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Stable Convergence Theorems for Products of Uniformly Continuous Mappings in Metric Spaces

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Abstract: We study the behavior of inexact products of uniformly continuous self-mappings of a complete metric space that is uniformly continuous and bounded on bounded sets. It is shown that previously established convergence theorems for products of non-expansive mappings continue to hold even under the presence of computational errors.

Keywords: complete metric space; inexact product; infinite product; uniformly continuous mapping

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1. Introduction

The study of fixed points and iterations of nonlinear mappings is a central topic in nonlinear functional analysis. See, for example, [1–23] and the references cited therein. This activity stems from Banach's classical theorem [24] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a non-expansive mapping to one of its fixed points. In particular, the convergence of infinite products of such mappings is important because of their many applications to the study of feasibility and optimization problems, which find important applications in engineering and medical sciences [19–22,25–30]. The book [14] contains several results that show the convergence of inexact orbits of a nonlinear self-mapping of a compete metric space to one of its fixed points. In the present paper, we establish a variant of these results for inexact products of uniformly continuous self-mappings of a complete metric space that is uniformly continuous and bounded on bounded sets. These mappings have a common invariant bounded set that attracts all the infinite products. It is shown that previously established convergence theorems for products of non-expansive mappings in [15] continue to hold even under the presence of computational errors. Our results also generalize the results of [23] obtained in the case when the common invariant set is a singleton and the results of [31] obtained for inexact powers of a single mapping when the invariant set is a singleton.

2. Main Results

Let (Z, ρ) be a complete metric space. For each $x \in Z$ and each $r > 0$ set

$$B(x, r) = \{z \in Z : \ \rho(x, z) \le r\}.$$

For each $x \in Z$ and each nonempty set $D \subset Z$, put

$$\rho(x, D) = \inf\{\rho(x, y) : \ y \in D\}.$$

Fix

$$\theta \in X.$$

Suppose that

$$F \subset Z$$

is a nonempty closed bounded set and that mappings $S_i : Z \to Z$, $i = 1, 2, \dots$ satisfy the following assumptions.

Assumption 1.

$$S_i(F) \subset F \text{ for all natural numbers } i.$$

Assumption 2. *For each nonempty bounded set $K \subset Z$ and each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\rho(S_i(x_1), S_i(x_2)) \leq \epsilon$$

for all natural numbers i and all pairs of points $x_1, x_2 \in K$ satisfying $\rho(x_1, x_2) \leq \delta$.

Assumption 3. *For each nonempty bounded set $K \subset Z$, there exists $M > 0$ such that*

$$S_i(K) \subset B(\theta, M)$$

for all natural numbers i.

Suppose that $\mathcal{R}$ is a collection of mappings $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$ such that the following assumptions hold.

Assumption 4. *For each $r \in \mathcal{R}$ and each integer $q \geq 1$ the mapping $t \to r(t + q)$, $t = 1, 2, \dots$ also belongs to $\mathcal{R}$.*

Assumption 5. *For each nonempty bounded set $K \subset Z$ and each $\epsilon > 0$, there exists a natural number $n(K, \epsilon)$ such that for each $x \in K$, each $r \in \mathcal{R}$ and each integer $n \geq n(K, \epsilon)$,*

$$\rho(\prod_{i=1}^{n} S_{r(i)}(x), F) \leq \epsilon.$$

In this paper, we prove the following results.

Theorem 1. *Let K be a nonempty, bounded subset of Z and let $\epsilon > 0$. Then there exist $\delta = \delta(\epsilon, K) > 0$ and a natural number N such that for each $r \in \mathcal{R}$, each natural number $n \geq N$ and each sequence $\{x_i\}_{i=0}^n \subset Z$, which satisfies*

$$x_0 \in K$$

and

$$\rho(S_{r(i+1)}(x_i), x_{i+1}) \leq \delta, \ i = 0, \dots, n-1$$

the inequality

$$\rho(x_i, F) \leq \epsilon$$

holds for all $i = N, \dots, n$.

