axioms

MDPI

Article

New Results on the SSIE with an Operator of the form $F_\Delta \subset \mathcal{E} + F'_x$ Involving the Spaces of Strongly Summable and Convergent Sequences Using the Cesàro Method

Bruno de Malafosse

Université du Havre, 76600 Le Havre, France; bdemalaf@wanadoo.fr

Abstract: Given any sequence $a = (a_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we can use E_a to represent the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/a = (y_n/a_n)_{n\geq 1} \in E$. In this paper, we use the spaces w_∞, w_0 and w of strongly bounded, summable to zero and summable sequences, which are the sets of all sequences y such that $\left(n^{-1}\sum_{k=1}^{n}|y_k|\right)_n$ is bounded and tends to zero, and such that $y - le \in w_0$, for some scalar$l$. These sets were used in the statistical convergence. Then we deal with the solvability of each of the SSIE $F_\Delta \subset \mathcal{E} + F'_x$, where $\mathcal{E}$ is a linear space of sequences, $F = c_0$, c, ℓ_∞, w_0, w or w_∞, and $F' = c_0$, c or ℓ_∞. For instance, the solvability of the SSIE $w_\Delta \subset w_0 + s_x^{(c)}$ relies on determining the set of all sequences $x = (x_n)_{n\geq 1} \in U^+$ that satisfy the following statement. For every sequence y that satisfies the condition $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}|y_k - y_{k-1} - l| = 0$, there are two sequences u and v, with $y = u + v$ such that $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}|u_k| = 0$ and $\lim_{n\to\infty}(v_n/x_n) = L$ for some scalars l and L.

Keywords: BK space; matrix transformations; multiplier of sequence spaces; sequence spaces inclusion equations

MSC: 40C05; 46A45

check for updates

Citation: de Malafosse, B. New Results on the SSIE with an Operator of the form $F_\Delta \subset \mathcal{E} + F'_x$ Involving the Spaces of Strongly Summable and Convergent Sequences Using the Cesàro Method. *Axioms* **2021**, *10*, 157. https://doi.org/10.3390/axioms10030157

Academic Editor: Silvestru Sever Dragomir

Received: 18 June 2021
Accepted: 15 July 2021
Published: 19 July 2021

1. Introduction

We write ω for the set of all complex sequences $y = (y_k)_{k\geq 1}$, ℓ_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively; and $\ell^p = \left\{y \in \omega : \sum_{k=1}^{\infty}|y_k|^p < \infty\right\}$ for $1 \leq p < \infty$. If $y, z \in \omega$, then we write $yz = (y_n z_n)_{n\geq 1}$. Let $U = \{y \in \omega : y_n \neq 0\}$; $U^+ = \{y \in \omega : y_n > 0\}$. We write $z/u = (z_n/u_n)_{n\geq 1}$ for all $z \in \omega$ and all $u \in U$, in particular, $1/u = e/u$, where e is the sequence with $e_n = 1$ for all n. Finally, if $a \in U^+$ and E is any subset of ω, then we put $E_a = (1/a)^{-1} * E = \{y \in \omega : y/a \in E\}$. Recall that the spaces w_∞ *and* w_0 of strongly bounded sequences that are summable to zero sequences using the Cesàro method, are the sets of all y such that $\left(n^{-1}\sum_{k=1}^{n}|y_k|\right)_n$ is bounded and tends to zero. In this way, Hardy and Littlewood [1], defined the set w *of strongly convergent sequences using the Cesàro method* for real numbers as follows. A sequence y is said to be *strongly Cesàro convergent to L, if* $y - Le \in w_0$. These spaces were studied by Maddox [2], Malkowsky and Rakočević [3] and Malkowsky and Başar in [4]. In [5], we gave some properties of well known operators defined by the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$. In this paper, we deal with special *sequence spaces inclusion equations (SSIE)* (cf. [5,6]), which are determined by an inclusion, for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $\left(E_{f(x)}\right)_T$ where f maps U^+ to itself, E is any linear space of sequences and T is a triangle. In [5], we dealt with the class of SSIE of the form $F \subset E_a + F'_x$, where $F \in \{c_0, \ell^p, w_0, w_\infty\}$, and E and F' are any of the sets c_0, c, s_1, ℓ^p, w_0 or w_∞ with $p \geq 1$. Then we stated some results on the solvability of the corresponding SSIE in the particular case of when $a = (r^n)_n$, and we dealt with the case of when $F = F'$. Then we dealt with the SSIE of the form $F \subset E_a + F'_x$ with $e \in F$, and we determined the solutions of these SSIE when $a = (r^n)_{n\geq 1}$, F is either c or s_1, and E and F' are any of the sets c_0, c, s_1, ℓ^p, w_0 or w_∞ with $p \geq 1$. Then we solved each of the SSIE $c \subset D_r * E_\Delta + c_x$, with $E \in \{c_0, c, s_1\}$, and the SSIE $s_1 \subset D_r * (s_1)_\Delta + s_x$. We also studied the SSIE $c \subset D_r * E_{C_1} + s_x^{(c)}$ with $E \in \{c, s_1\}$ and $s_1 \subset D_r * (s_1)_{C_1} + s_x$, where C_1 is the Cesàro operator defined by $(C_1)_n y = n^{-1}\sum_{k=1}^{n} y_k$ for all y, and we dealt with the solvability of the SSE associated with the previous SSIE and defined by $D_r * E_{C_1} + s_x^{(c)} = c$ with $E \in \{c_0, c, s_1\}$ and $D_r * E_{C_1} + s_x = s_1$ with $E \in \{c, s_1\}$. In [6], we dealt with the solvability of the SSIE of the form $\ell_\infty \subset \mathcal{E} + F'_x$ where $\mathcal{E}$ is a given linear space of sequences and F' is either c_0 or ℓ_∞. Then, for given linear space $\mathcal{E}$ of sequences, we solved each of the SSIE $c_0 \subset \mathcal{E} + s_x$ and $c \subset \mathcal{E} + s_x^{(c)}$ and the SSE $\mathcal{E} + s_x^{(c)} = c$.

In this paper, we use the difference sequence spaces $(c_0)_\Delta$, c_Δ and $(\ell_\infty)_\Delta$ introduced by Kizmaz (cf. [7]), and we deal with the solvability of each of the SSIE

$$F_\Delta \subset \mathcal{E} + F'_x,$$

where $F = c_0$, c, ℓ_∞, w_0, w_∞ or w; $F' = c_0$, c or ℓ_∞; and $\mathcal{E}$ is a linear space of sequences.

This paper is organized as follows. In Section 2, we recall some well known results on sequence spaces and matrix transformations. In Section 3, we recall some results on the multipliers of some sets. In Section 4, we recall some results used for the solvability of the SSIE. In Section 5, we deal with the solvability of the SSIE with an operator to solve each of the SSIE of the form $c_\Delta \subset \mathcal{E} + F'_x$, $(c_0)_\Delta \subset \mathcal{E} + F'_x$ and $(\ell_\infty)_\Delta \subset \mathcal{E} + F'_x$ with $F' = c_0$, c or ℓ_∞. In Section 6, we study each of the SSIE $(w_\infty)_\Delta \subset \mathcal{E} + F'_x$, where $F' = c_0$, c or ℓ_∞. Finally, in Section 7, we study the solvability of the SSIE $F_\Delta \subset \mathcal{E} + F'_x$ where $F = w_0$ or w, and $F' = c_0$, c or ℓ_∞.

