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Entire Analytic Functions of Unbounded Type on Banach Spaces and Their Lineability

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Abstract: In the paper we establish some conditions under which a given sequence of polynomials on a Banach space X supports entire functions of unbounded type, and construct some counter examples. We show that if X is an infinite dimensional Banach space, then the set of entire functions of unbounded type can be represented as a union of infinite dimensional linear subspaces (without the origin). Moreover, we show that for some cases, the set of entire functions of unbounded type generated by a given sequence of polynomials contains an infinite dimensional algebra (without the origin). Some applications for symmetric analytic functions on Banach spaces are obtained.

Keywords: analytic functions on Banach spaces; functions of unbounded type; symmetric polynomials on Banach spaces

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1. Introduction and Preliminaries

Let X be an infinite dimensional complex Banach space. A continuous function $f\colon X \to \mathbb{C}$ is said to be an *entire analytic function* (or just an *entire function*) if its restriction on any finite dimensional subspace is analytic. If an entire function f satisfies $f(\lambda x) = \lambda^n f(x)$ for every $x \in X$ and $\lambda \in \mathbb{C}$, then f is called an *n-homogeneous polynomial*. It is well known that for an n-homogeneous polynomial f there exists a unique symmetric n-linear mapping $B\colon X^n \to \mathbb{C}$ *associated with* f such that $f(x) = B(x,\dots,x)$. Each zero-homogeneous polynomial is a constant. A finite sum of homogeneous polynomials is a *polynomial*. The space of all entire analytic functions on X is denoted by $H(X)$, the space of all polynomials on X is denoted by $\mathcal{P}(X)$ and the space of all n-homogeneous polynomials on X is denoted by $\mathcal{P}(^nX)$. For every entire function f there exists a sequence of continuous n-homogeneous polynomials $\{f_n\}_{n=1}^{\infty}$ (so-called *Taylor polynomials*) such that

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \tag{1}$$

and the series converges for every $x \in X$. The Taylor series expansion (1) uniformly converges on the open ball rB centered at zero with radius $r = \varrho_0(f)$, where

$$\varrho_0(f) = \frac{1}{\limsup\limits_{n\to\infty} \|f_n\|^{1/n}}.$$

The radius $r = \varrho_0(f)$ is called the *radius of uniform convergence* of f at zero or the *radius of boundedness* of f at zero because the ball rB is the largest open ball at zero such that f is bounded on every closed subset of it. If $\varrho_0(f) = \infty$, then f is bounded on all bounded subsets of X and is called a function of *bounded type*. The algebra of all functions of bounded type on X is denoted by $H_b(X)$. Functions in $H(X) \setminus H_b(X)$ are called entire functions of *unbounded type*. Note that $\varrho_0(f) > 0$ for every $f \in H(X)$.

It is well-known that every infinite dimensional Banach space X admits entire functions of unbounded type. For example, for a given weak*-null sequence $\phi_n \in X^*$, $\|\phi\| = 1$, which always exists (see p. 157 [1]), the function

$$f(x) = \sum_{n=1}^{\infty} \phi_n^n(x) \tag{2}$$

is an entire function of unbounded type on X. We say that a sequence of functions g_n on X (not necessary linear) is *weak*-null* if $g_n(x) \to 0$ as $n \to \infty$ for every $x \in X$.

Entire functions of unbounded type were studied by many authors. In [2] Aron constructed an entire function f on a Banach space X such that for every $r > 0$ there is a point $x_0 \in X$ such that f is unbounded on the ball of radius r, centered at x_0. In [3,4] Ansemil, Aron, and Ponte constructed entire functions f on a Banach space which are bounded on any given finite collection of balls and unbounded on another given finite collection of balls. The set $H(X) \setminus H_b(X)$ is not linear and is not closed under multiplication of functions. However, Lopez–Salazar Codes in [5] show that for every infinite-dimensional Banach space X the set $H(X) \setminus H_b(X)$ contains an infinite-dimensional linear space (without zero) and even an infinite-dimensional algebra (without zero).

