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New Necessary Conditions for the Well-Posedness of Steady Bioconvective Flows and Their Small Perturbations

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Abstract: We introduce new necessary conditions for the existence and uniqueness of stationary weak solutions and the existence of the weak solutions for the evolution problem in the system arising from the modeling of the bioconvective flow problem. Our analysis is based on the application of the Galerkin method, and the system considered consists of three equations: the nonlinear Navier–Stokes equation, the incompressibility equation, and a parabolic conservation equation, where the unknowns are the fluid velocity, the hydrostatic pressure, and the concentration of microorganisms. The boundary conditions are homogeneous and of zero-flux-type, for the cases of fluid velocity and microorganism concentration, respectively.

Keywords: bioconvective flow; Navier–Stokes system; Galerkin estimates

MSC: 35Q35; 76Z05; 35Q30; 76D05

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1. Introduction

In this paper, we consider the analysis of the existence of solutions for the governing equations modeling the bioconvective flow problem. In order to define the system, we consider $\Omega \subset \mathbb{R}^3$ a bounded and regular domain with rigid boundary $\partial\Omega$ where the outward normal unitary vector to $\partial\Omega$ is given by $\mathbf{n} = (n_1, n_2, n_3)$. The flow induced by the upward swimming of certain microorganisms in Ω and during an interval of time $[0, T]$ is given by the following system [1]:

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla q = -km\mathbf{e}_3 + \mathbf{F}, && \text{in } Q_T = \Omega\times[0,T], && (1)\\
&\operatorname{div}(\mathbf{u}) = 0, && \text{in } Q_T, && (2)\\
&\frac{\partial m}{\partial t} - \theta\Delta m + \mathbf{u}\cdot\nabla m + U\frac{\partial m}{\partial x_3} = 0, && \text{in } Q_T, && (3)\\
&\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad m(x,0) = m_0(x), && \text{in } \Omega, && (4)\\
&\mathbf{u} = 0, && \text{on } \Gamma := \partial\Omega\times[0,T], && (5)\\
&\theta\frac{\partial m}{\partial \mathbf{n}} - Un_3 m = 0, && \text{on } \Gamma, && (6)
\end{aligned}
$$

where $\mathbf{u} = (u_1, u_2, u_3)^t$ is the velocity of the fluid; q is related to the pressure and defined by $q = p + gx_3$ with p the hydrostatic pressure and g the acceleration gravity constant; m is related to the local concentration of the microorganisms c and defined by $m = (g\bar{\rho}c)/k$ with k a positive constant; $\bar{\rho}$ is a positive constant defined as follows $\bar{\rho} = (\rho_0 - \rho_m)/\rho_m$

with ρ_0 and ρ_m the densities of the fluid and the microorganisms, respectively; $\mu > 0$ is the viscosity of the fluid; $\mathbf{e}_3 = (0,0,1)^t$ is the unit vector in the vertical direction; $\mathbf{F} = (F_1, F_2, F_3)^t$ is the external source; θ is a constant defining the diffusion rate of microorganisms; and U is the average velocity of swimming for the microorganisms. Moreover, the notations ∇, Δ, div , and $(\mathbf{u} \cdot \nabla)\mathbf{u}$ denote the gradient, Laplacian, divergence, and convection operators, respectively.

The system (1)–(6) was derived by Y. Moribe [2] and independently by M. Levandowsky, W. S. Hunter, and E. A. Spiegel [3] (see also [1,4,5] for the mathematical analysis). More recently, some new bioconvective flow models have been introduced, for instance: [6] (see also [7]) considered a generalized model with a nonconstant viscosity and the symmetric part of the deformation rate tensor, and Tuval et al. [8] constructed a mathematical model by considering as an additional unknown variable: the oxygen concentration (see also [9–11]).

In order to study the well-posedness of (1)–(6), we applied the Galerkin method twice. Firstly, we study the existence and uniqueness of solutions for the stationary problem by using the Galerkin method. Then, we study the existence of the evolution problem by combining the Galerkin method and perturbation arguments in a neighborhood where the stationary problem is well-posed. As a consequence of our analysis, we obtained the two main results: (i) we proved the existence and uniqueness of weak solutions for the stationary problem associated with (1)–(6) (see Theorem 1); (ii) we proved the existence of solutions of the evolution problem (see Theorem 3). We also proved the existence of weak solutions of a transformed problem defined as a change of variable for the stationary problem associated with (1)–(6) (see Theorem 1).

In this paper, we introduce two necessary conditions. The first one was assumed to obtain the uniqueness of weak solutions of the stationary problem associated with (1)–(6), and the second one is a necessary condition condition for the existence of weak solutions of the evolution problem (1)–(6). To be more specific, we proved the existence of stationary solutions assuming that the external force is of $L^2(\Omega)$ regularity and the coefficients satisfy the inequality:

$$2UC_P < \Theta, \tag{7}$$

with C_P defined in (13) for $p = 2$.

To prove the uniqueness of weak stationary solutions, in addition to (7), we assumed that the parameters U, θ, k, and μ are small enough such that, for any stationary solution $\mathbf{u}, m$ and some ε_0 independent of $\nabla \mathbf{u}$, the inequalities:

$$\left.\begin{aligned} &\theta - \frac{UC_P}{\theta - UC_P} - C_P^2 \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \geq \varepsilon_0 > 0, \\ &\mu - C_P^2 \|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} - \frac{kC_P^5}{(\theta - UC_P)\varepsilon_0} \|\nabla m\|_{L^2(\Omega)} > 0, \end{aligned}\right\} \tag{8}$$

are satisfied. To prove existence of weak solutions for the evolution problem, we considered that the stationary problem is solvable, the external force is of $L^2(0,T;L^2(\Omega))$ regularity, and the constants U, C_P, and θ are small enough such that:

$$0 < \frac{1}{\mu C_P^2} \left(\theta - \frac{UC_P}{\theta - UC_P} \right)^2 \leq 1. \tag{9}$$

The condition (7) is the standard assumption considered for instance in [1,6,7]. However, to the best of our knowledge, the conditions (8) and (9) have not been considered before. Moreover, we remark that the assumption (8) improves the recent result given in [6], where the authors obtained the uniqueness assuming that stationary solution $\mathbf{u}, m$ is small enough without a precise bound in terms of the parameters U, θ, k, μ and the constant C_P.

On the other hand, we mention two facts. The conditions (7)–(9) are useful in several situations, for instance in the implementation and analysis of the convergence for numerical methods approximating the stationary solution.However, we must clarify that there could be some situations where these conditions may not be valid. The authors of [12] deduced the existence of a weak solution for a generalized bioconvective model by uniquely considering the condition $UC_P^2 < \Theta$ instead of (7)–(9).

The paper is organized as follows. In Section 2, we introduce the notation, some previous results, and the general assumptions. In Section 3, we study the well-posedness and the stationary problem. In Section 4, we study the existence of weak solutions of the evolution problem. Finally, in Section 5, we give some conclusions and challenges.

2. Preliminaries

2.1. Functional Framework

We use the standard notation of functional spaces, which are used in the analysis of Navier–Stokes and the related equations of fluid mechanics; see for instance [13–15]. To be more precise, we considered the Lebesgue, Sobolev, and Bochner spaces. The Lebesgue space $L^p(\Omega)$ for $p \geq 1$ is defined by:

$$L^p(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \quad : \quad u \text{ is Lebesgue measurable, } \|u\|_{L^p(\Omega)} < \infty \Big\},$$

where:

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{1/p}, & p \in [1, \infty[, \\ \operatorname{ess\,sup}_\Omega |u(x)|, & p = \infty. \end{cases} \tag{10}$$

We recall that the spaces $L^p(\Omega)$ are Banach spaces with the norm given in (10) and $L^2(\Omega)$ is a separable Hilbert space. For $m \in \mathbb{N}$ and $p \geq 1$, the Sobolev spaces are defined as follows:

$$W^{m,p}(\Omega) := \Big\{ u \in L^q(\Omega) \ : \ D^{\boldsymbol{\alpha}} u \in L^q(\Omega), \, \forall |\boldsymbol{\alpha}| \leq m \Big\},$$

In particular, when $p = 2$, we use the notation $W^{m,2}(\Omega) = H^m(\Omega)$. The spaces of vector-valued functions are defined in the usual componentwise sense and are denoted by bold symbols, for instance $\mathbf{C}^\infty(\Omega) = [C^\infty(\Omega)]^3$, $\mathbf{L}^p(\Omega) = [L^p(\Omega)]^3$ and $\mathbf{W}^{m,p}(\Omega) = [W^{m,p}(\Omega)]^3$. Let X be a Banach space and $r \geq 1$. The Bochner spaces $L^r(0, T; X)$ are defined as follows:

$$L^r(0, T; X) = \Big\{ u : [0, T] \to X \quad : \quad u \text{ is strongly measurable, } \|u\|_{L^r(0,T;X)} < \infty \Big\},$$

where:

$$\|u\|_{L^r(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|^p dx\right)^{1/p}, & p \in [1, \infty[, \\ \operatorname{ess\,sup}_{[0,T]} \|u(t)\|, & p = \infty. \end{cases}$$

Moreover, we considered the following spaces and notation:

$$\left.\begin{array}{l} \mathbf{C}^\infty_{0,\sigma}(\Omega) = \Big\{ \mathbf{f} \in \mathbf{C}^\infty_0(\Omega) : \ \operatorname{div} \mathbf{f} = 0 \Big\}, \\ \mathbf{V} \text{ the completion of } \mathbf{C}^\infty_{0,\sigma}(\Omega) \text{ in } \mathbf{H}^1(\Omega), \\ \mathbf{H} \text{ the completion of } \mathbf{C}^\infty_{0,\sigma}(\Omega) \text{ in } \mathbf{L}^2(\Omega), \\ X \text{ the closed subspace of } L^2(\Omega) \text{ orthogonal to the constants}, \\ B = H^1(\Omega) \cap X. \end{array}\right\} \tag{11}$$

We notice that $\mathbf{C}^\infty_{0,\sigma}(\Omega)$ is the space of smooth solenoidal vector fields with compact support on Ω.