2. Preliminaries and Notation

An FK space is a *complete metric space*, for which convergence implies *coordinatewise convergence*. A *BK space* is a Banach space of sequences that is an *FK space*. A BK space E *is said to have AK if for every sequence* $y = (y_k)_{k\geq 1} \in E$, *then* $y = \lim_{p\to\infty}\sum_{k=1}^{p} y_k e^{(k)}$, *where* $e^{(k)} = (0, \dots, 1, \dots)$, 1 *being in the* $k-th$ *position.*

For a given infinite matrix $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ we define the operators $A_n = (\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^{\infty} \mathbf{a}_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed to be convergent for all n. Hence, we are led to the study of the operator A defined by $Ay = (A_n y)_{n\geq 1}$ mapping between sequence spaces. When A *maps* E *onto* F, *where* E *and* F *are subsets of* ω, *we write* $A \in (E, F)$ (cf. [2,8–10]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators* L *mapping onto* E, with norm $\|L\| = \sup_{y\neq 0}(\|L(y)\|_E / \|y\|_E)$, satisfies the identity $\mathcal{B}(E) = (E, E)$. We denote by ω, c_0, c and ℓ_∞ the sets of all sequences, and the sets of null, convergent and bounded sequences. For any subset F of ω, we write $F_A = \{y \in \omega \ : \ Ay \in F\}$, and for any subset E of ω we write $AE = \{y \in \omega \ :$ there is $x \in E$ such that $y = Ax\}$. Then, for given sequence $u \in \omega$ we define the diagonal matrix D_u by $[D_u]_{nn} = u_n$ for all n. It is interesting to rewrite the set E_u using a diagonal matrix. Let E be any subset of ω and $u \in U^+$ we have $E_u = D_u * E = \{y = (y_n)_{n\geq 1} \in \omega \ : y/u \in E\}$. We use the sets s_a^0, $s_a^{(c)}$ and s_a defined as follows (cf. [5], p. 160). For a given $a \in U^+$ we put $D_a * c_0 = s_a^0$, $D_a * c = s_a^{(c)}$ and $D_a * \ell_\infty = s_a$. We frequently write c_a instead of $s_a^{(c)}$ to simplify. Each of the spaces $D_a * E$, where $E \in \{c_0, c, \ell_\infty\}$ is a *BK space normed* by $\|y\|_{s_a} = \sup_{n\geq 1}(|y_n|/a_n)$ and s_a^0 *has AK*. If $a = (R^n)_{n\geq 1}$ with $R > 0$, then we write s_R, s_R^0 and $s_R^{(c)}$, for the sets s_a, s_a^0 and $s_a^{(c)}$, respectively. We can also write D_R for $D_{(R^n)_{n\geq 1}}$. When $R = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Recall that $S_1 = (s_1, s_1)$ is a *Banach algebra* and $(c_0, s_1) = (c, \ell_\infty) = (s_1, s_1) = S_1$. We have $A \in S_1$ if and only if $\sup_n(\sum_{k=1}^{\infty}|\mathbf{a}_{nk}|) < \infty$. Recall the Schur's result (cf. [10], Theorem 1.17.8, p. 15) on the class (s_1, c). We have $A \in (s_1, c)$ if and only if $\lim_{n\to\infty}\mathbf{a}_{nk} = l_k$ for some scalar l_k, $k = 1, 2, \dots$, and $\lim_{n\to\infty}\sum_{k=1}^{\infty}|\mathbf{a}_{nk}| = \sum_{k=1}^{\infty}|l_k|$, where the series $\sum_{k=1}^{\infty}|l_k|$ is convergent.

We also use the following known lemmas, where the infinite matrix $\mathcal{T}$ is said to be a triangle, if $\mathcal{T}_{nk} = 0$ for $k > n$ and $\mathcal{T}_{nn} \neq 0$ for all n.

Lemma 1. *Let $\mathcal{T}'$ and $\mathcal{T}''$ be any given triangles, and let $E, F \subset \omega$. Then, for any given operator $\mathcal{T}$ represented by a triangle we have $\mathcal{T} \in (E_{\mathcal{T}'}, F_{\mathcal{T}''})$ if and only if $\mathcal{T}''\mathcal{T}\mathcal{T}'^{-1} \in (E, F)$.*

By taking $T' = D_{1/a}$ and $\mathcal{T}'' = D_b$ for $a, b \in U^+$ we obtain the next well-known result.

Lemma 2. *Let $a, b \in U^+$, and let $E, F \subset \omega$ be any linear spaces. We have $A \in (E_a, F_b)$ if and only if $D_{1/b} A D_a \in (E, F)$.*

3. On the Triangle $C(\lambda)$ and on the Multipliers of Special Sets

In this section, we define the spaces of *strongly bounded and summable sequences by the Cesàro method*. Then we recall some results on the multipliers of sequence spaces involving the previous spaces.

3.1. On the Triangles $C(\lambda)$ and $\Delta(\lambda)$ and the Sets w_0, w and w_∞

For $\lambda \in U$, the infinite matrices $C(\lambda)$ and $\Delta(\lambda)$ are triangles defined as follows. We have $[C(\lambda)]_{nk} = 1/\lambda_n$ for $k \leq n$; this triangle was used, for instance, in [5]; see also the Rhaly matrix studied by [11,12]). Then, the nonzero entries of $\Delta(\lambda)$ are determined by $[\Delta(\lambda)]_{nn} = \lambda_n$ for all n, and $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$ for all $n \geq 2$. It can be shown that the matrix $\Delta(\lambda)$ is the inverse of $C(\lambda)$; that is, $C(\lambda)(\Delta(\lambda)y) = \Delta(\lambda)(C(\lambda)y) = y$ for all $y \in \omega$. If $\lambda = e$ we obtain the well known operator of the first difference represented by $\Delta(e) = \Delta$. We then have $\Delta_n y = y_n - y_{n-1}$ for all $n \geq 1$, with the convention $y_0 = 0$. We have $\Sigma = C(e)$ and then, we may write $C(\lambda) = D_{1/\lambda}\Sigma$. Note that $\Delta = \Sigma^{-1}$. The *Cesàro operator* is defined by $C_1 = C\big((n)_{n\geq 1}\big)$. In the following, we use the inverse of C_1 defined by $C_1^{-1} = \Delta(\lambda)$ where $\lambda = (n)_{n\geq 1}$. We use the set of sequences that are *$a-$strongly bounded and $a-$strongly convergent to zero,* defined for $a \in U^+$ by $W_a = \left\{y \in \omega : \sup_n \left(n^{-1}\sum_{k=1}^n |y_k|/a_k\right) < \infty\right\}$, and $W_a^0 = \left\{y \in \omega : \lim_{n\to\infty}\left(n^{-1}\sum_{k=1}^n |y_k|/a_k\right) = 0\right\}$ (cf. [5], p. 202). For $a = (r^n)_{n\geq 1}$ the sets W_a and W_a^0 are denoted by W_r and W_r^0. For $r = 1$ we obtain the well-known *spaces w_∞* and *w_0 of strongly bounded and strongly null sequences by the Cesàro method* (cf. [13]).

3.2. On the Multipliers of Some Sets

First, we need to recall some well known results. Let y and z be sequences, and let E and F be two subsets of ω. We then write $M(E, F) = \{y \in \omega : \ yz \in F \text{ for all } z \in E\}$; the set $M(E, F)$ is called the *multiplier space of E and F*. We will use the next lemmas.

Lemma 3. *Let E, $\widetilde{E}$, F and $\widetilde{F}$ be arbitrary subsets of ω. Then (i) $M(E, F) \subset M\Big(\widetilde{E}, F\Big)$ for all $\widetilde{E} \subset E$. (ii) $M(E, F) \subset M\Big(E, \widetilde{F}\Big)$ for all $F \subset \widetilde{F}$.*

Lemma 4. *Let $a, b \in U^+$ and let E and F be two subsets of ω. Then we have $D_a * E \subset D_b * F$ if and only if $a/b \in M(E, F)$.*

From Lemma 2 we obtain the next result.