Let $\mathbb{P} = \{P_1, P_2, \ldots, P_n, \ldots\}$ be a sequence of polynomials on X. We denote by $\mathcal{P}_{\mathbb{P}}(X)$ the smallest unital algebra containing all polynomials in $\mathbb{P}$. Let $H_{b\mathbb{P}}(X)$ be the closure of $\mathcal{P}_{\mathbb{P}}(X)$ in $H_b(X)$ with respect to the metrizable topology of the uniform convergence on bounded subsets of X, and $H_{\mathbb{P}}(X)$ is the subalgebra of all entire functions f on X such that their Taylor polynomials f_n are in $\mathcal{P}_{\mathbb{P}}(X)$. The algebras $H_{b\mathbb{P}}(X)$ and $\mathcal{P}_{\mathbb{P}}(X)$ were investigated in [6,7]. A typical example of $\mathcal{P}_{\mathbb{P}}(X)$ is the algebra of symmetric polynomials. Let S be a group of isometric operators on a Banach space X. A function f on X is *S-symmetric* if it is invariant with respect to the action of S. Symmetric polynomials and analytic functions on Banach spaces with respect to various groups were studied in [8–18].

Symmetric entire functions of unbounded type on ℓ_1 were studied in [19]. In [20] the authors considered the question: *Let $\mathcal{P}_0(^nX)$ be subspaces of $\mathcal{P}(^nX)$, $n \in \mathbb{N}$. Under which conditions is there a function $f = \sum_{n=0}^{\infty} f_n \in H(X) \setminus H_b(X)$ such that $f_n \in \mathcal{P}_0(^nX)$?* In this paper we show that some natural subspaces $\mathcal{P}_0(^nX)$ do not support entire functions of unbounded type. In particular, we show that there are no symmetric entire functions of unbounded type on $L_\infty[a;b]$.

In Section 2 we propose some conditions under which $\mathcal{P}_{\mathbb{P}}(X)$ supports entire functions of unbounded type and construct some counterexamples. In Section 3 we show that if X is an infinite dimensional Banach space, then $H(x) \setminus H_b(X)$ can be represented as a union of infinite dimensional linear subspaces (without the origin). Furthermore, we show that for some cases $H_{b\mathbb{P}}(X) \setminus H_{\mathbb{P}}(X)$ contains infinite dimensional algebras (without the origin). Some results of this paper were announced in [21].

We refer the reader to the books of Dineen [1] and Mujica [22] for extensive studies of analytic functions on Banach spaces.

2. Conditions of the Unboundedness

Proposition 1. *Let $\{Q_1, Q_2, \ldots, Q_n, \ldots\}$ be a weak*-null sequence of polynomials on X such that $\|Q_n\| = 1$ and $\deg Q_1 \le \deg Q_2 \le \ldots$. Then for every strictly increasing sequence of positive integers $\{k_n\}$, the function*

$$f(x) = \sum_{n=1}^{\infty} Q_n^{k_n}(x)$$

is a function of unbounded type.

Proof. Let $x \in X$. Since $Q_n(x) \to 0$ as $n \to \infty$, there exists a number m such that $|Q_n(x)| \leq \varepsilon$ for some $0 < \varepsilon < 1$, all $n > m$. Hence the series $\sum\limits_{n=1}^{\infty} |Q_n|_n^k$ converges. Thus

$$f(x) = \sum_{n=1}^{\infty} Q_n^{k_n}(x)$$

is well-defined for every $x \in X$. On the other hand, $\|Q_n^{k_n}\| = 1$ and so $\varrho_0(f) = 1$. Therefore, f is an entire function of unbounded type. $\square$

Let us notice the condition "$\{Q_n\}$ is a weak*-null sequence of polynomials on X such that $\|Q_n\| = 1$" is not sufficient to claim that $\sum\limits_{n=1}^{\infty} Q_n(x)$ is a function of unbounded type. For example, if $\deg Q_n = \deg Q_m$ for all $n, m \in \mathbb{N}$, then the series may be divergent.

Throughout the paper we will use the notations $a^{1/n}$ for the principal root of a and $\sqrt[n]{a}$ for the multi-valued root function of a.