2.2. Some Classical Inequalities

We use some classical inequalities and Sobolev embeddings with the appropriate notation. To be more precise, we use the notation in the following three inequalities:

(i) *The Young and Cauchy inequalities*. Let us consider $p, q \in]1, \infty[$ such that $p^{-1} + q^{-1} = 1$, then:

$$ab \leq \epsilon a^p + C_\epsilon b^q, \quad a, b \geq 0, \quad \epsilon > 0, \quad C_\epsilon = (p-1)\epsilon^{(1-q)} p^{-q}, \tag{12}$$

which are called the Young and Cauchy inequalities. We observe that, when $(p, q, \epsilon) = (2, 2, 1/2)$, the inequality (12) is reduced to the standard Cauchy inequality;

(ii) *The Poincaré inequality*. Let $\Omega \subset \mathbb{R}^3$ be a connected, bounded Lipschitz domain, then the estimate:

$$\|u\|_{L^q(\Omega)} \leq C_P \|\nabla u\|_{L^p(\Omega)}, \; u \in W_0^{1,p}(\Omega), \; p \in [1, 3[, \; q \in \left[1, \frac{3p}{3-p}\right], \tag{13}$$

is satisfied for a positive constant C_P depending only on p and Ω. For a generalized version of the Poincaré inequality on $W^{1,p}(\Omega)$, we refer to Proposition III.2.39 in [14];

(iii) *The Gagliardo–Nirenberg inequality*. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, then there exists a positive constant C_{gn} depending only on q and Ω such that:

$$\|\nabla u\|_{L^{2q/(q-2)(\Omega)}} \leq C_{gn} \|\nabla u\|_{L^2(\Omega)}^{1-3/q} \|u\|_{H^2(\Omega)}^{3/q}, \quad u \in H^2(\Omega), \quad q \in [2, \infty[. \tag{14}$$

Other forms of the Gagliardo–Nirenberg inequality were given for instance on Proposition III.2.35 of [14], and for a recent review, we refer to [16].

Moreover, we considered the continuous embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$ for some $\Omega \subset \mathbb{R}^3$ to be a bounded Lipschitz domain, or equivalently, we have that the estimate:

$$\|u\|_{L^\infty(\Omega)} \leq C_{iny}^{2,\infty} \|u\|_{H^2(\Omega)}, \quad u \in H^2(\Omega). \tag{15}$$

is satisfied for a positive constant $C_{iny}^{2,\infty}$.

2.3. The Stokes Operator and the Friedrichs Extension

The notation $A : D(A) := \mathbf{V} \cap \mathbf{H}^2(\Omega) \subset \mathbf{H} \to \mathbf{H}$ is used for the Stokes operator defined $A\mathbf{v} = P(-\Delta \mathbf{v})$ with P the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$ induced by the Helmholtz decomposition of $\mathbf{L}^2(\Omega)$. We recall that A has the following properties: linear, unbounded, positive, self-adjoint, and characterized by the identity:

$$(A\mathbf{w}, \mathbf{v}) = (\nabla \mathbf{w}, \nabla \mathbf{v}), \quad \forall \mathbf{w} \in D(A), \quad \mathbf{v} \in V, \tag{16}$$

where $(\cdot, \cdot)$ is the standard scalar product in $\mathbf{L}^2(\Omega)$.

The Friedrichs extension is denoted by A_1 and is defined from $D(A_1)$ to X by $A_1\phi = P_1(-\theta \Delta \phi)$ for all $\phi \in D(A_1)$ with:

$$D(A_1) := \left\{ \phi \in X \cap H^2(\Omega) : \theta \frac{\partial \phi}{\partial \mathbf{n}} - U n_3 \phi = 0 \text{ on } \partial\Omega \right\}$$

and P_1 the orthogonal projection of $L^2(\Omega)$ onto X. The operator A_1 is an unbounded linear and positive self-adjoint operator and satisfies the inequalities:

$$(A_1\phi, \phi) \;\geq\; (\theta - UC_P)\|\nabla \phi\|_{L^2(\Omega)}^2, \quad C_P\|A_1\phi\|_{L^2(\Omega)} \;\geq\; (\theta - UC_P)\|\nabla \phi\|_{L^2(\Omega)}, \tag{17}$$

for all $\phi \in D(A_1)$. We refer to [1] for other properties on A and A_1.

2.4. The Trilinear Forms B_0 and B_1

Let us consider B_0 from $\mathbf{V}\times\mathbf{V}\times\mathbf{V}$ to $\mathbb{R}$ and B_1 from $\mathbf{V}\times H^1(\Omega)\times H^1(\Omega)$ to $\mathbb{R}$. defined as follows:

$$B_0(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) = \int_\Omega \sum_{i,j=1}^N u_j(\mathbf{x})\frac{\partial v_i}{\partial x_j}(\mathbf{x})w_i(\mathbf{x})d\mathbf{x}, \tag{18}$$

$$B_1(\mathbf{u},\phi,\psi) = (\mathbf{u}\cdot\nabla\phi,\psi) = \int_\Omega \sum_{j=1}^N u_j(\mathbf{x})\frac{\partial \phi}{\partial x_j}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}. \tag{19}$$

The applications B_0 and B_1 are well-defined trilinear forms with the following properties:

$$B_0(\mathbf{u},\mathbf{v},\mathbf{w}) = -B_0(\mathbf{u},\mathbf{w},\mathbf{v}), \qquad B_1(\mathbf{u},\phi,\psi) = -B_1(\mathbf{u},\psi,\phi), \tag{20}$$

$$B_0(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \qquad B_1(\mathbf{u},\phi,\phi) = 0, \tag{21}$$

$$|B_0(\mathbf{u},\mathbf{v},\mathbf{w})| \le C_P^2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\Omega)}, \tag{22}$$

$$|B_1(\mathbf{u},\psi,\phi)| \le C_P^2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\|\nabla\psi\|_{\mathbf{L}^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}, \tag{23}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{V}$, and $\psi,\phi\in H^1(\Omega)$.

3. The Stationary Problem

The stationary problem associated with the bioconvective system (1)–(6) is defined as follows: given $\alpha>0$, find the functions $(\mathbf{u}_\alpha, m_\alpha, q_\alpha)$ such that:

$$-\mu\Delta\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + \nabla q_\alpha = -km\mathbf{e}_3 + \mathbf{F}_s \quad \text{in } \Omega, \tag{24}$$

$$\operatorname{div}\mathbf{u}_\alpha = 0 \quad \text{in } \Omega, \tag{25}$$

$$-\theta\Delta m_\alpha + \mathbf{u}_\alpha\cdot\nabla m_\alpha + U\frac{\partial m_\alpha}{\partial x_3} = 0 \quad \text{in } \Omega, \tag{26}$$

$$\mathbf{u}_\alpha = 0 \quad \text{on } \partial\Omega, \tag{27}$$

$$\theta\frac{\partial m_\alpha}{\partial\mathbf{n}} - Un_3 m_\alpha = 0 \quad \text{on } \partial\Omega, \tag{28}$$

$$\int_\Omega m_\alpha(\mathbf{x})d\mathbf{x} = \alpha. \tag{29}$$

Our analysis is based on recalling and adapting the results of Boldrini et al. [6] (see also [1]). Indeed, we introduce the change of variable:

$$\begin{cases} \widetilde{m} = m_\alpha - E \text{ with } E \text{ of the form } E(\mathbf{x}) = C_\alpha\exp(Ux_3/\theta) \text{ where:} \\ \text{the constant } C_\alpha \text{ is selected such that } \int_\Omega E(\mathbf{x})d\mathbf{x} = \alpha, \end{cases} \tag{30}$$

and we obtain that the problem (24)–(29) can be rewritten as follows:

$$-\mu\Delta\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + \nabla(q_\alpha + k\theta U^{-1}E) = -k\widetilde{m}\mathbf{e}_3 + \mathbf{F}_s \quad \text{in } \Omega, \tag{31}$$

$$\operatorname{div}\mathbf{u}_\alpha = 0 \quad \text{in } \Omega, \tag{32}$$

$$-\theta\Delta\widetilde{m} + \mathbf{u}_\alpha\cdot\nabla(\widetilde{m}+E) + U\frac{\partial\widetilde{m}}{\partial x_3} = 0 \quad \text{in } \Omega, \tag{33}$$

$$\mathbf{u}_\alpha = 0 \quad \text{on } \partial\Omega, \tag{34}$$

$$\theta\frac{\partial\widetilde{m}}{\partial\mathbf{n}} - Un_3\widetilde{m} = 0 \quad \text{on } \partial\Omega, \tag{35}$$

$$\int_\Omega \widetilde{m}(\mathbf{x})d\mathbf{x} = 0. \tag{36}$$

The deduction of (31)–(36) is straightforward by noticing that $-\theta\Delta E + U\partial_{x_3}E = 0$ in Ω and $\theta\partial_{\mathbf{n}}E - Un_3E = 0$ on $\partial\Omega$. Then, we use the concept of the weak solution for (31)–(36).