The following corollary is easily deduced from Theorem 1.

Corollary 1. *Assume that $r \in \mathcal{R}$, a sequence $\{x_i\}_{i=0}^\infty \subset Z$ has a bounded sub-sequence and that*

$$\lim_{i \to \infty} \rho(S_{r(i+1)}(x_i), x_{i+1}) = 0.$$

Then

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

Theorem 2. *Let $\epsilon > 0$. Then, there exists $\delta > 0$ such that for each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset Z$, which satisfies*

$$\rho(x_0, F) \leq \delta$$

and

$$\rho(S_{r(i+1)}(x_i), x_{i+1}) \leq \delta, \; i = 0, 1, \ldots$$

the inequality

$$\rho(x_i, F) \leq \epsilon$$

holds for all integers $i \geq 0$.

Theorem 3. *Let $M > 0$. Then, there exists $\bar{\delta} > 0$ such that for each $\epsilon > 0$ and each sequence*

$$\{\delta_i\}_{i=0}^{\infty} \subset (0, \bar{\delta}]$$

satisfying

$$\lim_{i \to \infty} \delta_i = 0$$

there exists a natural number n_0 such that the following assertion holds.

For each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset Z$, which satisfies

$$x_0 \in B(\theta, M)$$

and

$$\rho(S_{r(i+1)}(x_i), x_{i+1}) \leq \delta_i, \; i = 0, 1, \ldots,$$

the inequality

$$\rho(x_n, F) \leq \epsilon$$

holds for all integers $n \geq n_0$.

It should be mentioned that prototypes of our results were obtained in [15] when the mappings S_i, $i = 1, 2, \ldots$ are non-expansive, in [23] when the set F is a singleton and in [31] where the set F is a singleton and $S_i = S_1$ for all natural numbers i.

The paper is organized as follows. Section 3 contains auxiliary results. Theorem 1 is proved in Section 4. The proof of Theorem 2 is given in Section 5. Section 6 contains the proof of Theorem 2.

3. Auxiliary Results

Assumption 3 implies the following result.

Lemma 1. *Let K be a nonempty, bounded subset of Z, and let N be a natural number. Then, there exists $M_0 > 0$ such that*

$$K \subset B(\theta, M_0)$$

and that for each integer $n \in \{1, \ldots, N\}$ and each mapping $r : \{1, \ldots, n\} \to \{1, 2, \ldots\}$,

$$(\prod_{i=1}^{n} S_{r(i)})(K) \subset B(\theta, M_0).$$

Lemma 2. *Let K be a nonempty, bounded subset of Z, N be a natural number and let $\epsilon \in (0, 1)$. Then, there exists $\delta \in (0, \epsilon)$ such that for each $x, y \in K$ satisfying $\rho(x, y) \leq \delta$, each integer $n \in \{1, \ldots, N\}$ and each mapping $r : \{1, \ldots, n\} \to \{1, 2, \ldots\}$ the inequality*

$$\rho(\prod_{i=1}^{n} S_{r(i)}(x), \prod_{i=1}^{n} S_{r(i)}(y)) \leq \epsilon$$

holds.

Proof. Let $M_0 > 0$ be as guaranteed by Lemma 1. Then,

$$K \subset B(\theta, M_0) \tag{1}$$

and

$$(\prod_{i=1}^{n} S_{r(i)})(K) \subset B(\theta, M_0) \tag{2}$$

for each integer $n \in \{1, \dots, N\}$ and each mapping $r : \{1, \dots, n\} \to \{1, 2, \dots\}$.

