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# Approach to Multi-Attribute Decision-Making Methods for Performance Evaluation Process Using Interval-Valued T-Spherical Fuzzy Hamacher Aggregation Information

**Yun Jin [1], Zareena Kousar [2], Kifayat Ullah [2,*], Tahir Mahmood [3], Nimet Yapici Pehlivan [4] and Zeeshan Ali [3]**

1 Tourism Department, Wuxi Vocational College of Science and Technology, Wuxi 214028, China; yjin@nuaa.edu.cn
2 Department of Mathematics, Riphah Institute of Computing & Applied Sciences (RICAS), Riphah International University (Lahore Campus), Lahore 54000, Pakistan; zarinakausar046@gmail.com or 15662@students.riu.edu.pk
3 Department of Mathematics & Statistics, International Islamic University Islamabad, Islamabad 44000, Pakistan; tahirbakhat@iiu.edu.pk (T.M.); zeeshan.phdma102@iiu.edu.pk (Z.A.)
4 Department of Statistics, Science Faculty, Selçuk University, Konya 42031, Turkey; nimet@selcuk.edu.tr
* Correspondence: kifayat.khan.dr@gmail.com; Tel.: +92-334-909-0225

**Abstract:** Interval-valued T-spherical fuzzy set (IVTSFS) handles uncertain and vague information by discussing their membership degree (MD), abstinence degree (AD), non-membership degree (NMD), and refusal degree (RD). MD, AD, NMD, and RD are defined in terms of closed subintervals of $[0, 1]$ that reduce information loss compared to the T-spherical fuzzy set (TSFS), which takes crisp values from $[0, 1]$ intervals; hence, some information may be lost. The purpose of this manuscript is to develop some Hamacher aggregation operators (HAOs) in the environment of IVTSFSs. To do so, some Hamacher operational laws based on Hamacher t-norms (HTNs) and Hamacher t-conorms (HTCNs) are introduced. Using Hamacher operational laws, we develop some aggregation operators (AOs), including an interval-valued T-spherical fuzzy Hamacher (IVTSFH) weighted averaging (IVTSFHWA) operator, an IVTSFH-ordered weighted averaging (IVTSFHOWA) operator, an IVTSFH hybrid averaging (IVTSFHHA) operator, an IVTSFH-weighted geometric (IVTSFHWG) operator, an IVTSFH-ordered weighted geometric (IVTSFHOWG) operator, and an IVTSFH hybrid geometric (IVTSFHHG) operator. The validation of the newly developed HAOs is investigated, and their basic properties are examined. In view of some restrictions, the generalization and proposed HAOs are shown, and a multi-attribute decision-making (MADM) procedure is explored based on the HAOs, which are further exemplified. Finally, a comparative analysis of the proposed work is also discussed with previous literature to show the superiority of our work.

**Keywords:** T-spherical fuzzy set; interval-valued T-spherical fuzzy set; Hamacher aggregation operators; multi-attribute decision-making methods

**MSC:** 03B52; 47S40; 90B50



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## 1. Introduction

Multi-attribute decision making (MADM) is the most well-known branch of decision making that aims to select the most suitable alternative from a set of alternatives in the presence of multiple criteria that often conflict with each other. With the indecisiveness of decision-making (DM) topics and the fuzziness of DM conditions, MADM is accepted as an important technique due to its easy applicability. To solve such problems, where information is uncertain, Zadeh [1] put forward the theory of fuzzy sets (FSs), which describe (MD) information on a scale of $[0, 1]$ and provide a flexible platform to handle uncertainties. Atanassov [2] discovered intuitionistic FS (IFS) by coupling MD with an NMD under a restriction that the sum of both lies in $[0, 1]$. Although IFS provides better ground for handling uncertain information, it still restricts the decision-makers to a certain range and provides very little flexibility. Therefore, Yager [3,4] proposed the theories of Pythagorean FS (PyFS) and q-rung Orthopair FS (q-ROPFS), which improve the restrictions of IFSs and, hence, provide more flexible grounds in taking the MDs and NMDs. Keeping the advantages of expressing MD and NMD in terms of intervals instead of crisp values from $[0, 1]$, the theory of interval-valued IFS (IVIFS) was elaborated by Atanassov and Gargov [5] by improving the theory of interval-valued FS (IVFS), which was explored by Zadeh [6]. Moreover, the theory of interval-valued PyFS (IVPyFS) was elaborated by Peng and Yang [7], and the idea of interval-valued q-ROPFS (IVq-ROPFS) was introduced by Joshi et al. [8]. Garg and Rani [9,10] and Garg and Kumar [11] established some fruitful AOs of PyFSs and IFSs and investigated their applications in MADM. For a study on the theory and application of these concepts, one is referred to [12–18].

Theories of IFS, PyFS, and q-ROPFS cope with uncertain and complicated information in many practical situations of MADM and pattern recognition, but these duplets discussed only two phases of human opinion, i.e., MD and NMD, and AD and RD are ignored, which leads to the loss of information. To handle such issues, Cuong and Kreinovich [19] put forward the theory of picture FS (PFS), which is based on the MD, AD, NMD, and RD of information, with the condition that the sum of MD, AD, and NMD must lie in the $[0, 1]$ interval. Moreover, the theory of interval-valued PFS (IVPFS) was elaborated by Cuong and Kreinovich [19] and was further studied by Liu et al. [20]. Several scholars have applied the theory of IVPFS in numerous fields [21–23]. Mahmoud et al. [24] relaxed the strict condition of PFS and introduced the notion of spherical FS (SFS) and TSFS, where the range for assigning MD, AD, and NMD is limitless. Moreover, the theory of interval-valued TSFS (IVTSFS) was elaborated by Ullah et al. [25], where the closed subintervals of $[0, 1]$ are taken as MD, AD, NMD, and RD instead of crisp numbers from $[0, 1]$. Several scholars have applied the theory of TSFSs to numerous fields [26–31].