Proposition 2. *Let $\{Q_1, Q_2, \ldots, Q_n, \ldots\}$ be a sequence of polynomials on X such that $\|Q_n\| = 1$ and $\deg Q_1 < \deg Q_2 < \ldots$. The function*

$$f(x) = \sum_{n=1}^{\infty} Q_n(x)$$

is of unbounded type if and only if for every $x \in X$

$$|Q_n(x)|^{\frac{1}{\deg Q_n}} \to 0 \quad as \quad n \to \infty. \tag{3}$$

Proof. Let us suppose that $\{Q_n\}_{n=1}^{\infty}$ satisfies (3). Then for every $0 < \varepsilon < 1$ there is a number $n_0 \in \mathbb{N}$ such that for every $n > n_0$, $|Q_n(x)|^{\frac{1}{\deg Q_n}} < \varepsilon$. Thus,

$$\left| \sum_{n=n_0+1}^{\infty} Q_n(x) \right| \leq \sum_{n=n_0+1}^{\infty} |Q_n(x)| \leq \frac{1}{1-\varepsilon} < \infty.$$

Hence, $f(x)$ is well-defined for every $x \in X$. On the other hand, $\varrho_0(f) = 1$ and so f is an entire function of unbounded type.

Conversely, let $f \in H(X) \setminus H_b(X)$. Then for every $x_0 \in X$, $\|x_0\| = 1$ the series

$$f(\lambda x_0) = \sum_{n=1}^{\infty} Q_n(\lambda x_0) = \sum_{n=1}^{\infty} \lambda^{\deg Q_n} Q_n(x_0)$$

converges for every $\lambda \in \mathbb{C}$. This implies

$$|Q_n(x_0)|^{\frac{1}{\deg Q_n}} \to 0 \quad \text{as} \quad n \to \infty$$

and so

$$|\lambda||Q_n(x_0)|^{\frac{1}{\deg Q_n}} = |Q_n(\lambda x_0)|^{\frac{1}{\deg Q_n}} \to 0 \quad \text{as} \quad n \to \infty.$$

Since, every vector $x \in X$ can be represented by $x = \lambda x_0$, $\|x_0\| = 1$, $\lambda \in \mathbb{C}$, the proposition is proved. $\square$

In [20] the following theorem was proved.

Theorem 1. *Let us suppose that there is a dense subset $\Omega \subset X$ and a sequence of polynomials $P_n \in \mathcal{P}(^nX)$, $\limsup_{n\to\infty} \|P_n\|^{1/n} = c$, $0 < c < \infty$ such that for every $z \in \Omega$ there exists $m \in \mathbb{N}$ with the property that for every $y \in X$,*

$$B_{P_n}(\underbrace{z,\dots,z}_{k},\underbrace{y,\dots,y}_{n-k}) = 0$$

for all $k > m$ and $n > k$, where B_{P_n} is the symmetric n-linear mapping associated with P_n. Then

$$g(x) = \sum_{n=1}^{\infty} P_n(x) \in H(X) \setminus H_b(X).$$

It is not difficult to show that if a sequence of polynomials P_n satisfies the conditions of Theorem 1, then it is weak*-null. From Proposition 2 it follows that if P_n satisfies the conditions of Theorem 1, then $|P_n|^{1/n}$ is weak*-null.

For a given sequence of polynomials $\mathbb{P} = \{P_1, P_2, \dots, P_n, \dots\}$ on X, $\|P_n\| = 1$, $\deg P_n = n$, $n \in \mathbb{N}$ we denote by $\mathfrak{P}$ a multi-valued map from X to ℓ_∞ defined by

$$\mathfrak{P}(x) = \Big(P_1(x), \sqrt{P_2(x)}, \dots, \sqrt[n]{P_n(x)}, \dots\Big).$$

Let I_n be polynomials on ℓ_∞ defined by $I_n(z) = z_n^n$, $z = (z_1, \dots, z_n, \dots)$, $n \in \mathbb{N}$. The algebra, generated by polynomials $\{I_n\}$ was considered in [7]. Let us fix some evident properties of $\mathfrak{P}$.