Lemma 5. *(ref. [5], Corollary, 4.1, p. 161) Let $a, b \in U^+$. Then we have: (i) $M\big(s_a^0, \chi_b'\big) = s_{b/a}$ where χ' is any of the symbols s^0, $s^{(c)}$ or s. (ii) $M(\chi_a, s_b) = s_{b/a}$ where χ is any of the symbols $s^{(c)}$ or s. (iii) $M\Big(s_a, s_b^{(c)}\Big) = M\big(s_a, s_b^0\big) = s_{b/a}^0$ and $M\Big(s_a^{(c)}, s_b^{(c)}\Big) = s_{b/a}^{(c)}$.*

In the following, we use the results stated below (cf. [5], Lemma 5.7, p. 233).

Lemma 6. *We have: (i) (a) $M(c, c_0) = M(\ell_\infty, c) = M(\ell_\infty, c_0) = c_0$ and $M(c, c) = c$. (b) $M(E, \ell_\infty) = M(c_0, F) = \ell_\infty$ for $E, F = c_0$, c or ℓ_∞. (ii) (a) $M(w_\infty, \ell_\infty) = M(w_0, F) = s_{(1/n)_{n\geq 1}}$*

for $F = c_0$, c or ℓ_∞. (b) $M(w_\infty, c_0) = M(w_\infty, c) = s^0_{(1/n)_{n\geq 1}}$. (c) $M(E, w_0) = w_0$ for $E = s_1$ or c. (d) $M(E, w_\infty) = w_\infty$ for $E = c_0$, s_1 or c.

To state results on the multipliers involving the set w, we need the next elementary lemmas.

Lemma 7. *We have $w \subset s^0_{(n)_{n\geq 1}}$.*

Proof. Let $y \in w$. Then, by the inequality $n^{-1}|y_n - l| \leq n^{-1}\sum_{k=1}^n |y_k - l|$ for some scalar l and for all n, we deduce $n^{-1}|y_n - l| \to 0$ $(n \to \infty)$, and since $n^{-1}|y_n| \leq n^{-1}|y_n - l| + n^{-1}|l|$ we conclude $y \in s^0_{(n)_{n\geq 1}}$ and $w \subset s^0_{(n)_{n\geq 1}}$. □

Lemma 8. *We have $M(w, \ell_\infty) = M(w, c) = M(w, c_0) = s_{(1/n)_{n\geq 1}}$.*

Proof. By Lemma 7, we have $M\left(s^0_{(n)_{n\geq 1}}, c_0\right) \subset M(w, c_0)$ and by Part (i) of Lemma 5 we have $s_{(1/n)_{n\geq 1}} = M\left(s^0_{(n)_{n\geq 1}}, c_0\right) \subset M(w, c_0)$. Then, using Part (ii) (a) of Lemma 6, we conclude

$$s_{(1/n)_{n\geq 1}} \subset M(w, c_0) \subset M(w, c) \subset M(w, \ell_\infty) \subset M(w_0, \ell_\infty) = s_{(1/n)_{n\geq 1}},$$

This completes the proof. □

Remark 1. *It can easily be shown that $M(w_0, w_\infty) = M(w_\infty, w_\infty) = \ell_\infty$.*

4. On the Sequence Spaces Inclusions

In this section, we are interested in the study of the set of all positive sequences x that satisfy the inclusion $F \subset \mathcal{E} + F'_x$ where $\mathcal{E}$, F and F' are linear spaces of sequences. We may consider this problem as a *perturbation problem*. If we know the set $M(F, F')$, then the solutions of the *elementary inclusion* $F'_x \supset F$ are determined by $1/x \in M(F, F')$. Now, the question is: Let $\mathcal{E}$ be a linear space of sequences. What are the solutions of the *perturbed* inclusion $F'_x + \mathcal{E} \supset F$? An additional question may be the following one: what are the conditions on $\mathcal{E}$ under which the solutions of the elementary and the perturbed inclusions are the same ?

4.1. Some Definitions and Results Used for the Solvability of Some SSIE

In the following, we use the notation $\mathcal{I}(\mathcal{E}, F, F') = \{x \in U^+ : F \subset \mathcal{E} + F'_x\}$, where $\mathcal{E}$, F and F' are linear spaces of sequences and $a \in U^+$. We can state the next elementary results.

Lemma 9. *Let $\mathcal{E}$, $\mathcal{E}_1$, F, F', $\mathcal{F}$ and F'' be linear spaces of sequences. Then we have: (i) If $\mathcal{E}_1 \subset \mathcal{E}$, then $\mathcal{I}(\mathcal{E}_1, F, F') \subset \mathcal{I}(\mathcal{E}, F, F')$. (ii) If $\mathcal{F} \subset F$, then $\mathcal{I}(\mathcal{E}, F, F') \subset \mathcal{I}(\mathcal{E}, \mathcal{F}, F')$. (iii) If $F' \subset F''$, then $\mathcal{I}(\mathcal{E}, F, F') \subset \mathcal{I}(\mathcal{E}, F, F'')$.*

For any set χ of sequences we let $\overline{\chi} = \{x \in U^+ : 1/x \in \chi\}$. Then we write $\Phi = \{c_0, c, \ell_\infty, w_0, w, w_\infty\}$. By $c(1)$ we define the set of all sequences $\alpha \in U^+$ that satisfy the condition $\lim_{n\to\infty} \alpha_n = 1$. Then we consider the condition

$$G \subset G_{1/\alpha} \text{ for all } \alpha \in c(1), \tag{1}$$

for any given linear space G of sequences. Notice that condition (1) is satisfied for all $G \in \Phi$. Then we denote by U_1^+ the set of all sequences α with $0 < \alpha_n \leq 1$ for all n. We consider the condition

$$G \subset G_{1/\alpha} \text{ for all } \alpha \in U_1^+. \tag{2}$$

for any given linear space G of sequences. To show some results on the SSIE, we introduce a linear space of sequences H which contains the spaces E and F' and we will use the fact that if H satisfies the condition in (2) then we have $H_a + H_b = H_{a+b}$ for all $a, b \in U^+$

(cf. [5], Lemma 4.4, p. 162). Notice that c does not satisfy this condition, but each of the sets c_0, ℓ_∞, ℓ^p, w_0 and w_∞ satisfies the condition in (2). Thus we have, for instance, $s_a^0 + s_b^0 = s_{a+b}^0$ and $W_a + W_b = W_{a+b}$.

4.2. Some Properties of the Set $\mathcal{I}(\mathcal{E}, F, F')$

We need the next lemma involving the multiplier of F and F', which is an extension of Lemma 9.

Lemma 10. *Let $\mathcal{E}$, $\mathcal{E}_0$, F, $\mathcal{F}$ and F' be linear spaces of sequences. Then we have: (i) $\overline{M(F,F')} \subset \mathcal{I}(\mathcal{E}, F, F')$. (ii) If $\mathcal{I}(\mathcal{E}_0, F, F') \subset \overline{M(F,F')}$, for any linear space of sequences $\mathcal{E} \subset \mathcal{E}_0$, then $\mathcal{I}(\mathcal{E}, F, F') = \overline{M(F,F')}$. (iii) If $\mathcal{I}(\mathcal{E}, \mathcal{F}, F') \subset \overline{M(F,F')}$, for some linear space of sequences $\mathcal{F} \subset F$, then $\mathcal{I}(\mathcal{E}, F, F') = \overline{M(F,F')}$.*

Proof. (i) Let $x \in \overline{M(F,F')}$. Then, we successively obtain $1/x \in M(F,F')$, $F \subset F'_x$, $F \subset \mathcal{E} + F'_x$ and $x \in \mathcal{I}(\mathcal{E}, F, F')$. This implies $\overline{M(F,F')} \subset \mathcal{I}(\mathcal{E}, F, F')$, and (i) holds. (ii) We have $\mathcal{I}(\mathcal{E}, F, F') \subset \mathcal{I}(\mathcal{E}_0, F, F') \subset \overline{M(F,F')}$ and we conclude by (i) that $\mathcal{I}(\mathcal{E}, F, F') = \overline{M(F,F')}$. Part (iii) follows from the inclusions $\overline{M(F,F')} \subset \mathcal{I}(\mathcal{E}, F, F') \subset \mathcal{I}(\mathcal{E}, \mathcal{F}, F') \subset \overline{M(F,F')}$. □

5. On the Solvability of the SSIE with Operator of the form $F_\Delta \subset \mathcal{E} + F'_x$, Where F, $F' \in \{c_0, c, \ell_\infty\}$

In this section, we determine multipliers involving some difference sequence spaces. Then we state a general result on the solvability of the SSIE with operator $F_\Delta \subset \mathcal{E} + F'_x$ with $e \in F$. Then we apply these results to solve each of the SSIE $c_\Delta \subset \mathcal{E} + F'_x$ and $(c_0)_\Delta \subset \mathcal{E} + F'_x$ and $(\ell_\infty)_\Delta \subset \mathcal{E} + F'_x$ with $F' = c_0$, c or ℓ_∞.