HAOs are among the most influential AOs that are discussed in fuzzy frameworks, and a wide range of studies has been conducted on the theory of HAOs, which are based on HTN and HTCN [32] in different fuzzy environments. Numerous HAOs have been introduced and all these have different applications in different fields. Huang [33] introduced HAOs in IFS. Garg [34] introduced intuitionistic fuzzy HAOs with entropy weight and investigated their applications in MADM. Liu [35] studied the applications of HAOs of IVIFSs in MADM problems. Gao [36] familiarized prioritized Pythagorean fuzzy HAOs for MADM problems. Wei [37] established Pythagorean power HAOs and explored their applications in MADM. Darko and Liang [38] presented the notion of HAOs for q-ROPFs, while the HAOs of IVq-ROPFS were introduced by Donyatalab et al. [39]. A study of enterprise selection using MADM techniques was established by Jana and Pal [40]. The problems related to search and rescue robots was discussed using TSF HAOs by Ullah et al. [41]. Some other recent work on HAOs can be found in [42–46].

Ullah et al. [25] observed that processing uncertain and ambiguous information using IVTSFSs instead of TSFSs and defining MD, AD, NMD, and RD as an interval rather than crisp numbers taken from [0, 1] greatly reduced information loss. The aim of this paper is to introduce the notion of HAOs in the environment of IVTSFSs. The motivation for doing so is that the HAOs proposed in [32–38] can describe only two aspects of human opinion and lead to information loss. Further, the HAOs of PFSs and TSFSs proposed by Jana and Pal [39] and Ullah et al. [40] discuss the four aspects of human opinion but the MD, AD, NMD, and RD are described in terms of crisp numbers and, hence, lead to information loss. The paper focuses on the following points.

1. To introduce some novel Hamacher operational laws based on IVTSFSs.
2. By using Hamacher operational laws, novel IVTSFHWA and IVTSFHWG operators are developed.
3. An MADM procedure is explored based on the proposed HAOs using IVTSFSs.
4. To observe the consistency and validity of the presented approaches, some examples are examined.

5. A comparative analysis of the current and previous studies is developed.

This manuscript is organized as in Section 2; we recall some ideas like TSFS, IVTSFS, and some relevant concepts, including HTN and HTCN. In Section 3, we investigate some Hamacher operational laws for IVTSFSs. In Section 4, we explore the idea of an IVTSFHWA operator, an IVTSFHOWA operator, and an IVTSFHHA operator. In Section 5, the idea of IVTSFHWG, IVTSFHOWG, and IVTSFHHG operators is investigated and their properties are discussed. In Section 6, the superiority of the proposed HAOs is analyzed in view of some special cases. Section 7 is based on the MADM procedure and a comprehensive example, where the impact of the variable parameters is examined. A comparative study of the current and previous HAOs is set up in Section 8, while a thorough summary of the paper is presented in Section 9.

## 2. Preliminaries

Some basic definitions of TSFS and IVTSFS over set X, through some remarks, are defined in this section. Definitions of HTN and HTCN are also discussed.

**Definition 1 ([26]).** *A TSFS on set X is defined by* $I = (m(x), i(x), n(x) : x \in X)$, *where* $m(x)$, $i(x)$ *and* $n(x)$ *are mappings from X to* $[0,\ 1]$, *denoting MD, AD, and NMD with the condition* $0 \leq m^q(x) + i^q(x) + n^q(x) \leq 1$ *for* $q \in \mathbb{Z}^+$. *RD is defined by* $r(x) = (1 - (m^q(x) + i^q(x) + n^q(x)))^{\frac{1}{q}}$ and the triplet $(m(x),\ i(x),\ n(x))$ is known as the T-spherical fuzzy number (TSFN).

The superiority of TSFS can be understood from Remark 1.

**Remark 1 ([27]).** *From Definition 1, some existing special fuzzy sets can be derived from TSFS as follows:*

1. *$q$ −ROPFS for* $i(x) = 0$.
2. *SFS for* $q = 2$.
3. *PyFS for* $q = 2$ *and* $i(x) = 0$.
4. *PFS for* $q = 1$.
5. *IFS for* $q = 1$ *and* $i(x) = 0$.
6. *FS for* $q = 1$ *and* $i(x) = 0 = n(x)$.

**Definition 2 ([27]).** *An IVTSFS on set X is defined by* $I = (\left[m^l(x), m^u(x)\right], \left[i^l(x), i^u(x)\right], \left[n^l(x), n^u(x)\right] : x \in X)$, *where* $m(x), i(x)$ *and* $n(x)$ *are mappings from X to closed subintervals of* $[0,\ 1]$, *denoting the MD, AD and NMD with the condition* $0 \leq (m^u)^q(x) + (i^u)^q(x) + (n^u)^q(x) \leq 1$ *for* $q \in \mathbb{Z}^+$. *RD can be defined as:*

$$r(x) = \left(\left[r^l(x), r^u(x)\right]\right) = \left(\left[\begin{array}{c} (1 - ((m^u)^q(x) + (i^u)^q(x) + (n^u)^q(x)))^{\frac{1}{q}}, \\ \left(1 - \left(m^l\right)^q(x) + (i^l)^q(x) + \left(n^l\right)^q(x)\right)\right)^{\frac{1}{q}} \end{array}\right]\right)$$

*and the triplet* $\left(\left[m^l(x), m^u(x)\right], [i^l(x), i^u(x)], [n^l(x), n^u(x)\right]\right)$ *is defined as an interval-valued T-spherical fuzzy number (IVTSFN).*

From Definition 2, it is quite clear that existing fuzzy frameworks are derived from IVTSFS under some restrictions, given as follows:

**Theorem 1 ([27]).** *From Definition 2, an IVTSFS can be reduced to the following special fuzzy sets:*

1. *TSFS for* $m^l = m^u,\ i^l = i^u,\ n^l = n^u$.
2. *Interval-valued SFS (IVSFS) for* $q = 2$.

