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# On Self-Aggregations of Min-Subgroups

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**Abstract:** Preservation of structures under aggregation functions is an active area of research with applications in many fields. Among such structures, min-subgroups play an important role, for instance, in mathematical morphology, where they can be used to model translation invariance. Aggregation of min-subgroups has only been studied for binary aggregation functions . However, results concerning preservation of the min-subgroup structure under binary aggregations do not generalize to aggregation functions with arbitrary input size since they are not associative. In this article, we prove that arbitrary self-aggregation functions preserve the min-subgroup structure. Moreover, we show that whenever the aggregation function is strictly increasing on its diagonal, a min-subgroup and its self-aggregation have the same level sets.

**Keywords:** aggregation function; $T$-subgroup; strictly monotone function

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## 1. Introduction

Aggregation operators have become an important research topic in the last two decades. The motivation to use such functions comes from the need to summarize different pieces of information into a single object, which is a particularly challenging task when the incoming information is heterogeneous, imprecise, or incomplete. These operators are nowadays a fundamental tool of computer sciences with applications in classification, databases, control, decision making, or image processing among others. Recent monographs on this topic are [1–3].

An aggregation operator is a non-decreasing function $A : [0,1]^n \to [0,1]$ satisfying certain boundary conditions (see Definition 1). This construction allows one to aggregate not only numerical values but also any functions, or structures on a set that have output in the unit interval.

Min-subgroups were introduced by Rosenfeld in ([4]) as a fuzzy set $\mu$ whose domain is a group $G$ such that $\mu(x) = \mu(x^{-1})$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y$ in $G$. Note that from the definition, we immediately obtain $\mu(e) \geq \mu(x)$ for all $x$ in $G$, and hence the normalization condition $\mu(e) = 1$ is often added to the definition of fuzzy subgroup. Das studied min-subgroups thoroughly in [5], introducing a characterization in terms of level sets in which the level sets of $\mu$ correspond to crisp subgroups of $G$. Das also introduced an equivalence relation between fuzzy groups concerning level sets. Anthony and Sherwood (see [6]) extended Rosenfeld's definition using an arbitrary t-norm $T$ instead of the minimum. These groups are called $T$-subgroups. Formato and Gerla constructed a correspondence between $T$-indistinguishability operators on a set (relations that are reflexive, symmetric, and $T$-transitive) and $T$-subgroups related with the permutation group of the set further motivating the study of $T$-subgroups (see [7]).

Min-subgroups can be identified as indistinguishability operators that are invariant by translations (see [8]). This type of indistinguishability operator plays a fundamental role in some applications, notably in mathematical morphology (see [8–11]).

When the set of inputs of an aggregation function share a structure (i.e., they are all indistinguishability operators, min-subgroups, or other fuzzy relations with additional properties), the main problem is the preservation of that structure. In other words, the problem is determining conditions guarantee that the output has the same structure. Preservation of structures under aggregation has been widely studied in recent decades (see [12–21]).

In particular, preservation of the min-subgroup structure under binary aggregations was studied in [12]. However, these results cannot be immediately translated into $n$-ary aggregation functions since these operators are not necessarily associative. In this article, we obtain the first results concerning the preservation of the min-subgroup structure for aggregation of more than two min-subgroups. More concretely, we focus on the preservation of the min-subgroup structure under self-aggregation motivated by the central role they play in the binary case. Note that the minimum $t$-norm is the only $t$-norm that is idempotent, and it is characterized by its level-sets, which makes it very useful in certain contexts ([22]).

The remainder of the article is organized as follows. In Section 2, we introduce the relevant definitions and known facts. Section 3 contains our first new results. We show that the aggregations of an arbitrary number of min-subgroups are also min-subgroups. We also study the behavior of the fuzzy subgroup obtained from conjunctive, averaging, disjunctive, and mixed aggregation functions. Section 4 is devoted to investigate self-aggregations with respect to the equivalence classes of fuzzy subgroups given by its level sets. Our main result states that, for aggregation functions that are strictly increasing on their diagonal, the self-aggregation of a min-subgroup has the same level sets that the original min-subgroup. The article ends with some concluding remarks and future lines of research.