5.1. On the Multipliers of the form $M(X_\Delta, Y)$ Where $X, Y \in \{c_0, c, \ell_\infty\}$

In all that follows, for $a \in U^+$, we use the triangle $D_a\Sigma$, whose the nonzero entries are defined by $(D_a\Sigma)_{nk} = a_n$ for $k \le n$. We have $(D_a\Sigma)_n y = a_n \sum_{k=1}^n y_k$ for all $y \in \omega$ and for all n. This triangle is also called the Rhaly matrix (cf. [11,12]). We obtain some results on the multipliers involving the sets of the difference sequence spaces $(c_0)_\Delta$, c_Δ and $(\ell_\infty)_\Delta$ introduced by Kizmaz (cf. [7]; see also [14]), and stated in the next lemma.

Lemma 11. *(i) $M((c_0)_\Delta, Y) = s_{(1/n)_{n\ge1}}$ where $Y = c_0$, c or ℓ_∞. (ii) $M(c_\Delta, c_0) = s^0_{(1/n)_{n\ge1}}$, $M(c_\Delta, c) = s^{(c)}_{(1/n)_{n\ge1}}$ and $M(c_\Delta, \ell_\infty) = s_{(1/n)_{n\ge1}}$. (iii) $M((\ell_\infty)_\Delta, c_0) = M((\ell_\infty)_\Delta, c) = s^0_{(1/n)_{n\ge1}}$ and $M((\ell_\infty)_\Delta, \ell_\infty) = s_{(1/n)_{n\ge1}}$.*

Proof. Part (i) follows from the proof of [5], Proposition 6.8, p. 289. (ii) We have $a \in M(c_\Delta, c_0)$ if and only if $D_a\Sigma \in (c, c_0)$ and by the characterization of (c, c_0) we have $na_n \to 0$ $(n \to \infty)$ and $a \in s^0_{(1/n)_{n\ge1}}$. In the same way, we have $a \in M(c_\Delta, c)$ if and only if $D_a\Sigma \in (c, c)$, and by the characterization of (c, c) we obtain $a \in s^{(c)}_{(1/n)_{n\ge1}}$. The identity $M(c_\Delta, \ell_\infty) = s_{(1/n)_{n\ge1}}$ can be obtained using similar arguments. (iii) We show $M((\ell_\infty)_\Delta, c) \subset s^0_{(1/n)_{n\ge1}}$. For this, let $a \in M((\ell_\infty)_\Delta, c)$. Then we have $D_a\Sigma \in (\ell_\infty, c)$ which implies $D_a\Sigma \in (c, c)$ and $(na_n)_{n\ge1} \in c$. This implies $\lim_{n\to\infty} a_n = 0$, and by the Schur theorem we obtain $\lim_{n\to\infty}(|a_n| \sum_{k=1}^n 1) = 0$ and $a \in s^0_{(1/n)_{n\ge1}}$. Thus we have shown the inclusion $M((\ell_\infty)_\Delta, c) \subset s^0_{(1/n)_{n\ge1}}$. Now, it can easily be seen that $D_{(1/n)_{n\ge1}}\Sigma \in (\ell_\infty, \ell_\infty)$ which implies $(\ell_\infty)_\Delta \subset s_{(n)_{n\ge1}}$, and using Lemma 5, we obtain $s^0_{(1/n)_{n\ge1}} = M\left(s_{(n)_{n\ge1}}, c_0\right) \subset M((\ell_\infty)_\Delta, c_0)$. Thus we have shown the inclusions $s^0_{(1/n)_{n\ge1}} \subset M((\ell_\infty)_\Delta, c_0) \subset M((\ell_\infty)_\Delta, c) \subset s^0_{(1/n)_{n\ge1}}$ and we conclude $M((\ell_\infty)_\Delta, c_0) = M((\ell_\infty)_\Delta, c) = s^0_{(1/n)_{n\ge1}}$. Using (ii) and the inclusion $(\ell_\infty)_\Delta \subset s_{(n)_{n\ge1}}$, we can obtain

$$s_{(1/n)_{n\geq 1}} = M\left(s_{(n)_{n\geq 1}}, \ell_\infty\right) \subset M((\ell_\infty)_\Delta, \ell_\infty) \subset M(c_\Delta, \ell_\infty) = s_{(1/n)_{n\geq 1}}$$

and the identity $M((\ell_\infty)_\Delta, \ell_\infty) = s_{(1/n)_{n\geq 1}}$ holds. This completes the proof. □

5.2. General Result on the Solvability of the SSIE with Operator $F_\Delta \subset \mathcal{E} + F'_x$ with $e \in F$

In the following, we use the next result.

Theorem 1. *Let F, F' and $\mathcal{E}$ be linear spaces of sequences. Assume $e \in F$, $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ and that F' satisfies the condition in (1). Then, the set $\mathcal{I}(\mathcal{E}, F_\Delta, F')$ of all the positive solutions $x = (x_n)_{n\geq 1}$ of the SSIE $F_\Delta \subset \mathcal{E} + F'_x$ satisfies the inclusion $\mathcal{I}(\mathcal{E}, F_\Delta, F') \subset \overline{F'_{(1/n)_{n\geq 1}}}$. Moreover, if $F'_{(1/n)_{n\geq 1}} \subset M(F_\Delta, F')$ then*

$$\mathcal{I}(\mathcal{E}, F_\Delta, F') = \overline{F'_{(1/n)_{n\geq 1}}}. \tag{3}$$

Proof. Let $x \in \mathcal{I}(\mathcal{E}, F_\Delta, F')$. Then we have $F_\Delta \subset \mathcal{E} + F'_x$, and since $e \in F$, we have $(n)_{n\geq 1} \in F_\Delta$, and there are $\alpha \in \mathcal{E}$ and $\varphi \in F'$ such that $n = \alpha_n + x_n\varphi_n$ for all n. Then we have

$$\frac{n}{x_n}\left(1 - \frac{\alpha_n}{n}\right) = \varphi_n \text{ for all } n,$$

and the condition $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ implies $\lim_{n\to\infty} \alpha_n/n = 0$. Since F' satisfies the condition in (1), we obtain $(n/x_n)_{n\geq 1} \in F'$ and $x \in \overline{F'_{(1/n)_{n\geq 1}}}$. Thus we have shown the inclusion $\mathcal{I}(\mathcal{E}, F_\Delta, F') \subset \overline{F'_{(1/n)_{n\geq 1}}}$. Using Part (i) of Lemma 10, where $\overline{M(F_\Delta, F')} \subset \mathcal{I}(\mathcal{E}, F_\Delta, F')$, we conclude $\overline{F'_{(1/n)_{n\geq 1}}} \subset \mathcal{I}(\mathcal{E}, F_\Delta, F')$. This completes the proof. □

5.3. Solvability of the SSIE $c_\Delta \subset \mathcal{E} + F'_x$ Where $F' = c_0$, c or ℓ_∞

As a direct consequence of Theorem 1 and Lemma 11, we obtain the following results on the sets of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy each of the SSIE with operator $c_\Delta \subset \mathcal{E} + F'_x$ with $F' = c_0$, c or ℓ_∞.