3. *SFS for* $q = 2$ *and* $m^l = m^u, i^l = i^u, n^l = n^u$.
4. *IVPFS for* $q = 1$.
5. *PFS for* $q = 1$ *and* $m^l = m^u, i^l = i^u, n^l = n^u$.
6. *IVq-ROPFS for* $i^l = i^u = 0$.
7. $q$ *−ROPFS* $m^l = m^u, i^l = i^u = 0, n^l = n^u$.
8. *IVPyFS for* $q = 2$ *and* $i^l = i^u = 0$.
9. *PyFS for* $q = 2$ *and* $m^l = m^u, i^l = i^u = 0, n^l = n^u$.
10. *IVIFS for* $q = 1$ *and* $i^l = i^u = 0$.
11. *IFS for* $q = 1$ *and* $m^l = m^u, i^l = i^u = 0, n^l = n^u$.
12. *IVFS for* $q = 1$ *and* $n^l = n^u = i^l = i^u = 0$.
13. *FS for* $q = 1$ *and* $n^l = n^u = i^l = i^u = 0$ *and* $m^l = m^u$.

**Proof.** Trivial. □

In order to rank two or more IVTSFNs, the score function can be used given in Definition 3.

**Definition 3 ([25]).** *For an IVTSFN* $I = \left(\left[m^l, m^u\right], \left[i^l, i^u\right], \left[n^l, n^u\right]\right)$, *the score function is defined by:*

$$SC(I) = \frac{\left(m^l\right)^q\left(1 - \left(i^l\right)^q - \left(n^l\right)^q\right) + (m^u)^q(1 - (i^u)^q - (n^u)^q)}{3},\ SC(I) \in [0, 1]$$

**Definition 4 ([32]).** *HTN and HTCN are defined as follows, respectively:*

$$T_{hn}(a, b) = \frac{a.b}{\gamma + (1-\gamma)(a + b - ab)},\ \gamma > 0, (a,\ b) \in [0,\ 1]^2$$

$$T_{hcn}(a,\ b) = \frac{a + b - ab - (1-\gamma)ab}{1 - (1-\gamma)ab},\ \gamma > 0, (a,\ b) \in [0,\ 1]^2$$

*Further,* $T_{hn}(a, b)$ *is also considered a Hamacher product and* $T_{hcn}(a,\ b)$ *is known as the Hamacher sum.* $T_{hn}(a, b)$ *and* $T_{hcn}\ (a,\ b)$ *can be stated as follows, respectively:*

$$a \otimes b = \frac{a.b}{\gamma + (1-\gamma)(a + b - ab)},\ \gamma > 0\ (a,\ b) \in [0,\ 1]^2$$

$$a \oplus b = \frac{a + b - ab - (1-\gamma)ab}{1 - (1-\gamma)ab},\ \gamma > 0\ (a,\ b) \in [0,\ 1]^2$$

**Remark 2.** *Hamacher product and Hamacher sum are given in Definition 4; they are converted into algebraic product and algebraic sum for* $\gamma = 1$, *while they are converted into Einstein product and Einstein sum for* $\gamma = 2$.

## 3. Interval-Valued T-Spherical Fuzzy Hamacher Operations

The aim of this section is to develop some Hamacher operations in the framework of IVTSFSs. The Hamacher operations, including Hamacher sum and Hamacher product, are proposed. Then, some special cases of the TSF Hamacher operations are investigated.

**Definition 5.** *Let* $A = \left(\left[m_A^l\ ,\ m_A^u\right], \left[i_A^l\ ,\ i_B^u\right], \left[n_A^l, n_A^u\right]\right)$ *and* $B = \left(\left[m_B^l, m_B^u\right], \left[i_B^l, i_B^u\right], \left[n_B^l, n_B^u\right]\right)$ be two IVTSFNs for $\lambda,\ \gamma > 0$. The novel interval-valued T-spherical fuzzy Hamacher operations are expressed as:

$$A \oplus B = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[q]{\dfrac{{m_A^l}^q + {m_B^l}^q - {m_A^l}^q {m_B^l}^q - (1-\gamma){m_A^l}^q {m_B^l}^q}{1-(1-\gamma){m_A^l}^q {m_B^l}^q}}, \\ \sqrt[q]{\dfrac{{m_A^u}^q + {m_B^u}^q - {m_A^u}^q {m_B^u}^q - (1-\gamma){m_A^u}^q {m_B^u}^q}{1-(1-\gamma){m_A^u}^q {m_B^u}^q}} \end{array}\right] \\ \left[\begin{array}{c} \dfrac{i_A^l i_B^l}{\sqrt[q]{\gamma+(1-\gamma)\left({i_A^l}^q + {i_B^l}^q - {i_A^l}^q {i_B^l}^q\right)}}, \\ \dfrac{i_A^u i_B^u}{\sqrt[q]{\gamma+(1-\gamma)\left({i_A^u}^q + {i_B^u}^q - {i_A^u}^q {i_B^u}^q\right)}} \end{array}\right] \\ \left[\begin{array}{c} \dfrac{n_A^l n_B^l}{\sqrt[q]{\gamma+(1-\gamma)\left({n_A^l}^q + {n_B^l}^q - {n_A^l}^q {n_B^l}^q\right)}}, \\ \dfrac{n_A^u n_B^u}{\sqrt[q]{\gamma+(1-\gamma)\left({n_A^u}^q + {n_B^u}^q - {n_A^u}^q {n_B^u}^q\right)}} \end{array}\right] \end{array}\right) \tag{1}$$

$$A \oplus B = \left(\begin{array}{c} \left[\begin{array}{c} \dfrac{m_A^l m_B^l}{\sqrt[q]{\gamma+(1-\gamma)\left({m_A^l}^q + {m_B^l}^q - {m_A^l}^q {m_B^l}^q\right)}}, \\ \dfrac{m_A^u m_B^u}{\sqrt[q]{\gamma+(1-\gamma)\left({m_A^u}^q + {m_B^u}^q - {m_A^u}^q {m_B^u}^q\right)}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[q]{\dfrac{{i_A^l}^q + {i_B^l}^q - {i_A^l}^q {i_B^l}^q - (1-\gamma){i_A^l}^q {i_B^l}^q}{1-(1-\gamma){i_A^l}^q {i_B^l}^q}}, \\ \sqrt[q]{\dfrac{{i_A^u}^q + {i_B^u}^q - {i_A^u}^q {i_B^u}^q - (1-\gamma){i_A^u}^q {i_B^u}^q}{1-(1-\gamma){i_A^u}^q {i_B^u}^q}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[q]{\dfrac{{n_A^l}^q + {n_B^l}^q - {n_A^l}^q {n_B^l}^q - (1-\gamma){n_A^l}^q {n_B^l}^q}{1-(1-\gamma){n_A^l}^q {n_B^l}^q}}, \\ \sqrt[q]{\dfrac{{n_A^u}^q + {n_B^u}^q - {n_A^u}^q {n_B^u}^q - (1-\gamma){n_A^u}^q {n_B^u}^q}{1-(1-\gamma){n_A^u}^q {n_B^u}^q}} \end{array}\right] \end{array}\right) \tag{2}$$