## 2. Preliminary Facts

**Definition 1** ([1])**.** *An operation $A : [0,1]^n \longrightarrow [0,1]$ is called an aggregation function if it satisfies the following axioms:*

$(A1)$ *Monotonicity. If $x_i \leq y_i$ for each $i \in \{1,\dots,n\}$, then $A(x_1,\dots,x_n) \leq A(y_1,\dots,y_n)$.*
$(A2)$ *Boundary conditions. $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$.*

*Moreover, $A$ is called jointly strictly monotone if whenever $x_i < y_i$ for all $i \in \{1,\dots,n\}$, then $A(x_1,\dots,x_n) < A(y_1,\dots,y_n)$.*

Among the most relevant aggregation functions, we find the arithmetic mean, the geometric mean, the harmonic mean, and the quadratic mean (see [1,3]). Aggregation functions are classified into four broad classes: conjunctive, averaging, disjunctive, and mixed functions.

1. A conjunctive aggregation function $A$ is an aggregation function such that $A(r_1,\dots,r_n) \leq \min\{r_1,\dots,r_n\}$ for all $(r_1,\dots,r_n) \in [0,1]^n$. A prototypical example is any t-norm.
2. An averaging aggregation function $A$ is an aggregation function such that $\min\{r_1,\dots, r_n\} \leq A(r_1,\dots,r_n) \leq \max\{r_1,\dots,r_n\}$ for all $(r_1,\dots,r_n) \in [0,1]^n$. Ordered weighted averaging operators belong to this category.
3. A disjunctive aggregation function $A$ is an aggregation function such that $\max\{r_1,\dots, r_n\} \leq A(r_1,\dots,r_n)$ for all $(r_1,\dots,r_n) \in [0,1]^n$. One example is any t-conorm.
4. An aggregation function $A$ is called mixed if $A$ is not conjunctive, averaging, nor disjunctive. Uninorms belong to this type of aggregation functions.

Note that the averaging class is frequently called idempotent class since every averaging aggregation function $A$ satisfies $A(r,\dots,r) = r$ for all $(r,\dots,r) \in [0,1]^n$. Extensive information about aggregation functions can be found in [3].

**Definition 2** ([4])**.** *Let $(G,\cdot)$ be a group. We say that $\mu : G \longrightarrow [0,1]$ is a min-subgroup of $G$ if:*

$(G1)$ *For all $x \in G$, $\mu(x) \geq \mu(x^{-1})$.*

$(G2)$ *For all* $x, y \in G$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

Note that $G1$ is equivalent to $\mu(x) = \mu(x^{-1})$ for all $x \in G$. In the paper, $e$ denotes the neutral element of the group $G$.

**Definition 3** ([23]). *Let $\mu$ be a fuzzy subset of a given universe $X$. For each $t \in [0,1]$, the level set $\mu_t$ and strict level set $\mu^t$ are defined as follows:*

$$\mu_t = \{x \in X \mid \mu(x) \geq t\} \qquad \mu^t = \{x \in X \mid \mu(x) > t\}$$

*The support of $\mu$ is defined by* $\operatorname{supp} \mu = \mu^0$.

Level sets (or $\alpha$-cuts) have been studied extensively in fuzzy subgroups (see for instance [24,25]). P. Das used level sets to characterize the notion of min-subgroup ([5]).

**Proposition 1** ([5]). *Let $G$ be a group and $\mu$ a fuzzy set of $G$; then $\mu$ is a min-subgroup of $G$ if and only if all its non-empty level sets are subgroups of $G$.*

## 3. Self-Aggregation

Given an aggregation function $A$ and $n$ fuzzy subsets $\mu_1, \ldots, \mu_n$ of a group $G$, we consider the fuzzy set $A(\mu_1, \ldots, \mu_n)$ on $G$ defined by

$$A(\mu_1, \ldots, \mu_n)(x) = A(\mu_1(x), \ldots, \mu_n(x))$$

for each $x \in G$. We say that $A(\mu_1, \ldots, \mu_n)$ is the aggregation of $\mu_1, \ldots, \mu_n$ through $A$.

In this section, we will study the aggregation of $A(\mu, \ldots, \mu)$ whenever $\mu$ is a min-subgroup of a group $G$, i.e., the self-aggregation of $\mu$ through $A$.

Anthony and Sherwood (see [6]) introduced $T$-subgroups as an extension of min-subgroups using an arbitrary t-norm $T$ instead of the minimum.

The following theorem underlines the relevance of min-subgroups within $T$-subgroups since the minimum is the only $t$-norm that guarantees preservation of the $T$-subgroup structure for any binary self-aggregation process.

**Theorem 1** ([12]). *Let $G$ be a group with at least four elements and $T$ a t-norm satisfying $T \neq T_D$, where $T_D$ is the drastic t-norm. The following assertions are equivalent:*

1. $T = \min$.
2. *For each $T$-subgroup $\mu$ and each aggregation function $A$, $A(\mu, \mu)$ is a $T$-subgroup.*

Due to this result, given any aggregation function and any min-subgroup $\mu$, $A(\mu, \mu)$ is also a min-subgroup. However, since $A$ is not necessarily associative, the previous result does not guarantee that $A(\mu, \mu, \ldots, \mu)$ is also a min-subgroup. We establish that this is the case for arbitrarily sized aggregations.