Theorem 2. *Let $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ be a linear space of sequences. We have*

$$\mathcal{I}(\mathcal{E}, c_\Delta, F') = \begin{cases} \overline{s^0_{(1/n)_{n\geq 1}}} & \text{for } F' = c_0, \\ \overline{s^{(c)}_{(1/n)_{n\geq 1}}} & \text{for } F' = c, \\ \overline{s_{(1/n)_{n\geq 1}}} & \text{for } F' = \ell_\infty. \end{cases}$$

Proof. The result follows from Part (ii) of Lemma 11 and Theorem 1, where $F = c$, and $F' = c_0$, c and ℓ_∞ respectively. □

We may state some immediate applications of Theorem 2.

Example 1. *Using Lemma 10 and Theorem 2, it can easily be seen that the sets of the positive solutions $x = (x_n)_{n\geq 1}$ of each of the SSIE with operator, $c_\Delta \subset \ell_\infty + s^{(c)}_x$ and $c_\Delta \subset c + s^{(c)}_x$ and $c_\Delta \subset (c_0)_\Delta + s^{(c)}_x$, are determined by $(n/x_n)_{n\geq 1} \in c$. Then, the solutions of each of the SSIE $c_\Delta \subset (c_0)_\Delta + s^0_x$, $c_\Delta \subset \ell_\infty + s^0_x$ and $c_\Delta \subset c + s^0_x$ are determined by $n/x_n \to 0$ $(n \to \infty)$. In a similar way, the solutions of each of the SSIE $c_\Delta \subset (c_0)_\Delta + s_x$, $c_\Delta \subset \ell_\infty + s_x$ and $c_\Delta \subset c + s_x$ are determined by $(n/x_n)_{n\geq 1} \in \ell_\infty$.*

Example 2. *It can easily be seen that $w_0 \subset s^0_{(n)_{n\geq 1}}$. This implies that the set of all sequences $x = (x_n)_{n\geq 1} \in U^+$ that satisfy the SSIE with operator $c_\Delta \subset w_0 + s^0_x$ is determined by $n/x_n \to 0$ $(n \to \infty)$.*

Example 3. *The set of all positive sequences that satisfy the SSIE* $c_\Delta \subset c_{C_1} + s_x^0$ *is determined by* $\mathcal{I}(c_{C_1}, c_\Delta, c_0) = \overline{s^0_{(1/n)_{n\geq 1}}}$. *Then, the set of all positive sequences that satisfy the SSIE* $c_\Delta \subset c_{C_1} + s_x$ *is determined by* $\mathcal{I}(c_{C_1}, c_\Delta, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$.

5.4. Solvability of the SSIE of the Form $(c_0)_\Delta \subset \mathcal{E} + F'_x$.

In this part, Theorem 1 cannot be applied since $e \notin c_0$. Thus, we need to use some results stated in Section 4.

Theorem 3. *Let* $\mathcal{E} \subset s_\theta$ *for some* $\theta \in s^0_{(n)_{n\geq 1}}$ *be a linear space of sequences, and let* $F' = c_0$, c *or* ℓ_∞. *Then, the set of all the solutions of the SSIE* $(c_0)_\Delta \subset \mathcal{E} + F'_x$ *is determined by* $\mathcal{I}(\mathcal{E}, (c_0)_\Delta, F') = \overline{s_{(1/n)_{n\geq 1}}}$.

Proof. Let $x \in \mathcal{I}(\mathcal{E}, (c_0)_\Delta, F')$ where $F' = c_0$, c or ℓ_∞. Then we have $(c_0)_\Delta \subset \mathcal{E} + F'_x$, and since $F' \subset s_1$ and s_1 satisfies the condition in (2), we obtain $\mathcal{E} + F'_x \subset s_\theta + s_x = s_{\theta + x}$ and $(c_0)_\Delta \subset s_{\theta+x}$. Then we have $D_{1/(\theta+x)}\Sigma \in (c_0, s_1)$, and by the characterization of (c_0, s_1) we have $n/(\theta_n + x_n) = O(1)$ $(n \to \infty)$. Using the inclusion $\mathcal{E} \subset s_\theta$ with $\theta \in s^0_{(n)_{n\geq 1}}$, we have $n/x_n = O(1)$ $(n \to \infty)$, that is, $x \in \overline{s_{(1/n)_{n\geq 1}}}$. We conclude $\mathcal{I}(\mathcal{E}, (c_0)_\Delta, F') \subset \overline{s_{(1/n)_{n\geq 1}}}$. The converse follows from Theorem 1 and Part (i) of Lemma 11, where $M((c_0)_\Delta, s_1) = s_{(1/n)_{n\geq 1}}$. $\square$

Example 4. *By Theorem 3 with* $\theta = e$, *we deduce that the set of all positive sequences* $x = (x_n)_{n\geq 1}$ *that satisfy the SSIE* $(c_0)_\Delta \subset \ell_\infty + F'_x$ *is determined by* $\mathcal{I}(\ell_\infty, (c_0)_\Delta, F') = \overline{s_{(1/n)_{n\geq 1}}}$ *for* $F' = c_0$, c *or* ℓ_∞.

We consider another example, where $bv_p = \ell^p_\Delta$ with $p > 1$ is the set of $p-$bounded variations (cf. [14]).

Example 5. *Let* $p > 1$. *The set* $bv_p = \ell^p_\Delta$ *satisfies the inclusion* $bv_p \subset s_\theta$ *if and only if* $D_{1/\theta}\Sigma \in (\ell^p, s_1)$. *By the characterization of* (ℓ^p, s_1) *(cf. [3], Theorem 1.37, p. 161) we obtain* $\left(n/\theta_n^q\right)_{n\geq 1} \in \ell_\infty$. *We may take* $\theta_n = n^{1/q}$ *with* $q = p/(p-1)$, *which implies* $\theta \in s^0_{(n)_{n\geq 1}}$, *and by Theorem 3 we conclude that the set of all positive sequences* $x = (x_n)_{n\geq 1}$ *that satisfy the SSIE* $(c_0)_\Delta \subset bv_p + F'_x$ *is determined by* $\mathcal{I}(bv_p, (c_0)_\Delta, F') = \overline{s_{(1/n)_{n\geq 1}}}$ *for* $F' = c_0$, c *or* ℓ_∞.

5.5. Solvability of the SSIE of the Form $bv_\infty \subset \mathcal{E} + F'_x$

In this part, we use the notation bv_∞ for the difference sequence space $(\ell_\infty)_\Delta$ (cf. [14]) and we study each of the SSIE $bv_\infty \subset \mathcal{E} + F'_x$, where $F' \in \{c_0, c, \ell_\infty\}$.