$$\lambda A = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[q]{\dfrac{\left(1+(\gamma-1){m_A^l}^q\right)^\lambda - \left(1-{m_A^l}^q\right)^\lambda}{\left(1+(\gamma-1){m_A^l}^q\right)^\lambda + (\gamma-1)\left(1-{m_A^l}^q\right)^\lambda}}, \\ \sqrt[q]{\dfrac{\left(1+(\gamma-1){m_A^u}^q\right)^\lambda - \left(1-{m_A^u}^q\right)^\lambda}{\left(1+(\gamma-1){m_A^u}^q\right)^\lambda + (\gamma-1)\left(1-{m_A^u}^q\right)^\lambda}} \end{array}\right] \\ \left[\begin{array}{c} \dfrac{\sqrt[q]{\gamma}\left(i_A^l\right)^\lambda}{\sqrt[q]{\left(1+(\gamma-1)(1-{i_A^l}^q)\right)^\lambda + (\gamma-1)\left({i_A^l}^q\right)^\lambda}}, \\ \dfrac{\sqrt[q]{\gamma}\left(i_A^u\right)^\lambda}{\sqrt[q]{\left(1+(\gamma-1)(1-{i_A^u}^q)\right)^\lambda + (\gamma-1)\left({i_A^u}^q\right)^\lambda}} \end{array}\right] \\ \left[\begin{array}{c} \dfrac{\sqrt[q]{\gamma}\left(n_A^l\right)^\lambda}{\sqrt[q]{\left(1+(\gamma-1)(1-{n_A^l}^q)\right)^\lambda + (\gamma-1)\left({n_A^l}^q\right)^\lambda}}, \\ \dfrac{\sqrt[q]{\gamma}\left(n_A^u\right)^\lambda}{\sqrt[q]{\left(1+(\gamma-1)(1-{n_A^u}^q)\right)^\lambda + (\gamma-1)\left({n_A^u}^q\right)^\lambda}} \end{array}\right] \end{array}\right) \tag{3}$$

$$A^{\lambda} = \left( \begin{array}{c} \left[ \frac{\sqrt[q]{\gamma}\left(m_A^l\right)^{\lambda}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-m_A^{l\,q}\right)\right)^{\lambda}+(\gamma-1)\left(m_A^{l\,q}\right)^{\lambda}}}, \frac{\sqrt[q]{\gamma}\left(m_A^u\right)^{\lambda}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-m_A^{u\,q}\right)\right)^{\lambda}+(\gamma-1)\left(m_A^{u\,q}\right)^{\lambda}}} \right] \\ \left[ \sqrt[q]{\frac{\left(1+(\gamma-1)i_A^{l\,q}\right)^{\lambda}-\left(1-i_A^{l\,q}\right)^{\lambda}}{\left(1+(\gamma-1)i_A^{l\,q}\right)^{\lambda}+(\gamma-1)\left(1-i_A^{l\,q}\right)^{\lambda}}}, \sqrt[q]{\frac{\left(1+(\gamma-1)i_A^{u\,q}\right)^{\lambda}-\left(1-i_A^{u\,q}\right)^{\lambda}}{\left(1+(\gamma-1)i_A^{u\,q}\right)^{\lambda}+(\gamma-1)\left(1-i_A^{u\,q}\right)^{\lambda}}} \right] \\ \left[ \sqrt[q]{\frac{\left(1+(\gamma-1)n_A^{l\,q}\right)^{\lambda}-\left(1-n_A^{l\,q}\right)^{\lambda}}{\left(1+(\gamma-1)n_A^{l\,q}\right)^{\lambda}+(\gamma-1)\left(1-n_A^{l\,q}\right)^{\lambda}}}, \sqrt[q]{\frac{\left(1+(\gamma-1)n_A^{u\,q}\right)^{\lambda}-\left(1-n_A^{u\,q}\right)^{\lambda}}{\left(1+(\gamma-1)n_A^{u\,q}\right)^{\lambda}+(\gamma-1)\left(1-n_A^{u\,q}\right)^{\lambda}}} \right] \end{array} \right) \quad (4)$$

The Hamacher operations defined in Equations (1)–(4) are more effective than earlier Hamacher operations of IVIFSs, IVPyFSs, IVq-ROPFSs, and PFSs. The novel interval-valued T-spherical fuzzy Hamacher (IVTSFH) operations explain MD, AD, NMD, and RD with no restrictions because for every triplet $\left(\left[m_A^l, m_A^u\right], \left[i_A^l,\ i_A^u\ \right], \left[n_A^l, n_A^u\right]\right)$, there exist some $q \in \mathbb{Z}^+$ for which the triplet becomes an IVTSFN.

The Hamacher operations defined in Equations (1)–(4) can be reduced to some existing fuzzy sets, which can be described as:

1. For $q = 2$, IVTSFH operations become the Hamacher operations of the IVSFSs.
2. For $q = 1$, IVTSFH operations become the Hamacher operations of the IVPFSs.
3. For $i = 0$, IVTSFH operations become the Hamacher operations of the IVq-ROPFSs.
4. For $q = 2$, and $i = 0$, IVTSFH operations become the Hamacher operations of the PyFSs.
5. For $q = 1$ and $i = 0$, IVTSFH operations become the Hamacher operations of the IVIFSs.