Proposition 3. *For every sequence of polynomials $\mathbb{P} = \{P_n\}_{n=1}^{\infty}$ on X, $\|P_n\| = 1$, $\deg P_n = n$, $n \in \mathbb{N}$ the following statements hold:*

1. *The range $\mathfrak{P}(X)$ of X under $\mathfrak{P}$ is in ℓ_∞.*
2. *$\mathfrak{P}$ maps the ball $r\mathcal{B}_X$ into the ball $r\mathcal{B}_{\ell_\infty}$, $r \geq 0$.*
3. *If z is in the range of $\mathfrak{P}(x)$, then $P_n(x) = I_n(z)$.*

Proof. Since $\|P_n\| = 1$, $|P_n(x)|^{1/n} \leq \|x\|$ for every $x \in X$. So if $z_n \in \sqrt[n]{P_n(x)}$, then $z = (z_1, \dots, z_n, \dots) \in \ell_\infty$ and $\|z\|_{\ell_\infty} \leq \|x\|_X$. In addition, $I_n(z) = z_n^n = P_n(x)$. $\square$

Lemma 1. *Let $f \in H(\ell_\infty)$. Then the restriction f_0 of f to c_0 belongs to $H_b(c_0)$.*

Proof. According to the Aron–Berner result [23], a function $f_0 \in H(c_0)$ can be extended to an analytic function f on ℓ_∞ if and only if $f_0 \in H_b(c_0)$. $\square$

We say that the polynomial algebra $\mathcal{P}_{\mathbb{P}}(X)$ supports analytic functions of unbounded type if there exists a function $f \in H_{\mathbb{P}}(X) \setminus H_{b\mathbb{P}}(X)$.

Theorem 2. *Let $\mathbb{P} = \{P_n\}_{n=1}^{\infty}$ be as in Proposition 3.*

1. *If $\mathfrak{P}$ maps X onto ℓ_∞, then the algebra $\mathcal{P}_{\mathbb{P}}(X)$ does not support analytic functions of unbounded type.*
2. *If $\mathfrak{P}$ maps X to c_0, then*
$$\sum_{n=1}^{\infty} P_n(x) \in H(X) \setminus H_b(X).$$
3. *If there is $x \in X$ such that $\mathfrak{P}(x) \subset \ell_\infty \setminus c_0$, then*
$$\sum_{n=1}^{\infty} P_n(x) \notin H(X).$$

Proof. (1) Let us prove first that $\mathcal{P}_{\mathbb{I}}(\ell_\infty)$ does not support analytic functions of unbounded type on ℓ_∞, where $\mathcal{P}_{\mathbb{I}}(\ell_\infty)$ is $\mathcal{P}_{\mathbb{P}}(\ell_\infty)$ for $\mathbb{P} = \mathbb{I} = \{I_n\}_{n=1}^{\infty}$. Suppose that

$$f(z) = \sum_{n=0}^{\infty} f_n(z) \in H(\ell_\infty) \setminus H_b(\ell_\infty), \quad \text{and} \quad f_n \in \mathcal{P}_{\mathbb{I}}(\ell_\infty).$$

Since each f_n is an algebraic combination of $I_1, I_2, \ldots, I_n$ and $I_k(z) = z_k^k$, we have that $f_n(z)$ depends of finitely many coordinates $z_1, z_2, \ldots, z_n$. Thus, there is $s = (s_1, s_2, \ldots, s_n, 0, 0 \ldots)$ such that $\|f_n\| = |f_n(s)|$. In other words, the norm of f_n in ℓ_∞ is equal to the norm of the restriction of f_n on c_0. Let f_0 be the restriction of f on c_0. Hence we have that

$$\varrho_0(f) = \varrho_0(f_0).$$

So if f is a function of unbounded type, then f_0 is a function of unbounded type. However, it contradicts Lemma 1.

Let f now be an arbitrary function of unbounded type in $H_{\mathbb{P}}(X)$. Since all polynomials f_n belong to $\mathcal{P}_{\mathbb{P}}(X)$, there are polynomials q_n of n complex variables, $n \in \mathbb{N}$ such that

$$f_n(x) = q_n(P_1(x), P_2(x), \ldots, P_n(x)) = q_n(I_1(z), I_2(z), \ldots, I_n(z)) =: Q_n(z)$$

for every $z \in \mathfrak{P}(x)$. Clearly every Q_n is an n-homogeneous continuous polynomial. Since $\mathfrak{P}$ maps bounded sets to bounded sets,

$$g(z) = \sum_{n=0}^{\infty} Q_n(z)$$

must be a function of unbounded type. However, it is impossible because of the first part of the proof.