Theorem 4. *Let* $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ *be a linear space of sequences. Then, the sets of all positive sequences* $x = (x_n)_{n\geq 1}$ *that satisfy each of the SSIE* $bv_\infty \subset \mathcal{E} + s_x$, $bv_\infty \subset \mathcal{E} + s^0_x$ *and* $bv_\infty \subset \mathcal{E} + s^{(c)}_x$ *are determined by*

$$\mathcal{I}(\mathcal{E}, bv_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}} \text{ and } \mathcal{I}(\mathcal{E}, bv_\infty, c_0) = \mathcal{I}(\mathcal{E}, bv_\infty, c) = \overline{s^0_{(1/n)_{n\geq 1}}}.$$

Proof. First, we show the identities $\mathcal{I}(\mathcal{E}, bv_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$ and $\mathcal{I}(\mathcal{E}, bv_\infty, c_0) = \overline{s^0_{(1/n)_{n\geq 1}}}$. From Theorem 1, where $\mathcal{E} = s^0_{(n)_{n\geq 1}}$, $F = \ell_\infty$ and $F' = \ell_\infty$ and c_0, respectively, we obtain $\mathcal{I}(\mathcal{E}, bv_\infty, \ell_\infty) \subset \overline{s_{(1/n)_{n\geq 1}}}$ and $\mathcal{I}(\mathcal{E}, bv_\infty, c_0) \subset \overline{s^0_{(1/n)_{n\geq 1}}}$. Then, by Part (iii) of Lemma 11, we have $M(bv_\infty, \ell_\infty) = s_{(1/n)_{n\geq 1}}$ and $M(bv_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$ and we conclude by Part (iii) of Lemma 10. Now we show the identity $\mathcal{I}(\mathcal{E}, bv_\infty, c) = \overline{s^0_{(1/n)_{n\geq 1}}}$. For this, we let $x \in$

$\mathcal{I}(\mathcal{E},(\ell_\infty)_\Delta,c)$. Then we have $(\ell_\infty)_\Delta \subset s^0_{(n)_{n\geq1}} + s_x^{(c)}$, and by Theorem 1, where $\mathcal{E} = s^0_{(n)_{n\geq1}}$, $F = \ell_\infty$ and $F' = c$, we have $\mathcal{I}(\mathcal{E},(\ell_\infty)_\Delta,c) \subset s^{(c)}_{(1/n)_{n\geq1}}$ and $(n/x_n)_{n\geq1} \in c$. Now, we show the inclusion $(\ell_\infty)_\Delta \subset s^{(c)}_{(n+x_n)_{n\geq1}}$. We have $s^0_{(n)_{n\geq1}} \subset s^{(c)}_{(n+x_n)_{n\geq1}}$ since $n/(n+x_n) = O(1)$ $(n \to \infty)$.

Then we have

$$\frac{x_n}{n+x_n} = \frac{1}{\frac{n}{x_n}+1} \text{ for all } n,$$

and as we have just seen, we have $\lim_{n\to\infty} n/x_n = l$ for some scalar l and

$$\lim_{n\to\infty} \frac{1}{\dfrac{n}{x_n}+1} = \frac{1}{l+1} > 0.$$

Thus, we have shown the inclusion $s_x^{(c)} \subset s^{(c)}_{(n+x_n)_{n\geq1}}$. These statements imply the inclusions $(\ell_\infty)_\Delta \subset s^0_{(n)_{n\geq1}} + s_x^{(c)} \subset s^{(c)}_{(n+x_n)_{n\geq1}}$ and since $M((\ell_\infty)_\Delta,c) = s^0_{(1/n)_{n\geq1}}$ we obtain $(1/(n+x_n))_{n\geq1} \in s^0_{(1/n)_{n\geq1}}$. Then we have $n/(n+x_n) \to 0$ $(n\to\infty)$ and $(n/x_n)_{n\geq1} \in c_0$, and we have shown the inclusion $\mathcal{I}(\mathcal{E},(\ell_\infty)_\Delta,c) \subset s^0_{(1/n)_{n\geq1}}$. Finally, since $M((\ell_\infty)_\Delta,c) = s^0_{(1/n)_{n\geq1}}$, by Part (i) of Lemma 10, we conclude $\mathcal{I}(\mathcal{E},(\ell_\infty)_\Delta,c) = \overline{s^0_{(1/n)_{n\geq1}}}$. This completes the proof. □

We obtain the following result, where $bs = (\ell_\infty)_\Sigma$ is the set of all bounded series.

Example 6. *The solutions of each of the SSIE $bv_\infty \subset \ell_\infty + s_x^{(c)}$ and $bv_\infty \subset bs + s_x^{(c)}$ are determined by $\mathcal{I}(\ell_\infty,bv_\infty,c) = \mathcal{I}(bs,bv_\infty,c) = \overline{s^0_{(1/n)_{n\geq1}}}$.*

By using similar arguments as in Example 5, we obtain the following result.

Corollary 1. *Let $p \geq 1$. The solutions of the SSIE $bv_\infty \subset bv_p + s_x^{(c)}$ are determined by $\mathcal{I}\big(bv_p,bv_\infty,c\big) = \overline{s^0_{(1/n)_{n\geq1}}}$.*

6. Solvability of the SSIE of the Form $(w_\infty)_\Delta \subset \mathcal{E} + F'_x$

In this part, we deal with each of the SSIE with operators of the form $(w_\infty)_\Delta \subset \mathcal{E} + s_x^0$, $(w_\infty)_\Delta \subset \mathcal{E} + s_x$ and $(w_\infty)_\Delta \subset \mathcal{E} + s_x^{(c)}$. For instance, the solvability of the SSIE $(w_\infty)_\Delta \subset s^0_{(n)_{n\geq1}} + s_x$ consists of determining the set of all positive sequences $x = (x_n)_{n\geq1}$ that satisfy the next statement. For every y such that $n^{-1}\sum_{k=1}^n |y_k - y_{k-1}| = O(1)$ there are two sequences u and v with $y = u+v$ where $\lim_{n\to\infty} u_n/n = 0$ and $v_n/x_n = O(1)$ $(n\to\infty)$.

6.1. Determination of the Sets $M((w_\infty)_\Delta, Y)$ with $Y \in \{c_0, c, \ell_\infty\}$

We state the next Lemma.

Lemma 12. *We have (i) $M((w_\infty)_\Delta, s_1) = s_{(1/n)_{n\geq1}}$ and (ii) $M((w_\infty)_\Delta, c_0) = M((w_\infty)_\Delta, c) = s^0_{(1/n)_{n\geq1}}$.*

Proof. (i) We have $\Delta \in (w_\infty, w_\infty)$ which implies $w_\infty \subset (w_\infty)_\Delta$ and $M((w_\infty)_\Delta, s_1) \subset M(w_\infty, s_1) = s_{(1/n)_{n\geq1}}$. Then we have $w_\infty \subset (\ell_\infty)_{C_1}$ and $(w_\infty)_\Delta \subset \left[(\ell_\infty)_{C_1}\right]_\Delta$ and since $C_1\Delta = D_{(1/n)_{n\geq1}}\Sigma\Delta = D_{(1/n)_{n\geq1}} I = D_{(1/n)_{n\geq1}}$ we obtain $(w_\infty)_\Delta \subset (\ell_\infty)_{D_{(1/n)_{n\geq1}}} = s_{(n)_{n\geq1}}$. Then, by Part (ii) of Lemma 5, we obtain $s_{(1/n)_{n\geq1}} = M\Big(s_{(n)_{n\geq1}}, s_1\Big) \subset M((w_\infty)_\Delta, s_1)$.

Thus we have shown the identity $M((w_\infty)_\Delta, s_1) = s_{(1/n)_{n\geq 1}}$. (ii) First, we show the inclusion $s^0_{(1/n)_{n\geq 1}} \subset M((w_\infty)_\Delta, c_0)$. As we have just seen, we have $(w_\infty)_\Delta \subset s_{(n)_{n\geq 1}}$ and $s^0_{(1/n)_{n\geq 1}} = M\left(s_{(n)_{n\geq 1}}, c_0\right) \subset M((w_\infty)_\Delta, c_0)$. Then, by the inclusion if $w_\infty \subset (w_\infty)_\Delta$ we deduce $M((w_\infty)_\Delta, c_0) \subset M(w_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$, and we conclude that $M((w_\infty)_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$. Now, we show the identity $M((w_\infty)_\Delta, c) = s^0_{(1/n)_{n\geq 1}}$. As above, the inclusion of $w_\infty \subset (w_\infty)_\Delta$ implies $M((w_\infty)_\Delta, c) \subset M(w_\infty, c)$. Then, by Part (ii) (b) of Lemma 6, we have $M(w_\infty, c) = s^0_{(1/n)_{n\geq 1}}$ and we obtain $M((w_\infty)_\Delta, c) \subset s^0_{(1/n)_{n\geq 1}}$. Using the identity $M((w_\infty)_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$ and the inclusion of $M((w_\infty)_\Delta, c_0) \subset M((w_\infty)_\Delta, c)$, we obtain $M((w_\infty)_\Delta, c_0) = M((w_\infty)_\Delta, c) = s^0_{(1/n)_{n\geq 1}}$. This completes the proof. □

6.2. Application to the Solvability of the SSIE of the form $(w_\infty)_\Delta \subset \mathcal{E} + F'_x$.

In the following theorem, we solve each of the SSIE $(w_\infty)_\Delta \subset \mathcal{E} + F'_x$, where $F' \in \{c_0, c, \ell_\infty\}$.