Definition 1. *Let us consider* $\mathbf{F}_s \in \mathbf{H}$*. Then,* $(\mathbf{u}_\alpha, \widetilde{m}) \in \mathbf{V} \times B$ *is called a weak solution of* (31)–(36) *if the following identities:*

$$\mu(\nabla \mathbf{u}_\alpha, \nabla \mathbf{v}) + B_0(\mathbf{u}_\alpha, \mathbf{u}_\alpha, \mathbf{v}) + (k\widetilde{m}\mathbf{e}_3, \mathbf{v}) = (\mathbf{F}_s, \mathbf{v}), \tag{37}$$

$$\theta(\nabla \widetilde{m}, \nabla \phi) + B_1(\mathbf{u}_\alpha, \widetilde{m} + E, \phi) - U\left(\widetilde{m}, \frac{\partial \phi}{\partial x_3}\right) = 0, \tag{38}$$

are satisfied for all $(\mathbf{v}, \phi) \in \mathbf{V} \times B$.

Theorem 1. *Suppose that*

$$2UC_P < \theta. \tag{39}$$

If $\mathbf{F}_s \in \mathbf{H}$*, there is a weak solution of* (31)–(36) *in the sense of Definition 1.*

Proof. The proof is made by using the Galerkin method. Let us consider a Schauder basis $(\overline{\mathbf{w}}^j)_1^\infty$ for $\mathbf{V}$ and $(\overline{\phi}^j)_1^\infty$ for B. For each $n \in \mathbb{N}$, we define the spaces $\mathbf{W}_n = \operatorname{span}\{\overline{\mathbf{w}}^j \,:\, 1 \leq j \leq n\}$ and $M_n = \operatorname{span}\{\overline{\phi}^\ell \,:\, 1 \leq \ell \leq n\}$ and consider the Galerkin approximations:

$$\mathbf{u}_\alpha^n = \sum_{j=1}^n c_{n,j} \overline{\mathbf{w}}^j \in \mathbf{W}_n, \quad \text{and} \quad \overline{m}^n = \sum_{\ell=1}^n d_{n,\ell} \overline{\phi}^\ell \in M_n, \tag{40}$$

satisfying the approximate problem:

$$\mu(\nabla \mathbf{u}_\alpha^n, \nabla \overline{\mathbf{w}}^j) + B_0(\mathbf{u}_\alpha^n, \mathbf{u}_\alpha^n, \overline{\mathbf{w}}^j) + k(\widetilde{m}^n \mathbf{e}_3, \overline{\mathbf{w}}^j) = (\mathbf{F}_s, \overline{\mathbf{w}}^j), \qquad j = 1, \ldots, n, \tag{41}$$

$$\theta(\nabla \widetilde{m}^n, \nabla \overline{\phi}^\ell) + B_1(\mathbf{u}_\alpha^n, \widetilde{m}^n + E, \overline{\phi}^\ell) - U\left(\widetilde{m}^n, \frac{\partial \overline{\phi}^\ell}{\partial x_3}\right) = 0, \qquad \ell = 1, \ldots, n. \tag{42}$$

Applying the Galerkin method requires proving two facts: the existence of $(\mathbf{u}_\alpha^n, \widetilde{m}^n)$ satisfying (41) and (42) for each $n \in \mathbb{N}$ and the convergence of $(\mathbf{u}_\alpha^n, \widetilde{m}^n)$ along subsequences to the weak solution of (31)–(36).

We prove the existence of solutions for (41) and (42) by the application of Brouwer's fixed point theorem. Let $(\mathbf{z}, \xi) \in \mathbf{W}_n \times M_n$, and consider $(\mathbf{v}, \Psi) \in \mathbf{W}_n \times M_n$ satisfying the linearized equations:

$$\mu(\nabla \mathbf{v}, \nabla \overline{\mathbf{w}}^j) + B_0(\mathbf{z}, \mathbf{v}, \overline{\mathbf{w}}^j) + k(\Psi \mathbf{e}_3, \overline{\mathbf{w}}^j) = (\mathbf{F}_s, \overline{\mathbf{w}}^j), \qquad j = 1, \ldots, n, \tag{43}$$

$$\theta(\nabla \Psi, \nabla \overline{\phi}^\ell) + B_1(\mathbf{z}, \Psi + E, \overline{\phi}^\ell) - U\left(\Psi, \frac{\partial \overline{\phi}^\ell}{\partial x_3}\right) = 0, \qquad \ell = 1, \ldots, n. \tag{44}$$

We note that (43) and (44) is a linear system with $2n$ equations where the unknowns are the $2n$ coefficients of the expansion $\mathbf{v} = \sum_{j=1}^n c_j \overline{\mathbf{w}}^j$ and $\Psi = \sum_{\ell=1}^n d_\ell \overline{\phi}^\ell$. Thus, $(\mathbf{v}, \Psi)$ is uniquely defined, since $(\mathbf{v}, \Psi) = 0$ is the only solution of the homogeneous system, i.e., when $\mathbf{F}_s = 0$ and $E = 0$. To prove this fact, we consider that $(\mathbf{v}, \Psi)$ is a solution of the homogeneous system. Then, multiplying (43) by c_j and (44) by d_ℓ and summing on $j \in \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, n\}$, respectively, we obtain:

$$\mu(\nabla \mathbf{v}, \nabla \mathbf{v}) + B_0(\mathbf{z}, \mathbf{v}, \mathbf{v}) + k(\Psi \mathbf{e}_3, \mathbf{v}) = 0, \tag{45}$$

$$\theta(\nabla \Psi, \nabla \Psi) + B_1(\mathbf{z}, \Psi, \Psi) - U\left(\Psi, \frac{\partial \Psi}{\partial x_3}\right) = 0. \tag{46}$$

Then, using (21) and the Hölder and Poincaré–Friedrichs inequalities, we obtain:

$$\mu\|\nabla\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} \leq |-k(\Psi\mathbf{e}_3,\mathbf{v})| \leq k\|\Psi\|_{L^2(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \\ \leq k(C_P)^2\|\nabla\Psi\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)}, \tag{47}$$

$$\theta\|\nabla\Psi\|^2_{\mathbf{L}^2(\Omega)} \leq \left|U\left(\Psi,\frac{\partial\Psi}{\partial x_3}\right)\right| \leq UC_P\|\nabla\Psi\|^2_{L^2(\Omega)}. \tag{48}$$

Then, from (48) and the hypothesis (39), we obtain $\|\nabla\Psi\|_{\mathbf{L}^2(\Omega)} = 0$. This implies that Ψ is constant and by a reformulation of (36), we deduce, that $\Psi = 0$. Now, replacing $\|\nabla\Psi\|_{\mathbf{L}^2(\Omega)} = 0$ in (47), we obtain that $\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)} = 0$, which implies that $\mathbf{v} = 0$. Moreover, from (43) and (44), we obtain the following estimate:

$$\|\nabla\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\Psi\|^2_{\mathbf{L}^2(\Omega)} \leq C(\|\mathbf{F}_s\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla E\|^2_{\mathbf{L}^2(\Omega)}) \leq F_1; \tag{49}$$

see the deduction of (54) for the details. Then, (43) and (44) define a continuous application $\mathbb{T} : (\mathbf{z},\xi) \mapsto (\mathbf{v},\Psi)$ from $\mathbf{W}_n \times M_n$ to $\mathbf{W}_n \times M_n$ such that the closed convex set $\{(\mathbf{z},\xi) \in \mathbf{W}_n \times M_n : \|\nabla\mathbf{z}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\xi\|^2_{L^2(\Omega)} \leq F_1\}$ is invariant. Thus, by Brouwer's fixed point theorem, we deduce that $\mathbb{T}$ has at least one fixed point, which is the solution of (41) and (42).