Set

$$\delta_N = \epsilon/4. \tag{3}$$

By induction, using (A2), we define a sequence of positive numbers δ_i, $i = 0, \dots, N-1$ such that for each integer $i \in \{0, \dots, N-1\}$,

$$\delta_i < \delta_{i+1}/2 \tag{4}$$

and that for each $x, y \in B(\theta, M_0)$ satisfying $\rho(x, y) \leq \delta_i$ and each natural number j we have

$$\rho(S_j(x), S_j(y)) \leq \delta_{i+1}. \tag{5}$$

Set

$$\delta = \delta_0. \tag{6}$$

Assume that

$$x, y \in K \subset B(\theta, M_0) \tag{7}$$

(see (1)), $n \in \{1, \dots, N\}$, $r : \{1, \dots, n\} \to \{1, 2, \dots\}$ and that

$$\rho(x, y) \leq \delta = \delta_0 \tag{8}$$

(see (6)). In view of (2), (3) and (7), for each $j \in \{1, \dots, n\}$,

$$\prod_{i=1}^{j} S_{r(i)}(x),\ \prod_{i=1}^{j} S_{r(i)}(y) \in B(\theta, M_0). \tag{9}$$

By (5), (7), (8) and the choice of δ_0,

$$\rho(S_{r(1)}(x), S_{r(1)}(y)) \leq \delta_1. \tag{10}$$

We show by induction that for $j = 1, \dots, n$,

$$\rho(\prod_{i=1}^{j} S_{r(i)}(x), \prod_{i=1}^{j} S_{r(i)}(y)) \leq \delta_j. \tag{11}$$

In view of (10) our assumption holds for $j = 1$. Assume that $j \in \{1, \dots, n\} \setminus \{n\}$ and that (11) holds. It follows from (9), (11) and the choice of δ_j (see (5)) that

$$\rho(\prod_{i=1}^{j+1} S_{r(i)}(x), \prod_{i=1}^{j+1} S_{r(i)}(y)) = \rho(S_{r(j+1)} \prod_{i=1}^{j} S_{r(i)}(x), S_{r(j+1)} \prod_{i=1}^{j} S_{r(i)}(y)) \leq \delta_{j+1}$$

and the assumption made for j also holds for $j + 1$. Therefore, by induction, we showed that (11) holds for $j = 1, \dots, n$ and in particular

$$\rho(\prod_{i=1}^{n} S_{r(i)}(x), \prod_{i=1}^{n} S_{r(i)}(y)) \leq \delta_n \leq \epsilon/4.$$

Lemma 2 is proved. □

Lemma 3. *Let K be a nonempty, bounded subset of Z, N be a natural number and let $\epsilon \in (0,1)$. Then, there exists $\delta \in (0,\epsilon)$ such that for each integer $n \in \{1,\dots,N\}$, each mapping $r : \{1,\dots,n\} \to \{1,2,\dots\}$, each sequence $\{x_i\}_{i=0}^n \subset Z$, which satisfies*

$$x_0 \in K \tag{12}$$

and

$$\rho(S_{r(i+1)}(x_i), x_{i+1}) \leq \delta,\ i = 0,\dots,n-1 \tag{13}$$

and for a sequence $\{y_i\}_{i=0}^n \subset Z$ defined by

$$y_0 = x_0,$$

$$y_{i+1} = S_{r(i+1)}(y_i),\ i = 0,\dots,n-1$$

the inequality $\rho(x_i, y_i) \leq \epsilon$ holds for all $i = 1,\dots,n$.

Proof. Choose $M_0 > 1$ such that

$$K \subset B(\theta, M_0 - 1). \tag{14}.$$

By induction, using (A3), we define a sequence of numbers $M_i > 1$, $i = 1,2,\dots$ such that for each integer $i \geq 0$

$$M_{i+1} > M_i + 2 \tag{15}$$

and that for each natural number j,

$$S_j(B(\theta, M_i + 1)) \subset B(\theta, M_{i+1} - 1). \tag{16}$$

Set

$$\delta_N = \epsilon/4. \tag{17}$$

By induction, using (A2), we define a sequence of positive numbers δ_i, $i = 1,\dots,N$ such that (17) holds and that for each integer $i \in \{1,\dots,N\} \setminus \{N\}$,

$$\delta_i < \delta_{i+1}/4$$

and that for each $x, y \in B(\theta, M_N + 4)$ satisfying $\rho(x,y) \leq 2\delta_i$ and each natural number j we have