## 4. Interval Valued T-Spherical Fuzzy Hamacher Weighted Averaging (IVTSFHWA) Operators

The aim of this section is to develop IVTSFHWA operators based on the Hamacher operation introduced in Section 3. Note that from this section onward, $w_j = (w_1\ , w_2\ ,\ \ldots, w_l\ )^T$ represents the weight vector, where $w_j > 0$, and $\sum_1^l w_j = 1$, where $j,\ k \in J = \{1,\ 2,\ 3\ldots l\}$.

**Definition 6.** *Suppose* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are some IVTSFNs. Then, the IVTSFHWA operator* $T^l\ \rightarrow\ T$ *is defined as:*

$$IVTSFHWA\ (\text{Ҭ}_1,\ \text{Ҭ}_2,\ \text{Ҭ}_3,\ \ldots \text{Ҭ}_l) = \sum_{j=1}^{l} w_j \text{Ҭ}_j \quad (5)$$

By using previous results defined in Definition 5, we can obtain the subsequent result, as given in Theorem 2.

**Theorem 2.** *Consider* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are some IVTSFNs. Then, the IVTSFHWA operator is an IVTSFN and given by:*

$$IVTSFHWA(T_1, T_2, T_3, \ldots T_l) = \left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{lq}\right)^{w_j} - \prod_{j=1}^{l}\left(1-m_j^{lq}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{lq}\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{lq}\right)^{w_j}}} \\ , \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{uq}\right)^{w_j} - \prod_{j=1}^{l}\left(1-m_j^{uq}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{uq}\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{uq}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(i_j^l\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-i_j^{lq}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(i_j^{lq}\right)^{w_j}}} \\ , \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(i_j^u\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-i_j^{uq}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(i_j^{uq}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(n_j^l\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-n_j^{lq}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(n_j^{lq}\right)^{w_j}}} \\ , \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(n_j^u\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-n_j^{uq}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(n_j^{uq}\right)^{w_j}}} \end{array} \right] \end{array} \right) \quad (6)$$

**Proof.** To prove, the mathematical induction method is used.

Let $j = 2$

$$w_1 T_1 \oplus w_2 T_2 = \left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\left(1+(\gamma-1)m_1^{lq}\right)^{w_1} - \left(1-m_1^{lq}\right)^{w_1}}{\left(1+(\gamma-1)m_1^{lq}\right)^{w_1} + (\gamma-1)\left(1-m_1^{lq}\right)^{w_1}}}, \\ \sqrt[q]{\frac{\left(1+(\gamma-1)m_1^{uq}\right)^{w_1} - \left(1-m_1^{uq}\right)^{w_1}}{\left(1+(\gamma-1)m_1^{uq}\right)^{w_1} + (\gamma-1)\left(1-m_1^{uq}\right)^{w_1}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\left(i_1^l\right)^{w_1}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-i_1^{lq}\right)\right)^{w_1} + (\gamma-1)\left(i_1^{lq}\right)^{w_1}}} \\ \frac{\sqrt[q]{\gamma}\left(i_1^u\right)^{w_1}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-i_1^{uq}\right)\right)^{w_1} + (\gamma-1)\left(i_1^{uq}\right)^{w_1}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\left(n_1^l\right)^{w_1}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_1^{lq}\right)\right)^{w_1} + (\gamma-1)\left(n_1^{lq}\right)^{w_1}}} \\ \frac{\sqrt[q]{\gamma}\left(n_1^u\right)^{w_1}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_1^{uq}\right)\right)^{w_1} + (\gamma-1)\left(n_1^{uq}\right)^{w_1}}} \end{array} \right] \end{array} \right) \oplus \left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\left(1+(\gamma-1)m_2^{lq}\right)^{w_2} - \left(1-m_2^{lq}\right)^{w_2}}{\left(1+(\gamma-1)m_2^{lq}\right)^{w_2} + (\gamma-1)\left(1-m_2^{lq}\right)^{w_2}}}, \\ \sqrt[q]{\frac{\left(1+(\gamma-1)m_2^{uq}\right)^{w_2} - \left(1-m_2^{uq}\right)^{w_2}}{\left(1+(\gamma-1)m_2^{uq}\right)^{2} + (\gamma-1)\left(1-m_1^{uq}\right)^{w_2}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\left(i_2^l\right)^{w_2}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-i_2^{lq}\right)\right)^{w_2} + (\gamma-1)\left(i_2^{lq}\right)^{w_2}}} \\ \frac{\sqrt[q]{\gamma}\left(i_2^u\right)^{w_2}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-i_2^{uq}\right)\right)^{w_2} + (\gamma-1)\left(i_2^{uq}\right)^{w_2}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\left(n_2^l\right)^{w_2}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_2^{lq}\right)\right)^{w_2} + (\gamma-1)\left(n_2^{lq}\right)^{w_2}}} \\ \frac{\sqrt[q]{\gamma}\left(n_2^u\right)^{w_2}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_2^{uq}\right)\right)^{w_2} + (\gamma-1)\left(n_2^{uq}\right)^{w_2}}} \end{array} \right] \end{array} \right)$$

$$= \left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{2}\left(1+(\gamma-1)m_j^{lq}\right)^{w_j}-\prod_{j=1}^{2}\left(1-m_j^{lq}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(\gamma-1)m_j^{lq}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(1-m_j^{lq}\right)^{w_j}}}, \\ \sqrt[q]{\frac{\prod_{j=1}^{2}\left(1+(\gamma-1)m_j^{uq}\right)^{w_j}-\prod_{j=1}^{2}\left(1-m_j^{uq}\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(\gamma-1)m_j^{uq}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(1-m_j^{uq}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{2}\left(i_j^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{2}\left(1+(\gamma-1)(1-i_j^{lq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(i_j^{lq}\right)^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{2}\left(i_j^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{2}\left(1+(\gamma-1)(1-i_j^{uq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(i_j^{uq}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{2}\left(n_j^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{2}\left(1+(\gamma-1)(1-n_j^{lq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(n_j^{lq}\right)^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{2}\left(n_j^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{2}\left(1+(\gamma-1)(1-n_j^{uq})\right)^{w_j}-(\gamma-1)\prod_{j=1}^{2}\left(n_j^{uq}\right)^{w_j}}} \end{array} \right] \end{array} \right)$$

Hence, this result is true for $l = 2$.