(2) If $z = (z_1, z_2, \ldots) \in \mathfrak{P}(x)$, then

$$f(x) = \sum_{n=1}^{\infty} P_n(x) = \sum_{n=1}^{\infty} z_n^n = \sum_{n=1}^{\infty} I_n(z) \in H(c_0).$$

Thus f is well defined on X and so belongs to $H(X)$. Since $\|P_n\| = 1$, $\varrho_0(f) = 1$. Hence, $f \in H(X) \setminus H_b(X)$.

(3) If $\mathfrak{P}(x) \not\subset c_0$ for some fixed $x \in X$, then there exists a constant $c > 0$ and a subsequence $n_k \in \mathbb{N}$ such that $\left| \sqrt[n_k]{P_{n_k}(x)} \right| > c$ for all k. Let us consider the following function of one complex variable

$$\gamma(t) = \sum_{n=1}^{\infty} P_n(tx) = \sum_{n=1}^{\infty} t^n P_n(x).$$

The radius of convergence of this series satisfies $\varrho_0(\gamma(t)) \leq 1/c$, so if $t_0 > 1/c$, then the series

$$\sum_{n=1}^{\infty} P_n(tx)$$

diverges. Thus it does not belong to $H(X)$. $\square$

Remark 1. *Formally, we do not assume in Theorem 2 that X is infinite dimensional. However, any finite dimensional space does not admit entire functions of unbounded type. Thus, if $\mathfrak{P}(X) \subset c_0$, then X must be infinite dimensional.*

Example 1. *In [19] (see also [20]) it is shown that if $P_n(x) = n!G_n(x)$, where*

$$G_n(x) = \sum_{k_1 < k_2 < \cdots < k_n} x_{k_1} \cdots x_{k_n}, \quad x = (x_1, x_2, \ldots) \in \ell_1,$$

then

$$f(x)=\sum_{n=1}^{\infty}P_n(x)$$

is an entire function of unbounded type on ℓ_1. It is known [13] that $\|G_n\|=1/n!$. Thus, Theorem 2 implies that $\mathfrak{P}(x)\in c_0$ for every $x\in\ell_1$. The algebra $\mathcal{P}_{\mathbb{P}}(\ell_1)$ coincides with the algebra of all symmetric polynomials on ℓ_1 (see e.g., [12]) and admits another algebraic basis of homogeneous polynomials

$$F_n(x)=\sum_{k=1}^{\infty}x_k^n,\quad n=1,2,\ldots.$$

It is easy to see that $1\|F_n\|=1$ and $F_n(e_1)=1$ for every $n\in\mathbb{N}$, where $e_1=(1,0,0,\ldots)$. Hence, $\left(\sqrt[n]{F_n(e_1)}\right)_{n=1}^{\infty}\not\subset c_0$. However, as we observed, the algebra of symmetric polynomials supports entire functions of unbounded type. Therefore, if $c_0\not\supset\mathfrak{P}(X)\neq\ell_\infty$, then $\sum_{n=1}^{\infty}P_n\notin H(X)$, but $\mathcal{P}_{\mathbb{P}}(X)$ may still support entire functions of unbounded type.

Note that the existence of an isomorphism of $H_{b\mathbb{P}}(X)$ and $H_{b\mathbb{I}}(\ell_\infty)$ does not imply that $\mathcal{P}_{\mathbb{P}}(X)$ does not support analytic functions of unbounded type.

Example 2. *It is known [7] that there is an isomorphism $J\colon H_{b\mathbb{I}}(c_0)\to H_{b\mathbb{I}}(\ell_\infty)$ such that $J\colon I_n\mapsto I_n$ but $\mathcal{P}_{\mathbb{I}}(c_0)$ supports analytic functions of unbounded type. For example*

$$f(x)=\sum_{n=0}^{\infty}I_n(x)=\sum_{n=0}^{\infty}x_n^n\in H(c_0)\setminus H_b(c_0).$$

In other words, the isomorphism J can not be extended to an isomorphism between $H(c_0)$ and $H(\ell_\infty)$.

Let us recall that a function on $L_\infty[0,1]$ is *symmetric* if it is invariant with respect to measuring and measure preserving automorphisms of the interval $[0,1]$. The polynomials

$$R_n(x)=\int_{[0,1]}(x(t))^n\,dt,\quad x(t)\in L_\infty[0,1],\quad n\in\mathbb{N}$$

form an algebraic basis in the space of all symmetric polynomials on $L_\infty[0,1]$ Thus, the algebra of symmetric polynomials $\mathcal{P}_s(L_\infty[0,1])$ is a partial case of $\mathcal{P}_{\mathbb{P}}(X)$ if $X=L_\infty[0,1]$ and $P_n=R_n$. In [20] the authors asked: *Does an entire symmetric analytic function of unbounded type exist on $L_\infty[0,1]$?* Now we have a negative answer to this question.