$$\rho(S_j(x), S_j(y)) \leq \delta_{i+1}/2. \tag{18}$$

Set

$$\delta = \delta_1. \tag{19}$$

Assume that an integer $n \in \{1,\dots,N\}$, $r : \{1,\dots,n\} \to \{1,2,\dots\}$ and that a sequence $\{x_i\}_{i=0}^n \subset Z$ satisfies (12) and (13). By (12) and (14),

$$\rho(x_0, \theta) \leq M_0 - 1. \tag{20}$$

Set

$$y_0 = x_0, \tag{21}$$

$$y_{i+1} = S_{r(i+1)}(y_i),\ i = 0,\dots,n-1. \tag{22}$$

In view of (15), (16), (20)–(22) and the definition of M_i, $i = 1,2,\dots$,

$$y_i \in B(\theta, M_i - 1),\ i = 0,\dots,n. \tag{23}$$

By induction we show that for all $i = 1,\dots,n$,

$$\rho(x_i, y_i) \leq \delta_i. \tag{24}$$

Equations (13), (19) and (22) imply that

$$\rho(x_1, y_1) \le \rho(x_1, S_{r(1)}(x_0)) \le \delta = \delta_1$$

and our assumption holds for $i = 1$.

Assume that $p \in \{1, \dots, n\} \setminus \{n\}$ and that (24) is true for $i = 1, \dots, p$. In view of (15) and (23),

$$y_p \in B(\theta, M_p - 1) \subset B(\theta, M_N). \tag{25}$$

By (15), (17), (24) with $i = p$, (25) and the construction of δ_i, $i = 1, \dots, N$,

$$x_p \in B(\theta, M_p) \subset B(\theta, M_N), \tag{26}$$

$$\rho(x_p, y_p) \le \delta_p. \tag{27}$$

It follows from (25)–(27) and the choice of δ_p (see (18)) that

$$\rho(S_{r(p+1)}(x_p), S_{r(p+1)}(y_p)) \le \delta_{p+1}/2. \tag{28}$$

By (13), (19), (22) and (28),

$$\rho(x_{p+1}, y_{p+1}) \le \rho(x_{p+1}, S_{r(p+1)}(x_p)) + \rho(S_{r(p+1)}(x_p), y_{p+1})$$
$$\le \delta + \delta_{p+1}/2 \le \delta_{p+1}.$$

Therefore, the assumption made for p also holds for $p + 1$. Thus, by induction we showed that (24) holds for all $i = 0, 1, \dots, n$. Lemma 3 is proved. □

4. Proof of Theorem 1

We may assume without loss of generality that

$$\epsilon \le 4^{-1}$$

and that

$$\cup\{B(z, 4) : z \in F\} \subset K. \tag{29}$$

In view of Assumption 4, there exists a natural number $N \ge 4$ such that

$$\rho((\prod_{i=1}^{n} S_{r(i)})(x), F) \le \epsilon/4 \tag{30}$$

for each $x \in K$, each $r \in \mathcal{R}$ and each integer $n \ge N$.

Lemma 3 implies that there exists $\delta \in (0, \epsilon/4)$ such that the following property holds:

(a) for each integer $n \in \{1, \dots, 2N\}$, each mapping $r \in \mathcal{R}$, each sequence $\{z_i\}_{i=0}^{n} \subset Z$, which satisfies

$$z_0 \in K$$

and

$$\rho(S_{r(i+1)}(z_i), z_{i+1}) \le \delta,\ i = 0, \dots, n-1$$

and for a sequence $\{y_i\}_{i=0}^{n} \subset Z$ defined by

$$y_0 = x_0,$$

$$y_{i+1} = S_{r(i+1)}(y_i),\ i = 0, \dots, n-1$$

the inequality $\rho(z_i, y_i) \le \epsilon/4$ holds for all $i = 1, \dots, n$.