**Proposition 2.** *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function and $\mu$ a min-subgroup of a group $G$. Then, $A(\mu, \ldots, \mu)$ is also a min-subgroup of $G$.*

**Proof.** Take $x \in G$; we have that

$$A(\mu, \ldots, \mu)(x) = A(\mu(x), \ldots, \mu(x)) = A(\mu(x^{-1}), \ldots, \mu(x^{-1})) = A(\mu, \ldots, \mu)(x^{-1}).$$

Take $x, y \in G$. Without loss of generality, let us assume that $\mu(x) \leq \mu(y)$. Under this premise, using the fact that $A$ is a non-decreasing function, we have that

$$A(\mu, \ldots, \mu)(x) = \min\Big\{A(\mu, \ldots, \mu)(x), A(\mu, \ldots, \mu)(y)\Big\}. \tag{1}$$

Therefore,

$$A(\mu,\ldots,\mu)(xy) = A(\mu(xy),\ldots,\mu(xy)) \geq A\big(\min\{\mu(x),\mu(y)\},\ldots,\min\{\mu(x),\mu(y)\}\big).$$

Since $\mu(x) \leq \mu(y)$ and the monotonicity of $A$,

$$A\big(\min\{\mu(x),\mu(y)\},\ldots,\min\{\mu(x),\mu(y)\}\big) = A(\mu(x),\ldots,\mu(x)) = A(\mu,\ldots,\mu)(x).$$

Taking into account (1), the proof is completed. □

We proceed to study the comparison between $A(\mu,\ldots,\mu)$ and $\mu$ with respect to the usual order of fuzzy sets, that is, if $A(\mu,\ldots,\mu) \leq \mu$ or $A(\mu,\ldots,\mu) \geq \mu$. The following result shows sufficient conditions on $A$ in order to compare both of them.

**Proposition 3.** *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function and $\mu$ a min-subgroup of a group $G$.*

1. *If $A$ is a conjunctive aggregation function, then $A(\mu,\ldots,\mu) \leq \mu$.*
2. *If $A$ is an averaging aggregation function, then $A(\mu,\ldots,\mu) = \mu$.*
3. *If $A$ is a disjunctive aggregation function, then $A(\mu,\ldots,\mu) \geq \mu$.*

**Proof.** Let us consider $x \in G$.

1. $A(\mu,\ldots,\mu)(x) = A(\mu(x),\ldots,\mu(x)) \leq \min\{\mu(x),\ldots,\mu(x)\} = \mu(x)$.
2. On the one hand, $\mu(x) = \min\{\mu(x),\ldots,\mu(x)\} \leq A(\mu(x),\ldots,\mu(x)) = A(\mu,\ldots,\mu)(x)$. On the other hand, $A(\mu,\ldots,\mu)(x) = A(\mu(x),\ldots,\mu(x)) \leq \max\{\mu(x),\ldots,\mu(x)\} = \mu(x)$.
3. $A(\mu,\ldots,\mu)(x) = A(\mu(x),\ldots,\mu(x)) \leq \max\{\mu(x),\ldots,\mu(x)\} = \mu(x)$.

□

However, if $A$ is mixed, it is possible that $A(\mu,\ldots,\mu)$ is not comparable to $\mu$, and when it is, all the above inequalities can appear, as the following example shows.

**Example 1.** *Consider the group $G = (\mathbb{Z}_6,+)$ and the fuzzy sets $\mu,\eta,\nu,\sigma$ defined in the table below.*

| $G$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\mu$ | 0.9 | 0.5 | 0.5 | 0.9 | 0.5 | 0.5 |
| $\eta$ | 1 | 0.2 | 0.8 | 0.2 | 0.8 | 0.2 |
| $\nu$ | 0.4 | 0.3 | 0.3 | 0.4 | 0.3 | 0.3 |
| $\sigma$ | 1 | 0 | 0 | 0.5 | 0 | 0 |

*Clearly, they are min-subgroups of $G$ because their level sets are crisp subgroups of $G$. Let us consider the following binary aggregation function $A$, where $e = 0.5$ is the neutral element:*

$$A(x,y) = \begin{cases} y & \text{if} \quad x = e, \\ x & \text{if} \quad y = e, \\ 0 & \text{if} \quad x < e, y < e, \\ 1 & \text{if} \quad x > e, y > e, \\ e & \quad\text{otherwise.} \end{cases}$$