Corollary 1. *All entire symmetric functions on $L_\infty[0,1]$ are functions of bounded type.*

Proof. Let

$$f(x)=\sum_{n=0}^{\infty}f_n(x)$$

be an entire symmetric function on $L_\infty[0,1]$. Then each Taylor's polynomial f_n must be symmetric and so can be represented as an algebraic combination of polynomials $R_1,\ldots,R_n$. In [14] is proved that the map

$$x\mapsto\left(R_1(x),\sqrt{R_2(x)},\ldots,\sqrt[n]{R_n(x)},\ldots\right)$$

is onto ℓ_∞. Thus, by Theorem 2, the algebra of symmetric polynomials on $L_\infty[0;1]$ does not support entire functions of unbounded type. Hence $f(x)$ is of bounded type. □

3. Lineability of $H_{\mathbb{P}}(X) \setminus H_{b\mathbb{P}}(X)$

Theorem 3. *If $\mathcal{P}_{\mathbb{P}}(X)$ supports analytic functions of unbounded type, then for every $f \in H_{\mathbb{P}}(X) \setminus H_{b\mathbb{P}}(X)$ there exists an infinite dimensional linear subspace in $H_{\mathbb{P}}(X)$ which consists (excepting zero) of analytic functions of unbounded type and contains f.*

Proof. Let

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \in H_{\mathbb{P}}(X) \setminus H_{b\mathbb{P}}(X).$$

Then $\varrho_0(f) = r$ for some $0 < r < \infty$. Let $\delta > 0$ and $\delta < c = 1/r$. Denote by N_0 the subset of all nonnegative integers $\mathbb{Z}_+$ such that $\|f_m\|^{1/m} < \delta$. In other words, if $n \in \mathbb{Z}_+ \setminus N_0$, then $\delta \leq \|f_n\|^{1/n} \leq c$, that is, for every subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{Z}_+ \setminus N_0$

$$r \leq \varrho_0\left(\sum_{k=1}^{\infty} f_{n_k}\right) \leq \frac{1}{\delta} < \infty.$$

Let

$$\mathbb{Z}_+ = \coprod_{k=0}^{\infty} N_k$$

be a partition of the set $\mathbb{Z}_+$ into infinite many disjoint subsets N_k so that $|N_k| = \infty$ for $k > 0$ and N_0 is the finite or infinite set defined above. Let $N_0 = (j_1, j_2, \ldots)$. We denote

$$\mathcal{N}_k = \begin{cases} N_k \bigcup \{j_k\} & \text{if } k \leq |N_0| \\ N_k & \text{otherwise.} \end{cases}$$

Thus $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2, \ldots)$ is a partition of $\mathbb{Z}_+$ into infinitely many disjoint subsets of infinite cardinality.

For any bounded sequence of numbers $a = (a_1, a_2, \ldots)$ we assign a function

$$g_a(x) = \sum_{k=1}^{\infty} a_k \sum_{j \in \mathcal{N}_k} f_j(x).$$

For every $a \in \ell_\infty$, $a \neq 0$ the function g_a is well defined on X and is of unbounded type. Indeed, for the Taylor polynomials $(g_a)_n$ of g_a we have

$$\left|(g_a)_m(x)\right|^{1/m} \leq \|a\|_{\ell_\infty}^{1/m} |f_m(x)|^{1/m} \to 0 \quad \text{as} \quad m \to \infty.$$