Assume that $n \ge N$ is an integer, $r \in \mathcal{R}$ and that the sequence $\{x_i\}_{i=0}^{n} \subset Z$ satisfies

$$x_0 \in K, \tag{31}$$

$$\rho(S_{r(i+1)}(x_i), x_{i+1}) \leq \delta,\ i = 0, \ldots, n-1. \tag{32}$$

Define

$$y_0 = x_0,$$
$$y_{i+1} = S_{r(i+1)}(y_i),\ i = 0, \ldots, n-1. \tag{33}$$

Property (a) and (31)–(33) imply that

$$\rho(x_i, y_i) \leq \epsilon/4,\ i = 1, \ldots, \min\{n, 2N\}. \tag{34}$$

It follows from (30), (31), (33) and the choice of N that

$$\rho(y_i, F) \leq \epsilon/4,\ i = N, \ldots, n. \tag{35}$$

If $n \leq 2N$, then by (34) and (35), for $i = N, \ldots, n$

$$\rho(x_i, F) \leq \rho(x_i, y_i) + \rho(y_i, F) \leq \epsilon/2.$$

Assume that

$$n > 2N. \tag{36}$$

We show that

$$\rho(x_i, F) \leq \epsilon \text{ for all } i \in \{N, \ldots, n\}.$$

Assume the contrary. Then, there exists an integer

$$k \in (N, n] \tag{37}$$

such that

$$\rho(x_k, F) > \epsilon. \tag{38}$$

Equations (34)–(36) imply that for all $i = N, \ldots, 2N$,

$$\rho(x_i, F) \leq \rho(x_i, y_i) + \rho(y_i, F) \leq \epsilon/2. \tag{39}$$

Therefore, in view of (37)–(39),

$$k > 2N. \tag{40}$$

We may assume without loss of generality that

$$\rho(x_i, F) \leq \epsilon,\ i \in \{2N, \ldots, k-1\}. \tag{41}$$

Define

$$\tilde{r}(i) = r(i + k - N),\ i = 1, 2, \ldots.$$

In view of (40) and (A4),

$$\tilde{r} \in \mathcal{R}. \tag{42}$$

Define $\{\tilde{x}_i\}_{i=0}^{2N} \subset Z$ by

$$\tilde{x}_i = x_{i+k-N},\ i = 0, \ldots, N, \tag{43}$$

$$\tilde{x}_{i+1} = S_{\tilde{r}(i+1)}(\tilde{x}_i),\ i = N, \ldots, 2N-1. \tag{44}$$

Equations (32), (42) and (43) imply that for all integers $i = 0, \ldots, N-1$,

$$\rho(\tilde{x}_{i+1}, S_{\tilde{r}(i+1)}(\tilde{x}_i)) = \rho(x_{i+1+k-N}, S_{r(i+1+k-N)}(x_{i+k-N})) \leq \delta. \tag{45}$$

Set

$$\tilde{y}_0 = \tilde{x}_0,$$
$$\tilde{y}_{i+1} = S_{\tilde{r}(i+1)}(\tilde{y}_i),\ i = 0, \ldots, 2N-1. \tag{46}$$

It follows from (29), (39)–(41) and (43) that

$$\tilde{x}_0 = x_{k-N} \in K. \tag{47}$$

Property (a), (42) and (44)–(47) imply that

$$\rho(\tilde{x}_i, \tilde{y}_i) \leq \epsilon/4, \ i = 1, \ldots, 2N. \tag{48}$$

In view of (30), (46), (47) and the choice of N,

$$\rho(\tilde{y}_i, F) \leq \epsilon/4, \ i = N, \ldots, 2N. \tag{49}$$

By (43), (48) and (49),

$$\rho(x_k, F) = \rho(\tilde{x}_N, F) \leq \rho(\tilde{x}_N, \tilde{y}_N) + \rho(\tilde{y}_N, F) \leq \epsilon/2.$$

This contradicts (38). The contradiction we have reached proves Theorem 1.