Now, let us assume that it is true for $l = k$ and we have to prove that it is true for $l = k + 1$

$$IVTSFHWA(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_k) =$$

$$\left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{lq}\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^{lq}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{lq}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-m_j^{lq}\right)^{w_j}}}, \\ \sqrt[q]{\frac{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{uq}\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^{uq}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{uq}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-m_j^{uq}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(i_j^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-i_j^{lq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(i_j^{lq}\right)^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(i_j^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-i_j^{uq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(i_j^{uq}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(n_j^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-n_j^{lq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(n_j^{lq}\right)^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(n_j^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-n_j^{uq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(n_j^{uq}\right)^{w_j}}} \end{array} \right] \end{array} \right)$$

$$IVTSFHWA(\text{Ƭ}_1, \text{Ƭ}_2, \ldots, \text{Ƭ}_k, \text{Ƭ}_{k+1}) = IVTSFHWA(\text{Ƭ}_1, \text{Ƭ}_2, \ldots, \text{Ƭ}_k) \oplus \text{Ƭ}_{k+1}$$

$$= \left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{lq}\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^{lq}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{lq}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-m_j^{lq}\right)^{w_j}}}, \\ \sqrt[q]{\frac{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{uq}\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^{uq}\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)m_j^{uq}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-m_j^{uq}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(i_j^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-i_j^{lq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(i_j^{lq}\right)^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(i_j^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-i_j^{uq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(i_j^{uq}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(n_j^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-n_j^{lq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(n_j^{lq}\right)^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(n_j^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-n_j^{uq})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(n_j^{uq}\right)^{w_j}}} \end{array} \right] \end{array} \right)$$

$$\oplus\left(\begin{array}{c}\left[\begin{array}{c}\sqrt[q]{\frac{\left(1+(\gamma-1)m_{k+1}^{l}{}^{q}\right)^{w_{k+1}}-\left(1-m_{k+1}^{l}{}^{q}\right)^{w_{k+1}}}{\left(1+(\gamma-1)m_{k+1}^{l}{}^{q}\right)^{w_{k+1}}+(\gamma-1)\left(1-m_{k+1}^{l}{}^{q}\right)^{w_{k+1}}}},\\ \sqrt[q]{\frac{\left(1+(\gamma-1)m_{k+1}^{u}{}^{q}\right)^{w_{k+1}}-\left(1-m_{k+1}^{u}{}^{q}\right)^{w_{k+1}}}{\left(1+(\gamma-1)m_{k+1}^{u}{}^{q}\right)^{w_{k+1}}+(\gamma-1)\left(1-m_{k+1}^{u}{}^{q}\right)^{w_{k+1}}}}\end{array}\right]\\ \left[\begin{array}{c}\frac{\sqrt[q]{\gamma}\left(i_{k+1}^{l}\right)^{w_{k+1}}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-i_{k+1}^{l}{}^{q}\right)\right)^{w_{k+1}}+(\gamma-1)\left(i_{k+1}^{l}{}^{q}\right)^{w_{k+1}}}},\\ \frac{\sqrt[q]{\gamma}\left(i_{k+1}^{u}\right)^{w_{k+1}}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-i_{k+1}^{u}{}^{q}\right)\right)^{w_{k+1}}+(\gamma-1)\left(i_{k+1}^{u}{}^{q}\right)^{w_{k+1}}}}\end{array}\right]\\ \left[\begin{array}{c}\frac{\sqrt[q]{\gamma}\left(n_{k+1}^{l}\right)^{w_{k+1}}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_{k+1}^{l}{}^{q}\right)\right)^{w_{k+1}}+(\gamma-1)\left(n_{k+1}^{l}{}^{q}\right)^{w_{k+1}}}},\\ \frac{\sqrt[q]{\gamma}\left(n_{k+1}^{u}\right)^{w_{k+1}}}{\sqrt[q]{\left(1+(\gamma-1)\left(1-n_{k+1}^{u}{}^{q}\right)\right)^{w_{k+1}}+(\gamma-1)\left(n_{k+1}^{u}{}^{q}\right)^{w_{k+1}}}}\end{array}\right]\end{array}\right)$$

$$IVTSFHWA(\daleth_1,\daleth_2,\daleth_3,\ldots\daleth_{k+1})=$$

$$\left(\begin{array}{c}\left[\begin{array}{c}\sqrt[q]{\frac{\prod_{j=1}^{k+1}\left(1+(\gamma-1)m_j^{l\,q}\right)^{w_j}-\prod_{j=1}^{k+1}\left(1-m_j^{l\,q}\right)^{w_j}}{\prod_{j=1}^{k+1}\left(1+(\gamma-1)m_j^{l\,q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}\left(1-m_j^{l\,q}\right)^{w_j}}},\\ \sqrt[q]{\frac{\prod_{j=1}^{k+1}\left(1+(\gamma-1)m_j^{u\,q}\right)^{w_j}-\prod_{j=1}^{k+1}\left(1-m_j^{u\,q}\right)^{w_j}}{\prod_{j=1}^{k+1}\left(1+(\gamma-1)m_j^{u\,q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}\left(1-m_j^{u\,q}\right)^{w_j}}}\end{array}\right]\\ \left[\begin{array}{c}\frac{\sqrt[q]{\gamma}\prod_{j=1}^{k+1}\left(i_j^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k+1}\left(1+(\gamma-1)\left(1-i_j^{l\,q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}\left(i_j^{l\,q}\right)^{w_j}}},\\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k+1}\left(i_j^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k+1}\left(1+(\gamma-1)\left(1-i_j^{u\,q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}\left(i_j^{u\,q}\right)^{w_j}}}\end{array}\right]\\ \left[\begin{array}{c}\frac{\sqrt[q]{\gamma}\prod_{j=1}^{k+1}\left(n_j^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k+1}\left(1+(\gamma-1)\left(1-n_j^{l\,q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}\left(n_j^{l\,q}\right)^{w_j}}},\\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k+1}\left(n_j^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{k+1}\left(1+(\gamma-1)\left(1-n_j^{u\,q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}\left(n_j^{u\,q}\right)^{w_j}}}\end{array}\right]\end{array}\right)$$

It shows that it is true for $l=k+1$, and, hence, it holds for all values of $l$. □

Here, we define some main characteristics of the IVTSFHWA operator in Theorem 3.