*It is easy to check that $A$ is a mixed aggregation function. The self-aggregations of the previous min-subgroups are:*

| $G$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $A(\mu,\mu)$ | 1 | 0.5 | 0.5 | 1 | 0.5 | 0.5 |
| $A(\eta,\eta)$ | 1 | 0 | 1 | 0 | 1 | 0 |
| $A(\nu,\nu)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $A(\sigma,\sigma)$ | 1 | 0 | 0 | 0.5 | 0 | 0 |

*We can conclude that*

$$A(\mu,\mu) \geq \mu,$$
$$A(\nu,\nu) \leq \nu,$$
$$A(\sigma,\sigma) = \sigma,$$

*but* $A(\eta,\eta)$ *is not comparable to* $\eta$.

## 4. Self-Aggregation on the Equivalence Class

There are infinitely many min-subgroups that generate the same chain of subgroups. In order attempt any classification, it is natural to relate two such min-subgroups. P. Das introduced in [5] the following relation between min-subgroups of a group.

**Definition 4.** *Let G be a group and* $\mu,\eta$ *two min-subgroups of G. We say that* $\mu$ *is equivalent to* $\eta$*, written* $\mu \sim \eta$*, if* $\{\mu_t\}_{t\in\mu(G)} = \{\eta_s\}_{s\in\eta(G)}$ *where* $\mu(G)$ *and* $\eta(G)$ *are the ranges of* $\mu$ *and* $\eta$*, respectively. The class of an element* $\mu$ *will be denoted by* $[\mu]$.

There are other significant equivalences on min-subgroups [26–28]. A study on their connections has been recently presented in [29]. Our paper focuses only on the given one by P. Das, which is the most relevant in the literature. Many results can be transferred easily taking into account the implications diagram from [29]. A. Jain characterized the equivalence relation $\sim$ as follows.

**Proposition 4** ([30])**.** *Let G be a group and* $\mu,\eta$ *two min-subgroups of G. The following assertions are equivalent:*

1. $\mu(x) > \mu(y)$ *if and only if* $\eta(x) > \eta(y)$.
2. $\mu(x) \geq \mu(y)$ *if and only if* $\eta(x) \geq \eta(y)$.
3. $\{\mu_t\}_{t\in\mu(G)} = \{\eta_s\}_{s\in\eta(G)}$.
4. $\{\mu^t\}_{t\in\mu(G)} = \{\eta^s\}_{s\in\eta(G)}$.

We introduce the following example showing equivalence classes according to $\sim$ in order to illustrate how self-aggregation acts on the equivalence class.

**Example 2.** *Consider the min-subgroups* $\mu,\eta,\nu,\sigma$ *and the aggregation A presented in Example 1. We have:*

$$[\sigma] \neq [\mu] = [\nu] \neq [\eta] \text{ and } [\sigma] \neq [\eta].$$

*Moreover, self-aggregating each of these min-subgroups through A provides:*

$$[A(\mu,\mu)] = [\mu]$$
$$[A(\eta,\eta)] \neq [\eta]$$
$$[A(\nu,\nu)] \neq [\nu]$$
$$[A(\sigma,\sigma)] = [\sigma]$$

The example shows that self-aggregation does not preserve equivalence classes in general. We dedicate the last part of the section to finding conditions on an aggregation function $A$, which ensures that a min-subgroup and its self-aggregation by $A$ belong to the same equivalence class.

The following result is a straightforward consequence of Proposition 3.

**Proposition 5.** *If A is an averaging aggregation function and* $\mu$ *a min-subgroup of a group G, then* $[A(\mu,\ldots,\mu)] = [\mu]$.

The next proposition shows the relevance of jointly strictly monotone aggregation functions.

**Proposition 6.** *Let $G$ be a group and $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. If $A$ is jointly strictly monotone, then $[A(\mu,\ldots,\mu)] = [\mu]$ for each min-subgroup $\mu$ of $G$.*

**Proof.** We need to prove that $A(\mu,\ldots,\mu)$ and $\mu$ induce the same level sets. We will use the characterization of the Proposition 4. Let us take $x, y \in G$. Firstly, assume that $\mu(x) \geq \mu(y)$; by monotonicity of $A$, we have that $A(\mu,\ldots,\mu)(x) \geq A(\mu,\ldots,\mu)(y)$.