Let us prove the convergence of $(\mathbf{u}^n_\alpha, \widetilde{m}^n)$ along subsequences to the weak solution of (31)–(36). Multiplying (41) by $c_{n,j}$ and (42) by $d_{n,\ell}$, summing on $j \in \{1,\ldots,n\}$ and $\ell \in \{1,\ldots,n\}$, respectively, and adding E to the second result, we obtain:

$$\mu(\nabla\mathbf{u}^n_\alpha,\nabla\mathbf{u}^n_\alpha) + B_0(\mathbf{u}^n_\alpha,\mathbf{u}^n_\alpha,\mathbf{u}^n_\alpha) + k(\widetilde{m}^n\mathbf{e}_3,\mathbf{u}^n_\alpha) = (\mathbf{F}_s,\mathbf{u}^n_\alpha), \tag{50}$$

$$\theta(\nabla\widetilde{m}^n,\nabla(\widetilde{m}^n+E)) + B_1(\mathbf{u}^n_\alpha,\widetilde{m}^n+E,\widetilde{m}^n+E) - U\left(\widetilde{m}_\alpha,\frac{\partial}{\partial x_3}(\widetilde{m}^n+E)\right) = 0. \tag{51}$$

From (21) and using the Hölder and Poincaré–Friedrichs inequalities, we obtain:

$$\begin{aligned}
\mu\|\nabla\mathbf{u}^n_\alpha\|^2_{\mathbf{L}^2(\Omega)} &\leq |-k(\widetilde{m}^n\mathbf{e}_3,\mathbf{u}^n_\alpha) + (\mathbf{F}_s,\mathbf{u}^n_\alpha)| \\
&\leq k\|\widetilde{m}^n\|_{L^2(\Omega)}\|\mathbf{u}^n_\alpha\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{F}_s\|_{\mathbf{L}^2(\Omega)}\|\mathbf{u}^n_\alpha\|_{\mathbf{L}^2(\Omega)} \\
&\leq k(C_P)^2\|\nabla\widetilde{m}^n\|_{L^2(\Omega)}\|\nabla\mathbf{u}^n_\alpha\|_{\mathbf{L}^2(\Omega)} + C_P\|\mathbf{F}_s\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{u}^n_\alpha\|_{\mathbf{L}^2(\Omega)}, \\
\theta\|\nabla\widetilde{m}^n\|^2_{L^2(\Omega)} &\leq \left|-\theta(\nabla\widetilde{m}^n,\nabla E) + U\left(\widetilde{m}_\alpha,\frac{\partial\widetilde{m}^n}{\partial x_3}\right) + U\left(\widetilde{m}_\alpha,\frac{\partial E}{\partial x_3}\right)\right| \\
&\leq \theta\|\nabla\widetilde{m}^n\|_{L^2(\Omega)}\|\nabla E\|_{L^2(\Omega)} \\
&\qquad + UC_P\|\nabla\widetilde{m}^n\|^2_{L^2(\Omega)} + UC_P\|\nabla\widetilde{m}^n\|_{L^2(\Omega)}\|\nabla E\|_{L^2(\Omega)} \\
&\leq \Big(\theta + UC_P\Big)\|\nabla\widetilde{m}^n\|_{L^2(\Omega)}\|\nabla E\|_{L^2(\Omega)} + UC_P\|\nabla\widetilde{m}^n\|^2_{L^2(\Omega)}.
\end{aligned}$$

Then, by the application of the Young inequality, we obtain:

$$\left(\mu - \frac{1}{4\epsilon}\Big(k(C_P)^2 + C_P\Big)\right)\|\nabla\mathbf{u}^n_\alpha\|^2_{\mathbf{L}^2(\Omega)} \leq \epsilon k(C_P)^2\|\nabla\widetilde{m}^n\|^2_{L^2(\Omega)} + \epsilon C_P\|\mathbf{F}_s\|^2_{\mathbf{L}^2(\Omega)}, \tag{52}$$

$$\left(\theta - UC_P - \frac{1}{4\epsilon}\Big(\theta + UC_P\Big)\right)\|\nabla\widetilde{m}^n\|^2_{L^2(\Omega)} \leq \epsilon\Big(\theta + UC_P\Big)\|\nabla E\|^2_{L^2(\Omega)}, \tag{53}$$

for any $\epsilon > 0$. Now, by (39), we can select:

$$\epsilon^* > \max\left\{\frac{k(C_P)^2 + C_P}{4\mu}, \frac{\theta + UC_P}{4(\theta - UC_P)}\right\} > 0,$$

such that by applying (53), we deduce an estimate for $\|\nabla \widetilde{m}^n\|^2_{L^2(\Omega)}$, and using this result in (52), we can obtain an estimate for $\|\nabla \mathbf{u}^n_\alpha\|^2_{\mathbf{L}^2(\Omega)}$. Then, adding both estimates, we obtain:

$$\begin{aligned}
&\|\nabla \mathbf{u}^n_\alpha\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \widetilde{m}^n\|^2_{L^2(\Omega)} \\
&\le \frac{4(\epsilon^*)^2\left(\theta + UC_P\right)}{4\epsilon^*(\theta - UC_P) - (\theta + UC_P)}\left(\frac{4(\epsilon^* C_P)^2 k}{4\epsilon^*\mu - (k(C_P)^2 + C_P)} + 1\right)\|\nabla E\|^2_{L^2(\Omega)} \\
&\quad + \frac{4(\epsilon^*)^2 C_P}{4\epsilon^*\mu - (k(C_P)^2 + C_P)}\|\mathbf{F}_s\|^2_{\mathbf{L}^2(\Omega)}.
\end{aligned} \tag{54}$$

Thus, the sequence $\{(\mathbf{u}^n_\alpha, \widetilde{m}^n)\}$ is bounded in $\mathbf{V} \times B$. Now, using the fact that $\mathbf{V}$ is compactly immersed in $\mathbf{H}$ and B is compactly immersed in X, we can select a subsequence of $\{(\mathbf{u}^n_\alpha, \widetilde{m}^n)\}$ and $(\mathbf{u}_\alpha, \widetilde{m}) \in \mathbf{V} \times B$ such that:

$$\begin{aligned}
&\mathbf{u}^n_\alpha \to \mathbf{u}_\alpha \quad \text{weakly in } \mathbf{V} \text{ and strongly in } \mathbf{H}, \\
&\widetilde{m}^n \to \widetilde{m} \quad \text{weakly in } B \text{ and strongly in } X, \\
&\nabla \mathbf{u}^n_\alpha \to \nabla \mathbf{u}_\alpha \quad \text{weakly in } \mathbf{L}^2(\Omega), \\
&\nabla \overline{m}^n_\alpha \to \nabla \widetilde{m} \quad \text{weakly in } L^2(\Omega).
\end{aligned}$$

Thus, letting $n \to \infty$ in (41) and (42) and using the properties of B_0 and B_1, we conclude the proof of theorem. □

Theorem 2. *Suppose (39) is satisfied. If $\mathbf{F}_s \in \mathbf{H}$, there is the $\mathbf{u}_\alpha \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, $m_\alpha \in H^2(\Omega)$, and $q_\alpha \in H^1(\Omega)$ solution of the system (24)–(29). Moreover, if we consider $\mathbf{F}_s$ and E are such that the condition:*

$$\left.\begin{array}{l}
\text{the constants } U, \theta, k \text{ are small enough such that there is } \varepsilon_0 \\
\text{independent of } \nabla \mathbf{u}_\alpha \text{ such that } \Pi_1 \ge \epsilon_0 > 0 \text{ and } \Pi_2 > 0, \text{ where:} \\
\Pi_1 = \theta - \dfrac{UC_P}{\theta - UC_P} - C_P^2\|\nabla \mathbf{u}_\alpha\|_{\mathbf{L}^2(\Omega)}, \\
\Pi_2 = \mu - C_P^2\|\nabla \mathbf{u}_\alpha\|_{\mathbf{L}^2(\Omega)} - \dfrac{kC_P^5}{(\theta - UC_P)\varepsilon_0}\|\nabla m_\alpha\|_{L^2(\Omega)}, \\
\text{for any } \mathbf{u}_\alpha, m_\alpha \text{ satisfying (24)–(29)}
\end{array}\right\} \tag{55}$$

is satisfied, $\mathbf{u}_\alpha$ and m_α are uniquely defined, and q_α is uniquely defined up to an additive constant.

Proof. From Theorem 1, we have that there is $(\mathbf{u}_\alpha, \widetilde{m}) \in \mathbf{V} \times B$ satisfying the variational formulation (37) and (38). Then, using (30) and applying regularity arguments similar to the proof of Theorem 3.1 in [1], we follow the existence of the $\mathbf{u}_\alpha \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, $m_\alpha = \widetilde{m} + E \in H^2(\Omega)$ and $q_\alpha \in H^1(\Omega)$ solution of the system (24)–(29).

