$\begin{aligned} Au(x) &= u''(x) + u(x), \quad D(A) = C^2[0, \frac{\pi}{2}], \quad f(x) = 0, \\ Y_1(u(x)) &= \int_0^{\pi/2} u(x) dx, \quad Y_2(u(x)) = \int_0^{\pi/2} u(x) \sin x dx, \quad \beta_1 = \beta_2 = 1, \end{aligned}$$

and $\hat{A}$ and A_1 defined on

$$\begin{aligned} D(\hat{A}) &= \{u : u \in D(A), u(0) = u'(\pi/2) = 0\}, \\ D(A_1) &= \{u : u \in D(A), Y(u) = \mathbf{b}\}, \end{aligned}$$

respectively. Observe that the only solution of $\hat{A}u = 0$ is $u \equiv 0$, $z_1 = \cos x$ and $z_2 = \sin x$ are two linearly independent solutions of $Au = 0$, and the matrix

$$Y(\mathbf{z}) = \begin{pmatrix} Y_1(z_1) & Y_1(z_2) \\ Y_2(z_1) & Y_2(z_2) \end{pmatrix}$$

is non-singular. Thus, by applying Theorem 2, we get the unique solution

$$u(x) = \mathbf{z}Y^{-1}(\mathbf{z})\mathbf{b} = \frac{2\sin x + (\pi - 4)\cos x}{\pi - 2}.$$

Example 2. Let the second order ordinary differential equation

$$u''(x) - \frac{6}{x}u'(x) + \frac{12}{x^2}u(x) = 2x^4, \quad x \in (1, 2), \quad (30)$$

subjected to non-local boundary conditions

$$\begin{aligned} 8u\left(\frac{3}{2}\right) - \frac{3}{4}u(2) &= \frac{177}{22} \int_1^2 \frac{1}{t^3}u(t)dt, \\ u'\left(\frac{5}{4}\right) - \frac{12}{5}u\left(\frac{5}{4}\right) &= \frac{625}{768} \int_1^2 \left(\frac{1}{t^2}u'(t) - \frac{3}{t^3}u(t)\right)dt. \end{aligned} \quad (31)$$

Observe that this problem can be written as follows:

$$\begin{aligned} \left[\frac{d}{dx} - \frac{3}{x}\right]^2 u(x) &= 2x^4, \quad x \in (1, 2), \\ 8u\left(\frac{3}{2}\right) - \frac{3}{4}u(2) - \frac{177}{22} \int_1^2 \frac{1}{t^3}u(t)dt &= 0, \\ \left[\frac{d}{dx} - \frac{3}{x}\right]u(x)\Big|_{x=\frac{5}{4}} - \frac{625}{768} \int_1^2 \frac{1}{t^2} \left[\frac{d}{dt} - \frac{3}{t}\right]u(t)dt &= 0. \end{aligned}$$

Comparing now with (22), we take

$$\begin{aligned} Au(x) &= \left[\frac{d}{dx} - \frac{3}{x}\right]u(x), \quad D(A) = C^1[1, 2], \\ f(x) &= 2x^4, \\ \tilde{Y}(u) &= 8u\left(\frac{3}{2}\right) - \frac{3}{4}u(2) - \frac{177}{22} \int_1^2 \frac{1}{t^3}u(t)dt, \\ Y(Au) &= Au(x)\Big|_{x=\frac{5}{4}} - \frac{625}{768} \int_1^2 \frac{1}{t^2}Au(t)dt, \end{aligned}$$

and we set

$$\begin{aligned} \tilde{A}_0 u &= Au, \quad D(\tilde{A}_0) = \{u : u \in D(A), \tilde{Y}(u) = 0\}, \\ A_0 v &= Av, \quad D(A_0) = \{v : v \in D(A), Y(v) = 0\}. \end{aligned}$$

Further, let the correct operator $\hat{A}$ defined by

$$\hat{A}u = Au = f, \quad D(\hat{A}) = \{u : u \in D(A), u(1) = 0\},$$

and its inverse given by

$$\hat{A}^{-1}f(x) = e^{\int_1^x \frac{3}{s}ds} \int_1^x e^{-\int_1^t \frac{3}{s}ds} f(t)dt = x^3 \int_1^x \frac{1}{t^3}f(t)dt.$$

Finally, note that $z(x) = x^3$ is a fundamental solution of the homogeneous equation $Au = 0$ and that

$$Y(z) \neq 0, \quad \tilde{Y}(z) \neq 0.$$

Hence, from Theorem 4, it follows that the non-local boundary value problem (30), (31) has a unique solution, which after substituting into the formula (25), is

$$u(x) = \frac{x^3(x^3 - 1)}{3}.$$

6. Discussion

A method for constructing closed-form solutions to boundary value problems for ordinary differential equations with general multipoint and integral boundary conditions has been presented. Ready to use solution formulae in a symbolic form have been derived for some classes of boundary value problems. Specifically, we considered the following boundary value problems:

$$\begin{aligned} A_0 u &= f, \quad D(A_0) = \{u : u \in D(A), Y(u) = \mathbf{0}\}, \\ A_1 u &= f, \quad D(A_1) = \{u : u \in D(A), Y(u) = \mathbf{b}\}, \\ A_0^k u &= f, \quad D(A_0^k) = \{u : u \in D(A^k), Y(u) = \mathbf{0}, Y(Au) = \mathbf{0}, \dots, Y(A^{k-1}u) = \mathbf{0}\}, \\ A_0 \tilde{A}_0 u &= f, \quad D(A_0 \tilde{A}_0) = \{u : u \in D(A^2), \tilde{Y}(u) = \mathbf{0}, Y(Au) = \mathbf{0}\}, \end{aligned}$$

where the operators $A_0, A_1, \tilde{A}_0$ are restrictions of the n -th order linear differential operator A in (1) and $Y, \tilde{Y}$ are vectors of linear bounded functionals as in (4) and (23), respectively, describing general non-local boundary conditions.

The proposed methodology can be specialized to other categories of boundary value problems and extended to some classes of partial differential equations.

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