Theorem 5. *Let $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ be a linear space of sequences. Then,*

(i) The set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the SSIE $(w_\infty)_\Delta \subset \mathcal{E} + s_x$ is determined by $\mathcal{I}(\mathcal{E}, (w_\infty)_\Delta, s_1) = \overline{s_{(1/n)_{n\geq 1}}}$.

(ii) The sets of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy each of the SSIE $(w_\infty)_\Delta \subset \mathcal{E} + s^0_x$ and $(w_\infty)_\Delta \subset \mathcal{E} + s^{(c)}_x$ are determined by

$$\mathcal{I}(\mathcal{E}, (w_\infty)_\Delta, c_0) = \mathcal{I}(\mathcal{E}, (w_\infty)_\Delta, c) = \overline{s^0_{(1/n)_{n\geq 1}}}. \tag{4}$$

Proof. (i) By Part (i) of Theorem 4 and since $(\ell_\infty)_\Delta \subset (w_\infty)_\Delta$ we have $\mathcal{I}(\mathcal{E}, (w_\infty)_\Delta, s_1) \subset \mathcal{I}(\mathcal{E}, (\ell_\infty)_\Delta, s_1) = \overline{s_{(1/n)_{n\geq 1}}}$. Then, by Lemma 11 and Lemma 12, we have $M((w_\infty)_\Delta, s_1) = M((\ell_\infty)_\Delta, s_1) = s_{(1/n)_{n\geq 1}}$. We conclude by Part (i) of Lemma 10 that $\mathcal{I}(\mathcal{E}, (w_\infty)_\Delta, s_1) = \overline{s_{(1/n)_{n\geq 1}}}$. (ii) From Part (ii) of Theorem 4 and Lemma 12, we obtain the next two statements: $\overline{s^0_{(1/n)_{n\geq 1}}} = \overline{M((w_\infty)_\Delta, c_0)} \subset \mathcal{I}(\mathcal{E}, (w_\infty)_\Delta, c_0)$ and $\mathcal{I}(\mathcal{E}, (w_\infty)_\Delta, c_0) \subset \mathcal{I}(\mathcal{E}, (w_\infty)_\Delta, c) \subset \mathcal{I}(\mathcal{E}, (\ell_\infty)_\Delta, c) = \overline{s^0_{(1/n)_{n\geq 1}}}$. This implies the identities in (4) and completes the proof. □

Example 7. *Since $w_0 \subset s^0_{(n)_{n\geq 1}}$, the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the SSIE $(w_\infty)_\Delta \subset w_0 + s_x$ is determined by $x_n \geq Kn$ for all n and for some $K > 0$. Similarly, the sets of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the SSIE $(w_\infty)_\Delta \subset w_0 + s^0_x$ is determined by $\lim_{n\to\infty} x_n/n = \infty$.*

Example 8. *By the characterization of (c, c_0), we can see that $D_{(1/n)_{n\geq 1}} C_1^{-1} \in (c, c_0)$, which implies the inclusion $c_{C_1} \subset s^0_{(n)_{n\geq 1}}$. This implies that the solutions of the SSIE $(w_\infty)_\Delta \subset c_{C_1} + s^0_x$ are determined by $\lim_{n\to\infty} x_n/n = \infty$.*

In the following, we solve the SSIE $(w_\infty)_\Delta \subset W^0_r + s^{(c)}_x$, where $W^0_r = D_r w_0$ for $r > 0$. This solvability consists of determining the set of all sequences $x = (x_n)_{n\geq 1} \in U^+$ that satisfy the following statement. For every sequence $y = (y_n)_{n\geq 1}$ for which $n^{-1}\sum_{k=1}^n |y_k - y_{k-1}| \leq K$ for some $K > 0$ and for all n, there are two sequences u and v, with $y = u + v$ such that $n^{-1}\sum_{k=1}^n |u_k|/r^k \to 0$ $(n \to \infty)$ and $\lim_{n\to\infty}(v_n/x_n) = L$ for some scalar L.

Corollary 2. *Let $r > 0$. The set $\mathcal{I}^w_r$ of all the positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the SSIE $(w_\infty)_\Delta \subset W^0_r + s^{(c)}_x$ is determined by* $\mathcal{I}^w_r = \begin{cases} \overline{s^0_{(1/n)_{n\geq 1}}} & \text{if } r \leq 1, \\ U^+ & \text{if } r > 1. \end{cases}$

Proof. The inclusion $W_r^0 \subset s^0_{(n)_{n\geq 1}}$ holds if and only if $(r^n/n)_{n\geq 1} \in M(w_0, c_0)$, and from the identity $M(w_0, c_0) = s_{(1/n)_{n\geq 1}}$ this inclusion holds for all $r \leq 1$. Thus, by Theorem 5 we have $\mathcal{I}_r^w = \overline{s^0_{(1/n)_{n\geq 1}}}$ for all $r \leq 1$. Let $r > 1$. Then we have $r^{-n}\sum_{k=1}^n k = o(1)$ $(n \to \infty)$ and $D_{1/r}\Sigma \in \left(s_{(n)_{n\geq 1}}, c_0\right)$. Since $\left(s_{(n)_{n\geq 1}}, c_0\right) \subset (w_\infty, w_0)$ this implies $D_{1/r}\Sigma \in (w_\infty, w_0)$ and the inclusion $(w_\infty)_\Delta \subset W_r^0$ holds for all $r > 1$. This completes the proof. □

7. On the Solvability of the SSIE of the Form $F_\Delta \subset \mathcal{E} + F'_x$ Involving the Sets w_0, or w

In this section, we determine the multipliers $M(w_\Delta, Y)$ and $M((w_0)_\Delta, Y)$ where $Y = c_0, c$ or ℓ_∞. Then we apply these results to the solvability of the SSIE with operator $F_\Delta \subset \mathcal{E} + F'_x$ where $F = w_0$ or w and $F' = c_0, c$ or ℓ_∞.

7.1. On the Multipliers of the form $M(w_\Delta, Y)$ and $M((w_0)_\Delta, Y)$

In this part, we determine the multipliers $M(w_\Delta, Y)$ and $M((w_0)_\Delta, Y)$ where $Y = c_0$, c, or ℓ_∞.

Lemma 13. *(i) $M((w_0)_\Delta, Y) = s_{(1/n)_{n\geq 1}}$ for $Y = c_0, c$ or ℓ_∞. (ii) (a) $M(w_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$, (b) $M(w_\Delta, c) = s^{(c)}_{(1/n)_{n\geq 1}}$ and (c) $M(w_\Delta, \ell_\infty) = s_{(1/n)_{n\geq 1}}$.*