Let us consider that $(\mathbf{u}_{\alpha,i}, m_{\alpha,i}, q_{\alpha,i})$ for $i = 1, 2$ are two solutions of (24)–(29). By the application of Theorem 1, we have that $\mathbf{u}_{\alpha,i}$ and $\widetilde{m}_i = m_{\alpha,i} - E$ for $i = 1, 2$ are weak solutions of (31)–(36). Then, we have that the functions $\mathbf{z} = \mathbf{u}_{\alpha,1} - \mathbf{u}_{\alpha,2}$ and $\varphi = \widetilde{m}_1 - \widetilde{m}_2$, satisfy the identities:

$$\mu(\nabla \mathbf{z}, \nabla \mathbf{v}) + B_0(\mathbf{z}, \mathbf{u}_{\alpha,1}, \mathbf{v}) + B_0(\mathbf{u}_{\alpha,2}, \mathbf{z}, \mathbf{v}) + (k\varphi \mathbf{e}_3, \mathbf{v}) = 0, \tag{56}$$

$$\theta(\nabla \varphi, \nabla \phi) + B_1(\mathbf{z}, \widetilde{m}_1 + E, \phi) + B_1(\mathbf{u}_{\alpha,2}, \varphi, \phi) - U\left(\varphi, \frac{\partial \phi}{\partial x_3}\right) = 0, \tag{57}$$

for all $(\mathbf{v}, \phi) \in \mathbf{V} \times B$. In particular, considering $(\mathbf{v}, \phi) = (\mathbf{z}, -A_1\varphi)$, we deduce some useful estimates. Indeed, from (56), using the properties of B_0 given in (20)–(23) and (17), we deduce that:

$$\begin{aligned}\mu\|\nabla\mathbf{z}\|^2_{\mathbf{L}^2(\Omega)} &\le C_P^2\|\nabla\mathbf{z}\|^2_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{u}_{\alpha,1}\|_{\mathbf{L}^2(\Omega)} + k\|\varphi\|_{L^2(\Omega)}\|\mathbf{z}\|_{\mathbf{L}^2(\Omega)} \\ &\le \left\{C_P^2\|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{u}_{\alpha,1}\|_{\mathbf{L}^2(\Omega)} + k\frac{C_P^3}{\theta - UC_P}\|A_1\varphi\|_{\mathbf{L}^2(\Omega)}\right\}\|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{58}$$

Similarly, from (57) and the properties for B_1 given in (20)–(23) and (17), we obtain:

$$\begin{aligned}\theta\|A_1\varphi\|^2_{L^2(\Omega)} \quad &\le \quad C_P^2\|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)}\|\nabla\widetilde{m}_1 + \nabla E\|_{L^2(\Omega)}\|A_1\varphi\|_{L^2(\Omega)} \\ &\quad +C_P^2\|\nabla\mathbf{u}_{\alpha,2}\|_{\mathbf{L}^2(\Omega)}\|A_1\varphi\|^2_{L^2(\Omega)} + \frac{UC_P}{\theta - UC_P}\|A_1\varphi\|^2_{L^2(\Omega)},\end{aligned}$$

Now, using (55), we have the bound:

$$\|A_1\varphi\|_{L^2(\Omega)} \quad \le \quad \frac{C_P^2}{\varepsilon_0}\|\nabla\mathbf{z}\|_{\mathbf{L}^2(\Omega)}\|\nabla\widetilde{m}_1 + \nabla E\|_{L^2(\Omega)}. \tag{59}$$

Replacing (59) in (58) and simplifying, we obtain that $\Pi_2\|\nabla\mathbf{z}\|^2_{\mathbf{L}^2(\Omega)} \le 0$ or, equivalently, that $\|\nabla\mathbf{z}\|^2_{\mathbf{L}^2(\Omega)} = 0$. Consequently, (59) implies $|A_1\varphi| = 0$, and hence, $|\nabla\varphi| = 0$ by (17). Thus, from (34), (36), and $\|\nabla\mathbf{z}\|^2_{\mathbf{L}^2(\Omega)} = \|\nabla\varphi\|_{L^2(\Omega)} = 0$, we deduce that $(\mathbf{z},\varphi) = (0,0)$ or equivalently that $(\mathbf{u}_{\alpha,1}, m_{\alpha,1}) = (\mathbf{u}_{\alpha,2}, m_{\alpha,2})$. The uniqueness of q_α up to a constant follows by standard arguments in the Stokes equation; see [17] for the details. □

Remark 1. *By applying similar arguments to those used in the proof of Theorem 4.1 in* [6]*, we can deduce the existence of:*

$$(\mathbf{u}_\alpha, m_\alpha, q_\alpha) \in (\mathbf{V}\cap\mathbf{H}^2(\Omega)) \times (X\cap H^2(\Omega)) \times (H^1(\Omega)\cap L_0^2(\Omega))$$

with $L_0^2(\Omega) = \{h \in L^2(\Omega) \;:\; h(1,0) = 0\}$*, such that:*

$$P\Big[-\mu\Delta\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + km\mathbf{e}_3 - \mathbf{F}_s\Big] = 0 \qquad in \quad \mathbf{L}^2(\Omega), \tag{60}$$

$$P_1\Big[-\theta\Delta m_\alpha + \mathbf{u}_\alpha\cdot\nabla m_\alpha + U\frac{\partial m_\alpha}{\partial x_3}\Big] = 0 \qquad in \quad L^2(\Omega), \tag{61}$$

$$-\mu\Delta\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + km\mathbf{e}_3 - \mathbf{F}_s = -\nabla q_\alpha \qquad in \quad \Omega, \tag{62}$$

that is the existence of strong solutions of (24)–(29)*. To prove* (60) *and* (61)*, we introduce the following modifications in the proof of Theorem* 3*: (i) we consider that* $(\overline{\mathbf{w}}^j)_1^\infty$ *for* $\mathbf{V}$ *and* $(\overline{\phi}^j)_1^\infty$ *for* B *are given by the eigenfunctions of* A *and* A_1*, respectively; (ii) using the identities* $(A\mathbf{v},\mathbf{w}^j) = (\nabla\mathbf{v},\nabla\mathbf{w}^j)$ $(A_1\mathbf{v},\phi^\ell) = (\nabla\mathbf{v},\nabla\phi^\ell)$*, multiplying* (43) *by* c_j *and* (44) *by* d_ℓ*, and summing on* $j\in\{1,\ldots,n\}$ *and* $\ell\in\{1,\ldots,n\}$*, respectively, we deduce the system:*

$$\mu(A\mathbf{v},A\mathbf{v}) + B_0(\mathbf{z},\mathbf{v},A\mathbf{v}) + k(\Psi\mathbf{e}_3,A\mathbf{v}) = (\mathbf{F}_s,A\mathbf{v}), \tag{63}$$

$$\theta(A_1\Psi,A_1\Psi) + B_1(\mathbf{z},\Psi+E,A_1\Psi) - U\left(\frac{\partial\Psi}{\partial x_3},A_1\Psi\right) = 0; \tag{64}$$

(iii) from (63) *and* (64)*, applying the Hölder, Poincaré–Friedrichs, and Cauchy inequalities, and using the equivalence of the* $\mathbf{L}^2(\Omega)$*-norm of operator* A *(respectively the* $L^2(\Omega)$*-norm of operator* A_1*) and the norm of* $\mathbf{V}\cap\mathbf{H}^2(\Omega)$ *(respectively, the norm of* $X\cap H^2(\Omega)$*), we deduce the estimates:*

$$\mu\|A\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} \le C\Big(\|A\mathbf{z}\|_{\mathbf{L}^2(\Omega)} + 2k + 2\Big)\|A\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + 2k\|A_1\Psi\|^2_{L^2(\Omega)} + 2\|\mathbf{F}_s\|^2_{\mathbf{L}^2(\Omega)},$$

$$\Big(\theta - UC_P\Big)\|A_1\Psi\|^2_{L^2(\Omega)} \le C\|A\mathbf{z}\|_{\mathbf{L}^2(\Omega)}\Big(\|A_1\Psi\|^2_{L^2(\Omega)} + \|\nabla E\|^2_{L^2(\Omega)}\Big),$$

which implies that:

$$\begin{aligned}\|A\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|A_1\Psi\|^2_{L^2(\Omega)} \leq & C\Big(\|A\mathbf{z}\|_{\mathbf{L}^2(\Omega)} + 2k + 2\Big)\Big(\|A\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|A_1\Psi\|^2_{L^2(\Omega)}\Big) \\ & + C\|A\mathbf{z}\|_{\mathbf{L}^2(\Omega)}\|\nabla E\|^2_{L^2(\Omega)} + 2\|\mathbf{F}_s\|^2_{\mathbf{L}^2(\Omega)},\end{aligned} \tag{65}$$

with C a generic positive constant; (iv) using (65) we can define appropriately F_2, such that we can apply Brouwer's fixed point theorem to the operators $T_n : G_n \to G_n$ with $G_n = \{(\mathbf{z}, \xi) \in \mathbf{W}_n \times M_n : \|A\mathbf{z}\|^2_{\mathbf{L}^2(\Omega)} + \|A_1\xi\|^2_{L^2(\Omega)} \leq F_2\}$, and also, we can deduce that the approximate solutions of (41) and (42) are uniformly bounded in $\mathbf{H}^2(\Omega) \times H^2(\Omega)$; and (v) taking the limit when $n \to \infty$, we conclude the proof of the required statements. Meanwhile, we can deduce the equation in (62) by the standard arguments, which were given for instance in [17].