**Theorem 3.** *The HAOs of IVTSFNs satisfy the following properties:*

1. ***Idempotency.*** *If* $\daleth_j=\daleth=\left(\left[m_j^l,m_j^u\right],\left[i_j^l,i_j^u\right],\left[n_j^l,n_j^u\right]\right)\ \forall\ j=1,\ 2,\ 3,\ldots l$. Then, $IVTSFHWA(\daleth_1,\ \daleth_2,\ \daleth_3,\ \ldots\daleth_l)=\daleth$
2. ***Boundedness.*** *If* $\daleth^-=\left(\left[\overset{l}{\min}_j m_j^l,\overset{u}{\min}_j m_j^u\right],\left[\overset{l}{\max}_j i_j^l,\overset{u}{\max}_j i_j^u\right],\left[\overset{l}{\max}_j n_j^l,\overset{u}{\max}_j n_j^u\right]\right)$ *and* $\daleth^+=\left(\left[\overset{l}{\max}_j m_j^l,\overset{u}{\max}_j m_j^u\right],\left[\overset{l}{\min}_j i_j^l,\ \overset{u}{\min}_j i_j^u\right],\left[\overset{l}{\min}_j n_j^l,\ \overset{u}{\min}_j n_j^u\right]\right)$. *Then,* $\daleth^-\leq IVTSFHWA(\daleth_1,\ \daleth_2,\ \daleth_3,\ \ldots\daleth_l)\leq\daleth^+$
3. **Monotonicity.** *Let* $\daleth_j$ *and* $P_j$ *IVTSFNs such that* $\daleth_j\leq P_j\ \forall\ j$. *Then,* $IVTSFHWA(\daleth_1,\ \daleth_2,\ \daleth_3,\ \ldots\daleth_l)\leq IVTSFHWA(P_1,\ P_2,\ P_3,\ \ldots P_l)$

**Proof.** Trivial. □

The IVTSFHWA operator only evaluates IVTSFNs. To address situations where the ordering status of IVTSFNs is important in MADM problems, the IVTSFHOWA operator is defined as follows:

**Definition 7.** *Suppose that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are some IVTSFNs. Then, the IVTSFHOWA operator* $T^l \rightarrow T$ *is defined as:*

$$IVTSFHOWA(T_1,\ T_2,\ T_3,\ \ldots T_l) = \sum_{j=1}^{l} w_j T_{\sigma(j)} \tag{7}$$

*where* $T_{\sigma(j-1)} \geq T_{\sigma(j)}\ \forall\ j$ *is satisfied.*

**Theorem 4.** *Consider that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHOWA operator is an IVTSFN and given by:*

$$IVTSFHOWA(T_1, T_2, T_3, \ldots T_l) = \sum_{j=1}^{l} w_j T_{\sigma(j)} =$$

$$\left(\begin{array}{c}
\left[\begin{array}{c}
\sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_{\sigma(j)}^{l}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_{\sigma(j)}^{l}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\
,\sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_{\sigma(j)}^{u}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_{\sigma(j)}^{u}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}}
\end{array}\right] \\
\left[\begin{array}{c}
\frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(i_{\sigma(j)}^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-i_{\sigma(j)}^{l}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(i_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\
,\frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(i_{\sigma(j)}^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-i_{\sigma(j)}^{u}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(i_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}}
\end{array}\right] \\
\left[\begin{array}{c}
\frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(n_{\sigma(j)}^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-n_{\sigma(j)}^{l}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(n_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\
,\frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(n_{\sigma(j)}^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-n_{\sigma(j)}^{u}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(n_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}}
\end{array}\right]
\end{array}\right) \tag{8}$$

**Proof.** Trivial. □

**Remark 3.** *The IVTSFHOWA operator satisfies the three conditions of Idempotency, Monotonicity, and Boundedness, as defined in Theorem 3.*

The IVTSFHWA operator only evaluates IVTSFNs, while the IVTSFHOWA operator only aggregates the ordered position of IVTSFNs. When the ordered position and weight of the argument are important, the IVTSFHHA operator is proposed as follows:

**Definition 8.** *Suppose that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHHA operator* $T^l \rightarrow T$ *is defined as:*

$$IVTSFHHA(T_1,\ T_2,\ T_3,\ \ldots T_l) = \sum_{j=1}^{l} w_j \dot{T}_{\sigma(j)} \tag{9}$$

*where* $\dot{T}_{\sigma(j)}$ *is the* $j$th *largest value of the IVTSFNs* $\dot{T}_j = l\varpi_j T_j$ *with* $\varpi_j$ *as the weight of interval-valued T-Spherical fuzzy a*$\varpi$*rguments* $T_j$ *such that* $\varpi_j \in [0,1]$ *and* $\sum_1^n \varpi_j = 1$ *and the balancing coefficient is denoted by* $l$.