Proof. Part (i) follows from the proof of [5], Proposition 6.10, p. 291. (ii) (a) We show $M(w_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$. Since $c \subset w$, $c_\Delta \subset w_\Delta$, and by Part (i) we obtain $M(w_\Delta, c_0) \subset M(c_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$. Then, by Part (ii) of Lemma 12, we have $M((w_\infty)_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$ and by Part (iii) of Lemma 5, the inclusion of $w_\Delta \subset (w_\infty)_\Delta$ implies $s^0_{(1/n)_{n\geq 1}} = M((w_\infty)_\Delta, c_0)$ $\subset M(w_\Delta, c_0)$. Thus we have shown $M(w_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$. (ii) (b) We show $M(w_\Delta, c) = s^{(c)}_{(1/n)_{n\geq 1}}$. We have $c_\Delta \subset w_\Delta$, and by Part (ii) of Lemma 11, we obtain $M(w_\Delta, c) \subset M(c_\Delta, c) = s^{(c)}_{(1/n)_{n\geq 1}}$. Then we show the inclusion $s^{(c)}_{(1/n)_{n\geq 1}} \subset M(w_\Delta, c)$. We have $w \subset c_{C_1}$ and $w_\Delta \subset \left(c_{C_1}\right)_\Delta$, and since $C_1\Delta = D_{(1/n)_{n\geq 1}}$ we obtain $\left(c_{C_1}\right)_\Delta = s^{(c)}_{(n)_{n\geq 1}}$ and we conclude $w_\Delta \subset c_{D_{(1/n)_{n\geq 1}}} = s^{(c)}_{(n)_{n\geq 1}}$. Then, by Part (iii) of Lemma 5, we have $s^{(c)}_{(1/n)_{n\geq 1}} = M\left(s^{(c)}_{(n)_{n\geq 1}}, c\right) \subset$ $M(w_\Delta, c)$ and we have shown the identity $M(w_\Delta, c) = s^{(c)}_{(1/n)_{n\geq 1}}$. (ii) (c) From Part (i) and Lemma 12, we obtain

$$s_{(1/n)_{n\geq 1}} = M((w_\infty)_\Delta, \ell_\infty) \subset M(w_\Delta, \ell_\infty) \subset M((w_0)_\Delta, \ell_\infty) = s_{(1/n)_{n\geq 1}}.$$

This shows the identity $M(w_\Delta, \ell_\infty) = s_{(1/n)_{n\geq 1}}$. This completes the proof. □

7.2. Application to the Solvability of the SSIE $F_\Delta \subset \mathcal{E} + F'_x$ where $F = w_0$ or w and $F' = c_0$, c or ℓ_∞

In this part, under some conditions on $\mathcal{E}$ we solve each of the SSIE with operator (1) $(w_0)_\Delta \subset \mathcal{E} + s^0_x$, (2) $(w_0)_\Delta \subset \mathcal{E} + s^{(c)}_x$, (3) $(w_0)_\Delta \subset \mathcal{E} + s_x$ and (1′) $w_\Delta \subset \mathcal{E} + s^0_x$, (2′) $w_\Delta \subset \mathcal{E} + s^{(c)}_x$, (3′) $w_\Delta \subset \mathcal{E} + s_x$.

We can state the following theorem.

Theorem 6. *Let $\mathcal{E}$ be a linear space of sequences. Then we have:*

(i) Assume $\mathcal{E} \subset s_\theta$ for some $\theta \in s^0_{(n)_{n\geq 1}}$. Then $\mathcal{I}(\mathcal{E}, (w_0)_\Delta, F') = \overline{s_{(1/n)_{n\geq 1}}}$ for $F' = c_0$, c or ℓ_∞.

(ii) Assume $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$. Then (a) $\mathcal{I}(\mathcal{E}, w_\Delta, c_0) = \overline{s^0_{(1/n)_{n\geq 1}}}$, (b) $\mathcal{I}(\mathcal{E}, w_\Delta, c) = \overline{s^{(c)}_{(1/n)_{n\geq 1}}}$ and (c) $\mathcal{I}(\mathcal{E}, w_\Delta, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$.

Proof. (i) By Part (i) of Lemma 13 we have $s_{(1/n)_{n\geq 1}} = M((w_0)_\Delta, c_0)$, and by Part (i) of Lemma 10 we have $\overline{s_{(1/n)_{n\geq 1}}} \subset \mathcal{I}(\mathcal{E}, (w_0)_\Delta, c_0)$. Then, by the inclusion $(c_0)_\Delta \subset (w_0)_\Delta$ and using Theorem 3, we have $\mathcal{I}(\mathcal{E}, (w_0)_\Delta, \ell_\infty) \subset \mathcal{I}(\mathcal{E}, (c_0)_\Delta, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$. We conclude

$$\overline{s_{(1/n)_{n\geq 1}}} \subset \mathcal{I}(\mathcal{E}, (w_0)_\Delta, c_0) \subset \mathcal{I}(\mathcal{E}, (w_0)_\Delta, c) \subset \mathcal{I}(\mathcal{E}, (w_0)_\Delta, \ell_\infty) \subset \overline{s_{(1/n)_{n\geq 1}}}$$

and we have shown (i). Part (ii) follows from including $\overline{M(w_\Delta, F')} \subset \mathcal{I}(\mathcal{E}, w_\Delta, F') \subset \mathcal{I}(\mathcal{E}, c_\Delta, F') = \overline{M(c_\Delta, F')}$, and from Part (ii) of Lemma 13 and Part (ii) of Lemma 11, where we have $M(w_\Delta, F') = M(c_\Delta, F')$ for $F' = c_0, c$ or ℓ_∞. □

Example 9. *By Part (ii) of Theorem 6, the solutions of the SSIE* $w_\Delta \subset w_0 + s_x^{(c)}$ *are determined by* $(n/x_n)_{n\geq 1} \in c$*. As we have seen in Example 8, we have the inclusion* $c_{C_1} \subset s^0_{(n)_{n\geq 1}}$*, and by Part (ii) (b) of Theorem 6, the solutions of the SSIE* $w_\Delta \subset c_{C_1} + s_x^{(c)}$ *are determined by* $(n/x_n)_{n\geq 1} \in c$*.*

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Hardy G.H.; Littlewood, J.E. Sur la érie de Fourier d'une fonction à carré sommable. *C. R. Acad. Sci. Paris* **1913**, *156*, 1307–1309.
2. Maddox, I.J. *Infinite Matrices of Operators;* Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1980.
3. Malkowsky, E.; Rakočević, V. *An Introduction into the Theory of Sequence Spaces and Measure of Noncompactnes*; Zbornik Radova; Matematički Institut SANU: Belgrade, Serbia, 2000; Volume 9, pp. 143–243.
4. Başar, F.; Malkowsky, E.; Altay, B. Matrix transformations on the matrix domains of triangles in the spaces of strongly C_1 summable and bounded sequences. *Publ. Math.* **2008**, *73*, 193–213.
5. de Malafosse, B.; Malkowsky, E.; Rakočević, V. *Operators between Sequence Spaces and Applications*; Springer: Singapore, 2021. [CrossRef]
6. de Malafosse, B. Extension of some results on the (SSIE) and the (SSE) of the form $F \subset \mathcal{E} + F'_x$ and $\mathcal{E} + F_x = F$. *Fasc. Math.* **2017**, *59*, 107–123.
7. Kizmaz, H. On Certain Sequence Spaces. *Canad. Math. Bull.* **1981**, *24*, 169–176. [CrossRef]
8. Başar, F. *Summability Theory and Its Applications*; e-books; Bentham Science Publishers: Monographs, Istanbul, 2012.
9. Malkowsky, E. Recent results in the theory of matrix transformations in sequence spaces. *Mat. Vesnik* **1997**, *49*, 187–196.
10. Wilansky, A. *Summability through Functional Analysis*; North-Holland Mathematics Studies: Amsterdam, The Netherlands, 1984; Volume 85.
11. Rhaly, J.R.H.C. Terrace Matrices. *Bull. London Math. Soc.* **1989**, *21*, 399–406. [CrossRef]
12. Yildirim, M. On the spectrum and fine spectrum of the compact Rhaly operators. *Indian J. Appl. Math.* **2003**, *34*, 1443–1452.
13. Malkowsky, E. The continuous duals of the spaces $c_0(\Lambda)$ and $c(\Lambda)$ for exponentially bounded sequences Λ. *Acta Sci. Math. (Szeged)* **1995**, *61*, 241–250.
14. Başar, F.; Altay, B. On the space of sequences of p-bounded variation and related matrix mappings. *Ukr. Math. J.* **2003**, *55*, 135–147. [CrossRef]