4. The Evolution Problem

Let us consider $\{\mathbf{u}, m, q\}$ satisfying the bioconvective system (1)–(6) with the initial condition $\int_\Omega m_0 dx = \alpha$ and $\{\mathbf{u}_\alpha, m_\alpha, q_\alpha\}$ a solution of the stationary problem (24)–(29). Then, we analyze the relation of the evolution problem and the corresponding stationary problem by studying the perturbations of the stationary problem. Indeed, let us consider the change of variable:

$$\mathbf{v} = \mathbf{u} - \mathbf{u}_\alpha, \quad \eta = m - m_\alpha \tag{66}$$

which satisfies the following relations:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} - \mu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\mathbf{v} && \\
&\qquad + \nabla(q - q_\alpha) = -km\mathbf{e}_3 + \mathbf{F} - \mathbf{F}_s && \text{in} \quad Q_T, \quad (67)\\
&\operatorname{div}\mathbf{v} = 0 && \text{in} \quad Q_T, \quad (68)\\
&\frac{\partial \eta}{\partial t} - \theta\Delta\eta + \mathbf{v}\cdot\nabla\eta + \mathbf{v}\cdot\nabla m_\alpha + \mathbf{u}_\alpha\cdot\nabla\eta + U\frac{\partial\eta}{\partial x_3} = 0 && \text{in} \quad Q_T, \quad (69)\\
&\mathbf{v}(x,0) = \mathbf{v}_0(x) \quad \eta(x,0) = \eta_0(x) && \text{in} \quad \Omega. \quad (70)\\
&\mathbf{v} = 0, && \text{on} \quad \Gamma, \quad (71)\\
&\theta\frac{\partial\eta}{\partial\mathbf{n}} - Un_3\eta = 0 && \text{on} \quad \Gamma. \quad (72)
\end{aligned}$$

Definition 2. *Let us consider $\mathbf{F} \in L^2(0, T; \mathbf{H})$ and $\mathbf{F}_s \in \mathbf{H}$. Then:*

$$(\mathbf{v}, \eta) \in L^2(0, T, \mathbf{V}) \cap L^\infty(0, T, \mathbf{H}) \times L^2(0, T, B) \cap L^\infty(0, T, X)$$

is called a weak solution of (67)–(72) if the following identities:

$$\begin{aligned}\left(\frac{\partial\mathbf{v}}{\partial t}, \mathbf{w}\right) + \mu(\nabla\mathbf{v}, \nabla\mathbf{w}) + B_0(\mathbf{v}, \mathbf{v}, \mathbf{w}) + B_0(\mathbf{v}, \mathbf{u}_\alpha, \mathbf{w}) \\ + B_0(\mathbf{u}_\alpha, \mathbf{v}, \mathbf{w}) + (k\eta\mathbf{e}_3, \mathbf{w}) = (\mathbf{F} - \mathbf{F}_s, \mathbf{w}),\end{aligned} \tag{73}$$

$$\begin{aligned}\left(\frac{\partial\eta}{\partial t}, \phi\right) + \theta(\nabla\eta, \nabla\phi) + B_1(\mathbf{v}, \eta, \phi) + B_1(\mathbf{v}, m_\alpha, \phi) + B_1(\mathbf{u}_\alpha, \eta, \phi) \\ - U\left(\eta, \frac{\partial\phi}{\partial x_3}\right) = 0,\end{aligned} \tag{74}$$

are satisfied for all $(\mathbf{w}, \phi) \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; B)$.

Theorem 3. *Suppose* $\mathbf{F} \in L^2(0,T;\mathbf{H})$, $\mathbf{F}_s \in \mathbf{H}$, $\alpha \in \mathbb{R}^+$, *the hypotheses* (39), (55) *are satisfied, and* $\{\mathbf{u}_\alpha, m_\alpha, q_\alpha\}$ *is the solution of the stationary problem* (24)–(29) *corresponding to* $\alpha = \int_\Omega m_0 d\mathbf{x}$. *Moreover, we assume the constants* U, C_P *and* θ *are small enough such that:*

$$0 < \frac{1}{\mu C_P^2}\left(\theta - \frac{UC_P}{\theta - UC_P}\right)^2 \leq 1, \tag{75}$$

Then, for all $(\mathbf{v}_0, \eta_0) \in \mathbf{H} \times X$, *there is* $(\mathbf{v}, \eta)$, *a weak solution of* (67)–(72). *Furthermore, the weak solution is such that* $\|\mathbf{v}(\cdot,t) - \mathbf{v}_0\|_{\mathbf{H}} \to 0$ *and* $\|\eta(\cdot,t) - \eta_0\|_X \to 0$ *when* $t \to 0^+$.

Proof. The proof is performed by using the Galerkin method. Indeed, let us consider a Schauder basis $(\overline{\mathbf{w}}^j)_1^\infty$ for $\mathbf{V}$ and $(\overline{\phi}^j)_1^\infty$ for $\mathbf{B}$. For each $n \in \mathbb{N}$, we define the spaces $\mathbf{W}_n = \text{span}\{\overline{\mathbf{w}}^j : 1 \leq j \leq n\}$ and $M_n = \text{span}\{\overline{\phi}^\ell : 1 \leq \ell \leq n\}$ and consider the Galerkin approximations:

$$\mathbf{v}^n(t,\mathbf{x}) = \sum_{j=1}^n c_{n,j}(t)\overline{\mathbf{w}}^j(\mathbf{x}) \in \mathbf{W}_n, \quad \text{and} \quad \eta^n(t,\mathbf{x}) = \sum_{\ell=1}^n d_{n,\ell}(t)\overline{\phi}^\ell(\mathbf{x}) \in M_n,$$

satisfying the approximate problem:

$$\begin{aligned}\left(\frac{\partial \mathbf{v}^n}{\partial t}, \overline{\mathbf{w}}^j\right) &+ \mu(\nabla \mathbf{v}^n, \nabla \overline{\mathbf{w}}^j) + B_0(\mathbf{v}^n, \mathbf{v}^n, \overline{\mathbf{w}}^j) + B_0(\mathbf{v}^n, \mathbf{u}_\alpha, \overline{\mathbf{w}}^j) \\ &+ B_0(\mathbf{u}_\alpha, \mathbf{v}^n, \overline{\mathbf{w}}^j) + (k\eta^n \mathbf{e}_3, \overline{\mathbf{w}}^j) = (\mathbf{F} - \mathbf{F}_s, \overline{\mathbf{w}}^j),\end{aligned} \tag{76}$$

$$\begin{aligned}\left(\frac{\partial \eta^n}{\partial t}, \overline{\phi}^\ell\right) &+ \theta(\nabla \eta^n, \nabla \overline{\phi}^\ell) + B_1(\mathbf{v}^n, \eta^n, \overline{\phi}^\ell) + B_1(\mathbf{v}^n, m_\alpha, \overline{\phi}^\ell) \\ &+ B_1(\mathbf{u}_\alpha, \eta^n, \overline{\phi}^\ell) - U\left(\eta^n, \frac{\partial \overline{\phi}^\ell}{\partial x_3}\right) = 0,\end{aligned} \tag{77}$$

$$\mathbf{v}^n(\mathbf{x},0) = P_n \mathbf{v}_0, \quad \eta^n(\mathbf{x},0) = \overline{P}_n \eta_0, \tag{78}$$

where P_n and $\overline{P}_n$ are orthogonal projections on $\mathbf{W}_n$ and M_n, respectively. We note that the system (76) and (77) is a system of ordinary differential equations for the coefficients $c_{n,j}$ and $d_{n,\ell}$ with the initial conditions $c_{n,j}(0) = (\mathbf{v}_0, \overline{\mathbf{w}}^j)$ and $d_{n,\ell}(0) = (\eta_0, \overline{\phi}^\ell)$. The initial value problem for $c_{n,j}$ and $d_{n,\ell}$ has a maximal solution on the interval $[0, t_n]$ for some $t_n \leq T$. Moreover, we note that we can choose $t_n = T$ as a consequence of the fact that the properties:

$$\{(\mathbf{v}^n, \eta^n)\} \text{ is bounded in } L^2(0,T,\mathbf{V}) \cap L^\infty(0,T,\mathbf{H}) \times L^2(0,T,B) \cap L^\infty(0,T,X); \tag{79}$$

$$\left\{\left(\frac{\partial \mathbf{v}^n}{\partial t}, \frac{\partial \eta^n}{\partial t}\right)\right\} \text{ is bounded in } L^1(0,T,\mathbf{V}') \times L^1(0,T,B'). \tag{80}$$

are satisfied. Indeed, we detail the proofs of (79) and (80).