5. Proof of Theorem 2

We may assume without loss of generality that $\epsilon < 4^{-1}$. Let

$$K = \cup\{B(z, 4) : \ z \in F\}. \tag{50}$$

By Theorem 1, there exist $\delta_0 \in (0, \epsilon)$ and a natural number N such that the following property holds.

(a) For each natural number $n \geq N$, each $r \in \mathcal{R}$ and each sequence $\{z_i\}_{i=0}^n \subset Z$, which satisfies

$$z_0 \in K$$

and

$$\rho(S_{r(i+1)}(z_i), z_{i+1}) \leq \delta_0, \ i = 0, \ldots, n-1$$

we have

$$\rho(z_i, F) \leq \epsilon, \ i = N, \ldots, n.$$

Lemma 3 implies that there exists $\delta_1 \in (0, \epsilon/4)$ such that the following property holds: (b) for each mapping $r \in \mathcal{R}$, each sequence $\{z_i\}_{i=0}^N \subset Z$, which satisfies

$$z_0 \in K$$

and

$$\rho(S_{r(i+1)}(z_i), z_{i+1}) \leq \delta_1, \ i = 0, \ldots, N-1$$

and for a sequence $\{y_i\}_{i=0}^N \subset Z$ defined by

$$y_0 = x_0,$$

$$y_{i+1} = S_{r(i+1)}(y_i), \ i = 0, \ldots, N-1$$

the inequality $\rho(x_i, y_i) \leq \epsilon/4$ holds for all $i = 1, \ldots, N$.

Lemma 2 implies that there exists $\delta_2 \in (0, \epsilon/4)$ such that the following property holds: (c) for each $x, y \in K$ satisfying $\rho(x, y) \leq 2\delta_2$ and each mapping $r \in \mathcal{R}$,

$$\rho(\prod_{i=1}^n S_{r(i)}(x), \prod_{i=1}^n S_{r(i)}(y)) \leq \epsilon/4, \ n = 1, \ldots, N.$$

Set

$$\delta = 2^{-1} \min\{\delta_0, \delta_1, \delta_2\}. \tag{51}$$

Assume that $r \in \mathcal{R}$ and that a sequence $\{x_i\}_{i=0}^{\infty} \subset Z$ satisfies

$$\rho(x_0, F) \le \delta, \tag{52}$$

$$d(x_{i+1}, S_{r(i+1)}(x_i)) \le \delta,\ i = 0, 1, \dots. \tag{53}$$

Property (a) and Equations (50)–(53) imply that

$$\rho(x_i, F) \le \epsilon \tag{54}$$

for all integers $i \ge N$.

Set

$$y_0 = x_0,$$

$$y_{i+1} = S_{r(i+1)}(y_i),\ i = 0, 1, \dots. \tag{55}$$

Property (b) and Equations (50)–(53) and (55) imply that

$$\rho(x_i, y_i) \le \epsilon/4,\ i = 0, \dots, N. \tag{56}$$

It follows from and Equations (50)–(52) that there exists

$$\xi_0 \in F \tag{57}$$

such that

$$\rho(y_0, \xi_0) < 2\delta \le \delta_2 < \epsilon. \tag{58}$$

By property (c) and Equations (50), (55), (57) and (58), for $n = 1, \dots, N$,

$$\rho(\prod_{i=1}^{n} S_{r(i)}(\xi_0), y_n) \le \epsilon/4. \tag{59}$$

In view of (56) and (59), for all $n = 1, \dots, N$,

$$\rho(\prod_{i=1}^{n} S_{r(i)}(\xi_0), x_n) \le \epsilon/2.$$

Combined with (57) and Assumption 1, this implies that

$$\rho(x_i, F) \le \epsilon/2,\ i = 1, \dots, N.$$

This completes the proof of Theorem 2.