Conversely, assume that $A(\mu,\ldots,\mu)(x) \geq A(\mu,\ldots,\mu)(y)$. We must check that $\mu(x) \geq \mu(y)$. By contradiction, $\mu(x) < \mu(y)$. Since $A$ is jointly strictly monotone, we conclude that $A(\mu(x),\ldots,\mu(x)) < A(\mu(y),\ldots,\mu(y))$; equivalently, $A(\mu,\ldots,\mu)(x) < A(\mu,\ldots,\mu)(y)$, obtaining the desired contradiction. □

We proceed with the main result of the article. Let us recall that an aggregation function $A$ is strictly increasing on its diagonal if for each $x, y \in [0,1]$, satisfying $x < y$; then

$$A(x,\ldots,x) < A(y,\ldots,y).$$

**Theorem 2.** *Let $G$ be a group and $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The following assertions are equivalent:*

1. *$A$ is a strictly increasing function on its diagonal.*
2. *$A(\mu,\ldots,\mu)$ and $\mu$ induce the same level sets.*

**Proof.** 1 $\Longrightarrow$ 2. We will use the characterization of the Proposition 4. Let us take $x, y \in G$. Assume that $\mu(x) \geq \mu(y)$; by monotonicity of $A$, we have that $A(\mu,\ldots,\mu)(x) \geq A(\mu,\ldots,\mu)(y)$.

Conversely, assume that $A(\mu,\ldots,\mu)(x) \geq A(\mu,\ldots,\mu)(y)$. We must check that $\mu(x) \geq \mu(y)$. By contradiction, suppose that $\mu(x) < \mu(y)$. Since $A$ is a strict increasing function on its diagonal, we conclude that $A(\mu(x),\ldots,\mu(x)) < A(\mu(y),\ldots,\mu(y))$, and equivalently, $A(\mu,\ldots,\mu)(x) < A(\mu,\ldots,\mu)(y)$, which is a contradiction.

2 $\Longrightarrow$ 1. We prove that if $A$ is not strictly increasing on its diagonal, then there is a min-subgroup $\mu \in G$ such that $A(\mu,\ldots,\mu)$ and $\mu$ do not have the same level sets. Under this premise, there are $a, b \in [0,1]$ such that

$$a < b \text{ and } A(a,\ldots,a) \geq A(b,\ldots,b).$$

By monotonicity, we have that $A(a,\ldots,a) = A(b,\ldots,b)$. Let us create the fuzzy set $\mu : G \longrightarrow [0,1]$, satisfying $\mu(e) = b$ and $\mu(x) = a$ whenever $x \neq e$. (We remember that $e$ denotes the neutral element of $G$.) Clearly, $\mu$ is a min-subgroup of $G$ according to Proposition 1. Therefore, considering an element $x \neq e$, we conclude that

$$A(\mu(x),\ldots,\mu(x)) = A(a,\ldots,a) = A(b,\ldots,b) = A(\mu(e),\ldots,\mu(e)).$$

Since $\mu(x) < \mu(e)$, they induce different level sets. □

As a direct consequence of the previous theorem, we have obtained the desired characterization.

**Corollary 1.** *Let $\mu$ be a min-subgroup of a group $G$. If $A$ is a strict t-norm or a strict t-conorm, then $A(\mu,\ldots,\mu)$ belongs to the same equivalence class as $\mu$.*

## 5. Concluding Remarks

Let $A$ be a generic aggregation function, $G$ a group, $\mu$ a min-subgroup of $G$, and $[\mu]$ the Das class of $\mu$.

Firstly we have shown that the structure of min-subgroup is preserved by arbitrary self-aggregation functions—i.e., $A(\mu, \ldots, \mu)$ is a min-subgroup—and we have studied when $A(\mu, \ldots, \mu)$ is comparable to $\mu$.

Secondly, we have shown an example of an aggregation function $A$ and a fuzzy subgroup $\mu$ satisfying $[A(\mu, \ldots, \mu)] \neq [\mu]$. Hence, the Das class of a min-subgroup is not necessarily preserved by an arbitrary aggregation function. We have shown that this class is preserved if $A$ is an averaging or a jointly strictly monotonous aggregation function.

Thirdly, our main results states that $A(\mu, \ldots, \mu)$ and $\mu$ induce the same level sets if and only if $A$ is a strictly increasing function on its diagonal. This result implies that if A is a strict $t$-norm or a strict $t$-conorm, $A(\mu, \ldots, \mu)$ belong to the same equivalence class as $\mu$.

Future research could examine under what conditions the Lukasiewicz and product subgroup structures are preserved by arbitrary self-aggregation functions and explore the implications of the migrativity property ([31]) for the preservation of these subgroup structures under self-aggregation functions.

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