Proof of (79). Multiplying (76) by $c_{n,j}$ and (77) by $d_{n,\ell}$ and summing on $j \in \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, n\}$, respectively, we obtain:

$$\begin{aligned}\left(\frac{\partial \mathbf{v}^n}{\partial t}, \mathbf{v}^n\right) &+ \mu(\nabla \mathbf{v}^n, \nabla \mathbf{v}^n) + B_0(\mathbf{v}^n, \mathbf{v}^n, \mathbf{v}^n) \\ &+ B_0(\mathbf{v}^n, \mathbf{u}_\alpha, \mathbf{v}^n) + B_0(\mathbf{u}_\alpha, \mathbf{v}^n, \mathbf{v}^n) + (k\eta^n \mathbf{e}_3, \mathbf{v}^n) = (\mathbf{F} - \mathbf{F}_s, \mathbf{v}^n),\end{aligned} \tag{81}$$

$$\left(\frac{\partial \eta^n}{\partial t}, \eta^n\right) + \theta(\nabla \eta^n, \nabla \eta^n) + B_1(\mathbf{v}^n, \eta^n, \eta^n) \\ + B_1(\mathbf{v}^n, m_\alpha, \eta^n) + B_1(\mathbf{u}_\alpha, \eta^n, \eta^n) - U\left(\eta^n, \frac{\partial \eta^n}{\partial x_3}\right) = 0. \tag{82}$$

Using the properties of B_0 and B_1 given in (20)–(23) and the Poincaré inequality, we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} + \mu\|\nabla \mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad \le C_P^2\|\nabla \mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)}\|\nabla \mathbf{u}_\alpha\|_{\mathbf{L}^2(\Omega)} + kC_P\|\eta^n(\cdot,t)\|_{L^2(\Omega)}\|\nabla \mathbf{v}^n(\cdot,t)\|_{\mathbf{L}^2(\Omega)} \\ &\qquad + C_P\|\mathbf{F}(\cdot,t) - \mathbf{F}_s(\cdot)\|_{\mathbf{L}^2(\Omega)}\|\nabla \mathbf{v}^n(\cdot,t)\|_{\mathbf{L}^2(\Omega)}, \end{aligned} \tag{83}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\eta^n(\cdot,t)\|^2_{L^2(\Omega)} + \theta\|\nabla \eta^n(\cdot,t)\|^2_{L^2(\Omega)} \\ &\quad \le C_P^2\|\nabla \mathbf{v}^n(\cdot,t)\|_{\mathbf{L}^2(\Omega)}\|\nabla m_\alpha\|_{L^2(\Omega)}\|\nabla \eta^n(\cdot,t)\|_{L^2(\Omega)} + UC_P\|\nabla \eta^n(\cdot,t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{84}$$

By applying Young's inequality, we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} + \gamma_1(\epsilon)\|\nabla \mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad \le \frac{\epsilon k C_P}{2}\|\eta^n(\cdot,t)\|^2_{L^2(\Omega)} + \frac{\epsilon C_P}{2}\|\mathbf{F}(\cdot,t) - \mathbf{F}_s(\cdot)\|^2_{\mathbf{L}^2(\Omega)} \end{aligned} \tag{85}$$

$$\frac{1}{2}\frac{d}{dt}\|\eta^n(\cdot,t)\|^2_{L^2(\Omega)} + \gamma_2(\epsilon)\|\nabla \eta^n(\cdot,t)\|^2_{L^2(\Omega)} \le \frac{\epsilon C_P^2}{2}\|\mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)}, \tag{86}$$

for any $\epsilon > 0$, where:

$$\gamma_1(\epsilon) = \mu - C_P^2\|\nabla \mathbf{u}_\alpha\|^2_{\mathbf{L}^2(\Omega)} - \frac{(k+1)C_P^2}{2\epsilon}, \quad \gamma_2(\epsilon) = \theta - UC_P - \frac{C_P^2}{2\epsilon}\|\nabla m_\alpha\|^2_{L^2(\Omega)}.$$

Now, noticing that $\Pi_1 > 0$ in (55) and (75) implies that $\mu > C_P^2\|\nabla \mathbf{u}_\alpha\|^2_{\mathbf{L}^2(\Omega)}$, we can select:

$$\epsilon = \bar{\epsilon} = \max\left\{\frac{(k+1)C_P^2}{2(\mu - C_P^2\|\nabla \mathbf{u}_\alpha\|^2_{\mathbf{L}^2(\Omega)})}, \frac{C_P^2\|\nabla m_\alpha\|^2_{\mathbf{L}^2(\Omega)}}{2(\theta - UC_P)}\right\} > 0,$$

such that $\gamma_1(\bar{\epsilon}) > 0$ and $\gamma_2(\bar{\epsilon}) > 0$. Consequently, by adding (85) and (86), we deduce that:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} + \|\eta^n(\cdot,t)\|^2_{L^2(\Omega)}\right) \\ &\quad + \min\left\{\gamma_1(\bar{\epsilon}), \gamma_2(\bar{\epsilon})\right\}\left(\|\nabla \mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \eta^n(\cdot,t)\|^2_{L^2(\Omega)}\right) \\ &\quad \le \frac{\bar{\epsilon} C_P}{2}\max\{k, C_P\}\left(\|\mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} + \|\eta^n(\cdot,t)\|^2_{L^2(\Omega)}\right) \\ &\qquad + \frac{\bar{\epsilon} C_P}{2}\|\mathbf{F}(\cdot,t) - \mathbf{F}_s(\cdot)\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{87}$$

Then, from the Gronwall inequality, the initial condition (78) and recalling that P and $\overline{P}$ are orthogonal projections, i.e., $\|P_n\| \le 1$ and $\|\overline{P}_n\| \le 1$, we have that:

$$\begin{aligned} &\|\mathbf{v}^n(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)} + \|\eta^n(\cdot,t)\|^2_{L^2(\Omega)} \le \exp\left(\frac{\bar{\epsilon} C_P}{2}\max\{k, C_P\}t\right) \\ &\quad \times \left[\|\mathbf{v}^n(\cdot,0)\|^2_{\mathbf{L}^2(\Omega)} + \|\eta^n(\cdot,0)\|^2_{L^2(\Omega)} + \frac{\bar{\epsilon} C_P}{2}\int_0^t \|\mathbf{F}(\cdot,s) - \mathbf{F}_s(\cdot)\|^2_{\mathbf{L}^2(\Omega)} ds\right] \\ &\quad \le G(t), \end{aligned} \tag{88}$$

with $G : [0, T] \to \mathbb{R}^+$ a continuous function independent of n and defined by:

$$G(t) = \exp\left(\frac{\overline{\epsilon}C_P}{2}\max\{k, C_P\}t\right) \times \left[\|\mathbf{v}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\eta_0\|^2_{L^2(\Omega)} + \frac{\overline{\epsilon}C_P}{2}\int_0^t \|\mathbf{F}(\cdot, s) - \mathbf{F}_s\|^2_{\mathbf{L}^2(\Omega)} ds\right].$$

Thus, the estimate (88) implies (79).

Proof of (80). Let us consider the orthogonal projectors P_n and $\overline{P}_n$ defined on $L(\mathbf{V}, \mathbf{V})$ and $L(B, B)$, respectively, with norms less than or equal to one, and also, we define the following operators:

$$(\mathbb{A}(\mathbf{u}), \mathbf{w}) = (\mu\nabla\mathbf{u}, \nabla\mathbf{w}), \qquad (\mathbb{B}_0(\mathbf{u}, \mathbf{v}), \mathbf{w}) = B_0(\mathbf{u}, \mathbf{v}, \mathbf{w}),$$
$$(\mathbb{B}_1(\mathbf{u}, m), \phi) = B_1(\mathbf{u}, m, \phi), \qquad (\mathbb{H}(m), \mathbf{w}) = (km\mathbf{e}_3, \mathbf{w}) + (\mathbf{F} - \mathbf{F}_s, \mathbf{w}),$$
$$(\overline{\mathbb{A}}(m), \phi) = (\theta\nabla m, \nabla\phi) - \theta\int_{\partial\Omega}\frac{\partial m}{\partial\mathbf{n}}\phi dS - U\int_\Omega m\frac{\partial\phi}{\partial x_3}d\mathbf{x} + U\int_{\partial\Omega} mn_3 dS.$$