Moreover, since $a \neq 0$ there is a number j such that $a_j \neq 0$. Thus

$$\limsup_{m \to \infty} \left\|(g_a)_m\right\|^{1/m} \geq \limsup_{m \in \mathcal{N}_j} \left\|(g_a)_m\right\|^{1/m} = \limsup_{m \in \mathcal{N}_j} |a_j|^{1/m} \|f_m\|^{1/m} \geq \delta$$

and so

$$\varrho_0(g_a) \leq \frac{1}{\delta} < \infty.$$

Since $f \in H_{\mathbb{P}}(X)$, $g_a \in H_{\mathbb{P}}(X)$ for every $a \in \ell_\infty$. On the other hand, the set which depends on the choice of f, δ, and $\mathcal{N}$

$$\mathcal{V}_{f,\delta,\mathcal{N}} = \{g_a \colon a \in \ell_\infty\}$$

is a linear space because $g_a + \lambda g_b = g_{a+\lambda b}$ for all $a, b \in \ell_\infty$, $\lambda \in \mathbb{C}$. Clearly $f = g_a$ for $a = (1, 1, \ldots)$. $\square$

Note that the subspace $\mathcal{V}_{f,\delta,\mathcal{N}}$ is not maximal. Indeed, if $\mathcal{N}'$ is a subpartition of $\mathcal{N}$, then $\mathcal{V}_{f,\delta,\mathcal{N}'} \supset \mathcal{V}_{f,\delta,\mathcal{N}}$. It is easy to deduce by the Zorn Lemma that there is a maximal linear subspace in $H(x) \setminus H_b(X)$ containing a given function of unbounded type. So we have the following corollary.

Corollary 2. *Let X be an infinite dimensional Banach space. The set $H(x) \setminus H_b(X)$ can be represented as a union of infinite dimensional linear subspaces (without the origin).*

It is known (see [5]) that for every infinite dimensional Banach space X there are sequences $\{e_k\}_{k=1}^{\infty} \subset X$ and $\{\varphi_k\}_{k=1}^{\infty} \subset X^*$ such that

1. $\lim_{k\to\infty} \varphi_k(x) = 0$ for every $x \in X$,
2. $\|\varphi_k\| = 1$, $k \in \mathbb{N}$,
3. $\sup_{k\in\mathbb{N}} \|e_k\| < \infty$,
4. $\varphi_k(e_j) = \delta_{kj}$, where $k, j \in \mathbb{N}$ and δ_{kj} is the Kronecker delta.

In Theorem 2 of [5], actually it was proved that if the functionals φ_k are as above, then for every strictly increasing sequence of prime numbers $\{a_j\}_{j=1}^{\infty}$ the following functions

$$f_j = \sum_{k=1}^{\infty} a_j^k \varphi_k^k$$

generate an infinite dimensional algebra $\mathcal{A}$ such that every nonzero element h in $\mathcal{A}$ is an entire function of unbounded type and $\sup_n |h(e_n)| = \infty$. In particular, it is so if $X = c_0$, $\{e_k\}_{k=1}^{\infty}$ is the basis in c_0 and $\{\varphi_k\}_{k=1}^{\infty}$ is the sequence of coordinate functionals.

Theorem 4. *Let $\{P_n\}_{n=1}^{\infty}$, $\|P_n\| = 1$, $n \in \mathbb{N}$ be a sequence of n-homogeneous polynomials on a Banach space X such that $\mathfrak{P}(X) \subset c_0$ and there exists a sequence $\{z_k\}_{k=1}^{\infty}$ in X such that $\sup_k \|z_k\| < \infty$ and $P_n(z_k) = \delta_{nk}$. Then for every strictly increasing sequence of prime numbers $\{a_j\}_{j=1}^{\infty}$ the functions*

$$g_j = \sum_{k=1}^{\infty} a_j^k P_k$$

generate an infinite-dimensional algebra $\mathcal{B}$ such that every nonzero element in $u \in \mathcal{B}$ is an entire function of unbounded type and $\sup_n |u(z_n)| = \infty$.

Proof. Let us consider the algebra $\mathcal{A}$ generated by functions

$$f_j(x) = \sum_{k=1}^{\infty} a_j^k \big(\varphi_k(x)\big)^k = \sum_{k=1}^{\infty} a_j^k x_k^k, \quad \text{where} \quad x = \sum_{n=1}^{\infty} x_n e_n \in c_0 \quad \text{and} \quad j \in \mathbb{N}.$$