6. Proof of Theorem 3

We may assume without loss of generality that

$$\cup\{B(z, 4) :\ z \in F\} \subset B(\theta, M). \tag{60}$$

By Theorem 1 and Assumption 3, there exist

$$\bar{\delta} \in (0, 1] \text{ and } M_1 > M$$

such that the following property holds:

(a) each $r \in \mathcal{R}$ and each sequence $\{x_i\}_{i=0}^{n} \subset Z$, which satisfies

$$x_0 \in B(\theta, M)$$

and

$$\rho(S_{r(i+1)}(x_i), x_{i+1}) \le \bar{\delta},\ i = 0, 1, \dots$$

we have

$$x_i \in B(\theta, M_1),\ i = 0, 1, \ldots.$$

Assume that $\epsilon > 0$ and a sequence

$$\{\delta_i\}_{i=0}^{\infty} \subset (0, \bar{\delta}] \tag{61}$$

satisfies

$$\lim_{i \to \infty} \delta_i = 0. \tag{62}$$

By Theorem 1, there exist $\delta \in (0, \bar{\delta}]$ and a natural number N such that the following property holds:

(b) for each $r \in \mathcal{R}$ and each sequence $\{z_i\}_{i=0}^{n} \subset Z$, which satisfies

$$z_0 \in B(\theta, M_1)$$

and

$$\rho(S_{r(i+1)}(z_i), z_{i+1}) \leq \delta,\ i = 0, 1, \ldots$$

the inequality

$$\rho(z_i, F) \leq \epsilon$$

holds for all integers $i \geq N$.

In view of (61), there exists an integer $n_1 \geq N$ such that

$$\delta_i < \delta \text{ for all integers } i \geq n_1. \tag{63}$$

Set

$$n_0 = n_1 + N. \tag{64}$$

Assume that $r \in \mathcal{R}$, $\{x_i\}_{i=0}^{\infty} \subset Z$,

$$x_0 \in B(\theta, M),$$

$$\rho(S_{r(i+1)}(x_i), x_{i+1}) \leq \delta_i,\ i = 0, 1, \ldots. \tag{65}$$

Property (a) and Equations (61), (64) and (65) imply that

$$x_i \in B(\theta, M_1) \text{ for all integers } i \geq 1. \tag{66}$$

It follows from property (b) and Equations (63), (65) and (66) that for all integers $i \geq n_1 + N = n_0$,

$$\rho(x_i, F) \leq \epsilon.$$

Theorem 3 is proved.

7. An Application

Let $(Z, \langle \cdot, \cdot \rangle)$ be a Hilbert space equipped with an inner product that induces a complete norm $\| \cdot \|$. For each $x, y \in Z$ set $\rho(x, y) = \|x - y\|$.

Let m be a natural number, $C_i \subset X$, $i = 1, \ldots, m$ be nonempty closed convex sets and let $P_i : X \to C_i$, $i = 1, \ldots, m$ be projections. Set

$$F = \cap_{i=1}^{m} C_i.$$

We suppose that $F \neq \emptyset$. Our goal is to find a point $x \in F$. This is a well-known feasibility problem that finds important applications in engineering and medical sciences [19–22,25–30]. Fix a natural number $\bar{N} \geq m$ and denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \ldots\} \to \{1, \ldots, m\}$ such that for each number j,

$$\{1, \ldots, m\} \subset \{r(j), \ldots, r(j + \bar{N} - 1)\}.$$

Choose $x \in Z$ and $r \in \mathcal{R}$. It is well-known that under certain mild assumptions,

$$\rho(\prod_{i=1}^{n} P_{r(i)}(x), F) \to 0$$

as $n \to \infty$.

It is not difficult to see that this feasibility problem is a particular case of the general problem that is considered in this paper. Evidently, Assumptions 1–4 hold, while Assumption 5 holds if the family of sets possesses the following bounded regularity property:

for each $\epsilon > 0$ and each $M > 0$, there exists $\delta > 0$ such that if $x \in B(0, M)$ satisfies $\rho(x, C_i) \leq \delta$ for all $i = 1, \ldots, m$, then $\rho(x, F) \leq \epsilon$.

See, for example, Theorems 2.14, 2.15 and 3.8 of [21].

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