Then, from (76) and (77) and using the arguments of Lions [18], we obtain:

$$\frac{\partial\mathbf{v}^n}{\partial t} = -P_n^*\Big(\mathbb{A}(\mathbf{v}^n) + \mathbb{B}_0(\mathbf{v}^n, \mathbf{v}^n) + \mathbb{B}_0(\mathbf{v}^n, \mathbf{u}_\alpha) + \mathbb{B}_0(\mathbf{u}_\alpha, \mathbf{v}^n) - \mathbb{H}(\eta^n)\Big),$$
$$\frac{\partial\eta^n}{\partial t} = -\overline{P}_n^*\Big(\overline{\mathbb{A}}(\eta^n) + \mathbb{B}_1(\mathbf{v}^n, \eta^n) + \mathbb{B}_1(\mathbf{v}^n, m_\alpha) + \mathbb{B}_1(\mathbf{u}_\alpha, \eta^n)\Big),$$

where P_n^* and $\overline{P}_n^*$ are the adjoint operators of P_n^* and $\overline{P}_n^*$, respectively. Now, noticing that:

$$\|\mathbb{A}(\mathbf{u})\|_{\mathbf{V}'} \le \mu\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)},$$
$$\|\mathbb{B}_0(\mathbf{u}, \mathbf{v})\|_{\mathbf{V}'} \le C_P^2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)},$$
$$\|\mathbb{B}_1(\mathbf{u}, m)\|_{B'} \le C_P^2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\|\nabla m\|_{L^2(\Omega)},$$
$$\|\mathbb{H}(m)\|_{\mathbf{V}'} \le C_P\Big(k\|m\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{F} - \mathbf{F}_s\|_{\mathbf{L}^2(\Omega)}\Big),$$
$$\|\overline{\mathbb{A}}(m)\|_{B'} \le \theta\|\nabla m\|_{L^2(\Omega)} + U\|m\|_{L^2(\Omega)},$$

we deduce that (80) is satisfied.

From (79) and (80) and applying Corollary 6 of [19], we conclude that the sequence $\{(\mathbf{v}^n, \eta^n)\}$ is relatively compact in $L^q(0, T, \mathbf{H}) \times L^{q'}(0, T, X)$ for all $q, q' \in [1, \infty[$. Then, there is a subsequence of $\{(\mathbf{v}^n, \eta^n)\}$ labeled again by $\{(\mathbf{v}^n, \eta^n)\}$ and $(\mathbf{v}, \eta)$ such that:

$$(\mathbf{v}, \eta) \in L^2(0, T, \mathbf{V}) \cap L^\infty(0, T, \mathbf{H}) \times L^2(0, T, B) \cap L^\infty(0, T, X)$$
$$(\mathbf{v}^n, \eta^n) \to (\mathbf{v}, \eta) \quad \text{weakly in } L^2(0, T; \mathbf{V} \times B)\,,$$
$$(\mathbf{v}^n, \eta^n) \to (\mathbf{v}, \eta) \quad \text{weakly * in } L^\infty(0, T; \mathbf{H} \times X),$$
$$(\mathbf{v}^n, \eta^n) \to (\mathbf{v}, \eta) \quad \text{strongly in } L^2(0, T; \mathbf{H} \times X),$$
$$(\nabla\mathbf{v}^n, \nabla\eta^n) \to (\nabla\mathbf{v}, \eta) \quad \text{weakly in } L^2(0, T; \mathbf{L}^2(\Omega) \times L^2(\Omega)),$$
$$\eta^n \to \eta \quad \text{c.t.p. in } \Omega \times [0, T].$$

Moreover, we can deduce that $(\mathbf{v}, \eta)$ is a weak solution of (67)–(72). Indeed, multiplying (76) and (77) by a function $\psi \in C^1([0, T])$ such that $\psi(T) = 0$ and integrating on $[0, T]$, we have that:

$$
\begin{aligned}
&-\int_0^T \left(\mathbf{v}^n, \overline{\mathbf{w}}^j\right)\psi' dt + \mu \int_0^T (\nabla \mathbf{v}^n, \nabla \overline{\mathbf{w}}^j)\psi dt + \int_0^T B_0(\mathbf{v}^n, \mathbf{v}^n, \overline{\mathbf{w}}^j)\psi dt \\
&\quad + \int_0^T B_0(\mathbf{v}^n, \mathbf{u}_\alpha, \overline{\mathbf{w}}^j)\psi dt + \int_0^T B_0(\mathbf{u}_\alpha, \mathbf{v}^n, \overline{\mathbf{w}}^j)\psi dt + \int_0^T (k\eta^n \mathbf{e}_3, \overline{\mathbf{w}}^j)\psi dt \\
&\qquad = \int_0^T (\mathbf{F} - \mathbf{F}_s, \overline{\mathbf{w}}^j)\psi dt + (P_n \mathbf{v}_0, \overline{\mathbf{w}}^j)\psi(0), \\
&-\int_0^T \left(\eta^n, \overline{\phi}^\ell\right)\psi' dt + \theta \int_0^T (\nabla \eta^n, \nabla \overline{\phi}^\ell)\psi dt + \int_0^T B_1(\mathbf{v}^n, \eta^n, \overline{\phi}^\ell)\psi dt \\
&\qquad + \int_0^T B_1(\mathbf{v}^n, m_\alpha, \overline{\phi}^\ell)\psi dt + \int_0^T B_1(\mathbf{u}_\alpha, \eta^n, \overline{\phi}^\ell)\psi dt - U \int_0^T \left(\eta^n, \frac{\partial \overline{\phi}^\ell}{\partial x_3}\right)\psi dt \\
&\qquad = (P_n \eta_0, \overline{\phi}^\ell)\psi(0).
\end{aligned}
$$

Then, letting $n \to \infty$ and using the standard density arguments, we obtain that $(\mathbf{v}, \eta)$ satisfy the variational relations in Definition 2.

On the other hand, for $n = 3$, we have that $\mathbf{v}(\cdot, t) \in C([0, T]; \mathbf{V}')$. Then:

$$
\mathbf{v}(\cdot, t) \to \mathbf{v}_0 \text{ weakly in } \mathbf{H} \text{ when } t \to 0^+. \tag{89}
$$

Similarly:

$$
\eta(\cdot, t) \to \eta_0 \text{ weakly in } X \text{ when } t \to 0^+. \tag{90}
$$

From (88), we have that $\|\mathbf{v}(\cdot, t)\|^2_{\mathbf{L}^2(\Omega)} + \|\eta(\cdot, t)\|^2_{L^2(\Omega)} \leq G(t)$. Then, $\limsup \|\mathbf{v}(\cdot, t)\|^2_{\mathbf{L}^2(\Omega)} + \|\eta(\cdot, t)\|^2_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{v}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\eta_0\|^2_{L^2(\Omega)}$ when $t \to 0^+$. Moreover, we recall that $\mathbf{H}$ is a uniformly convex space, since the square of the norm is a semicontinuous functional in the weak topology. Then, we conclude that $\|\mathbf{v}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\eta_0\|^2_{L^2(\Omega)} \leq \liminf \|\mathbf{v}(\cdot, t)\|^2_{\mathbf{L}^2(\Omega)} + \|\eta(\cdot, t)\|^2_{L^2(\Omega)}$ when $t \to 0^+$. Thus, $\|\mathbf{v}(\cdot, t)\|^2_{\mathbf{L}^2(\Omega)} + \|\eta(\cdot, t)\|^2_{L^2(\Omega)} \to \|\mathbf{v}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\eta_0\|^2_{L^2(\Omega)}$; see [20]. Thus, by (89) and (90), we obtain that $(\mathbf{v}(\cdot, t), \eta(\cdot, t)) \to (\mathbf{v}_0, \eta_0)$ strongly in $\mathbf{H} \times X$ when $t \to 0^+$. $\square$

Remark 2. *Recently, in* [12]*, the authors considered a generalized bioconvective problem and obtained the existence and uniqueness over two-dimensional domains and the existence over three-dimensional domains. They applied the Galerkin method without using the perturbation of steady states and deduced their existence result requiring uniquely that $UC_P^2 < \theta$ (see Theorem 3.5 in* [12]*).*

5. Conclusions and Future Work

This paper presented new necessary conditions to obtain the existence and uniqueness of stationary weak solutions and the existence of the weak solutions for the evolution problem of the bioconvective system introduced in [1]. The new conditions were formulated in terms of the Poincaré constant, the coefficients, and the external force of the system (see (7)–(9)). The conditions introduced in the paper generalize the assumptions given in [6] for the stationary problem and differ from the hypothesis considered in [1] for the evolution problem. In [1], the authors considered a different smallness condition for the solution of the stationary problem and did not consider any relation between the parameters of the system as those considered in (75). The relation (75) is easy to verify, for instance, in the case of the implementation of numerical methods.

In our future work, we plan to continue the research of bioconvective system in at least three ways. First, we plan to study other analogous new conditions for the case of the generalized bioconvective system studied in [7]. The second idea is to apply the Galerkin methodology to the generalized bioconvective system introduced in [8] (see also [9] for stationary problem results). Moreover, we expect to develop an analysis of the bioconvective system of this paper, under the methodology and the conditions considered in [12].

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