Note that $\Big(\varphi_k\big(\mathfrak{P}(z_k)\big)\Big)^k = e_k$ for every evaluation of $\mathfrak{P}(z_k)$. From Theorem 2 of [5] mentioned above, it follows that if $h \in \mathcal{A}$ and $h \neq 0$, then $\sup_n |h(e_n)| = \infty$. Every function $u \in \mathcal{B}$ can be represented as a finite algebraic combination of functions g_j,

$$\begin{aligned} u(x) &= \sum_{j_1 < \cdots < j_m < N} \lambda_{j_1 \ldots j_m} \big(g_{j_1}(x)\big)^{p_{j_1}} \cdots \big(g_{j_m}(x)\big)^{p_{j_m}} \\ &= \sum_{j_1 < \cdots < j_m < N} \lambda_{j_1 \ldots j_m} \Big(f_{j_1}\big(\mathfrak{P}(x)\big)\Big)^{p_{j_1}} \cdots \Big(f_{j_m}\big(\mathfrak{P}(x)\big)\Big)^{p_{j_m}}, \end{aligned}$$

where $N \in \mathbb{N}$, $\lambda_{j_1 \ldots j_m} \in \mathbb{C} \setminus \{0\}$ and $p_{j_k} \in \mathbb{N}$ for all j_k. In other words,

$$u(x) = h\big(\mathfrak{P}(x)\big) \quad \text{for some} \quad h \in \mathcal{A} \quad \text{and} \quad u(z_k) = h(e_k).$$

Hence, $\sup_n |u(z_n)| = \infty$ and so all functions in $\mathcal{B} \setminus \{0\}$ are unbounded on some bounded subsets. On the other hand, since $\mathfrak{P}(X) \subset c_0$, all functions g_j by Theorem 2 are well-defined on X and so their finite algebraic combinations are well-defined on X too. Thus, $\mathcal{B} \setminus \{0\} \subset H(X) \setminus H_b(X)$. $\square$

Corollary 3. *Let $P_n(x) = n!G_n(x)$, $x \in \ell_1$ be the basis of symmetric polynomials on ℓ_1 as in Example 1. Then for every strictly increasing sequence of prime numbers $\{a_j\}_{j=1}^{\infty}$ the functions*

$$g_j = \sum_{k=1}^{\infty} (-1)^{k+1} a_j^k P_k$$

generate an infinite-dimensional subalgebra in the algebra of symmetric analytic functions comprising (excepting zero) of analytic functions of unbounded type.

Proof. We need to construct a sequence $\{z_n\}_{n=1}^{\infty}$, biorthogonal to $\{P_k\}_{k=1}^{\infty}$. Let us define

$$z_n = \frac{1}{(n!)^{1/n}}(\alpha_1, \ldots, \alpha_n, 0, 0, \ldots),$$

where $\{\alpha_1, \ldots, \alpha_n\} = \sqrt[n]{1}$ are the roots of the unity. From the Vieta formulas it follows that

$$P_n(z_n) = \frac{n!}{n!}\alpha_1 \cdots \alpha_n = (-1)^{n+1}$$

and

$$P_k(z_n) = \frac{n!}{(n!)^{k/n}} G_k(z_n) = 0 \quad \text{if} \quad k < n.$$

If $k > n$, then $G_k(z_n) = 0$ because z_n has only n nonzero coordinates. Thus $P_k(z_n) = \delta_{kn}$. In addition, using the Stirling formula, we can estimate

$$\|z_n\| = \frac{n}{(n!)^{1/n}} \le n\left(\frac{e^n}{n^n}\right)^{1/n} = e < \infty.$$

Thus, we can apply Theorem 4 for the sequence of polynomials $\{(-1)^{k+1}P_k\}_{k=1}^{\infty}$. $\square$

4. Discussion and Conclusions

One of the main results of the paper is that not every infinitely generated algebra of polynomials on a Banach space supports entire functions of unbounded type. We found some necessary conditions and some sufficient conditions of this property but we have no conditions which are simultaneously necessary and sufficient. The mapping $\mathfrak{P}$ allows us to reduce this question to polynomial algebras on subsets of ℓ_∞. However, we do not know whether $\mathcal{P}_{\mathbb{I}}(c)$ supports entire functions of unbounded type, where c is the space of all convergent sequences. Moreover, we do not know if there exists a supersymmetric entire function of unbounded type (supersymmetric analytic functions and their properties were considered in [16]).

The theorems on linear subspaces and subalgebras are interesting in the context of the general question about linear structures in nonlinear sets [24] and are extensions of Lopez–Salazar Codes' results [5] for more special cases.

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