**Theorem 5.** *Consider that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right) \; \forall \; j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHHA operator is an IVTSFN and given by:*

$$IVTSFHHA(T_1, T_2, T_3, \ldots T_l) = \left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{m}_{\sigma(j)}^{l}{}^{q}\right)^{w_j} - \prod_{j=1}^{l}\left(1-\dot{m}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{m}_{\sigma(j)}^{l}{}^{q}\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(1-\dot{m}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ , \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{m}_{\sigma(j)}^{u}{}^{q}\right)^{w_j} - \prod_{j=1}^{l}\left(1-\dot{m}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{m}_{\sigma(j)}^{u}{}^{q}\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(1-\dot{m}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(i_{\sigma(j)}^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-i_{\sigma(j)}^{l}{}^{q}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(i_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ , \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(i_{\sigma(j)}^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-i_{\sigma(j)}^{u}{}^{q}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(i_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(\dot{n}_{\sigma(j)}^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-\dot{n}_{\sigma(j)}^{l}{}^{q}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(\dot{n}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ , \frac{\sqrt[q]{\gamma}\prod_{j=1}^{k}\left(\dot{n}_{\sigma(j)}^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-\dot{n}_{\sigma(j)}^{u}{}^{q}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{l}\left(\dot{n}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array} \right] \end{array} \right) \tag{10}$$

**Proof.** Trivial. □

**Remark 4.** *Theorem 5 reduces to the IVTSFHWA operator for* $w_j = \left(\frac{1}{l}, \frac{1}{l}, \frac{1}{l}, \ldots \frac{1}{l}\right)^T$. *Then, it reduces into the IVTSFHOWA operator for* $\mathcal{w}_j = \left(\frac{1}{l}, \frac{1}{l}, \frac{1}{l}, \ldots \frac{1}{l}\right)^T$.

**Remark 5.** *The IVTSFHOWA operator satisfies the three conditions of Idempotency, Monotonicity, and Boundedness defined in Theorem 3.*

## 5. Interval-Valued T-Spherical fuzzy Hamacher Weighted Geometric (IVTSFHWG) Operators

The aim of this section is to develop IVTSFHWG operators based on the Hamacher operations introduced in Section 3. Some basic characteristics of the IVTSFHWG operators are also investigated.

**Definition 9.** *Suppose that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right) \; \forall \; j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHWG operator* $T^l \; \rightarrow \; T$ *is defined as:*

$$IVTSFHWG\left(T_1, \; T_2, \; T_3, \; \ldots T_l\right) = \sum_{j=1}^{l} T_j^{w_j} \tag{11}$$

By using previous results defined in Definition 5, we can obtain the subsequent result, as given in Theorem 6.

**Theorem 6.** *Consider that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHWG operator is an IVTSFN and given by:*

$$IVTSFHWG(\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \ldots \mathrm{T}_l) = \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(m_j^l\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-m_j^{l\,q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(m_j^{l\,q}\right)^{w_j}}} \\ , \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(m_j^u\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-m_j^{u\,q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(m_j^{u\,q}\right)^{w_j}}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)i_j^{l\,q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-i_j^{l\,q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)i_j^{l\,q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-i_j^{l\,q}\right)^{w_j}}} \\ , \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)i_j^{u\,q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-i_j^{u\,q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)i_j^{u\,q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-i_j^{u\,q}\right)^{w_j}}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{l\,q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^{l\,q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{l\,q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^{l\,q}\right)^{w_j}}} \\ , \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{u\,q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^{u\,q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{u\,q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^{u\,q}\right)^{w_j}}} \end{array}\right] \end{array}\right) \tag{12}$$

**Proof.** This result can be proven similar to Theorem 2. It can be noted that this operator also fulfilled the conditions of Monotonicity, Idempotency, and Boundedness. □

**Definition 10.** *Suppose that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHWG operator* $T^l \rightarrow T$ *is defined as:*

$$IVTSFHOWG(\mathrm{T}_1,\ \mathrm{T}_2,\ \mathrm{T}_3,\ \ldots \mathrm{T}_l) = \sum_{j=1}^{l} \mathrm{T}_{\sigma(j)}^{w_j} \tag{13}$$

*where* $\mathrm{T}_{\sigma(j-1)} \geq \mathrm{T}_{\sigma(j)}\ \forall\ j$ *is satisfied.*

**Theorem 7.** *Consider that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHOWG operator is an IVTSFN and given by:*

$$IVTSFHOWG(\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \ldots \mathrm{T}_l) = \sum_{j=1}^{l} \mathrm{T}_{\sigma(j)}^{w_j} = \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(m_{\sigma(j)}^l\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-m_{\sigma(j)}^{l}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(m_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ , \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(m_{\sigma(j)}^u\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)\left(1-m_{\sigma(j)}^{u}{}^{q}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(m_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)i_{\sigma(j)}^{l}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-i_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)i_{\sigma(j)}^{l}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-i_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ , \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)i_{\sigma(j)}^{u}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-i_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)i_{\sigma(j)}^{u}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-i_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_{\sigma(j)}^{l}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_{\sigma(j)}^{l}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ , \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_{\sigma(j)}^{u}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_{\sigma(j)}^{u}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array}\right] \end{array}\right) \tag{14}$$

**Proof.** Trivial. □

**Remark 6.** *The IVTSFHOWG operator satisfies the three conditions of Idempotency, Monotonicity, and Boundedness defined in Theorem 3.*

The IVTSFHWG operator only evaluates the IVTSFNs, while the IVTSFHOWG operator only aggregates the ordered position of the IVTSFNs. When both ordered position and weight of the argument becomes important, the IVTSFHHG operator is as follows:

**Definition 11.** *Suppose that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHHG operator* $T^l \rightarrow T$ *is defined as:*

$$IVTSFHHG(\mathrm{T}_1,\ \mathrm{T}_2,\ \mathrm{T}_3,\ \ldots \mathrm{T}_j) = \sum_{j=1}^{l} \dot{\mathrm{T}}_{\sigma(j)}^{w_j} \tag{15}$$

*where* $\dot{\mathrm{T}}_{\sigma(j)}$ *is the* $j$th *largest of the IVTSFNs* $\dot{\mathrm{T}}_j = l\boldsymbol{w}_j \mathrm{T}_j$, *with* $\boldsymbol{w}_j$ *as the weight of interval-valued T-Spherical fuzzy arguments* $T_j$ *such that* $\boldsymbol{w}_j \in [0, 1]$ *and* $\sum_1^n \boldsymbol{w}_j = 1$ *and the balancing coefficient is denoted by* $l$.

**Theorem 8.** *Consider that* $T_j = \left(\left[m_j^l, m_j^u\right], \left[i_j^l, i_j^u\right], \left[n_j^l, n_j^u\right]\right)\ \forall\ j = 1, 2, 3, \ldots, l$ *are IVTSFNs. Then, the IVTSFHHG operator is an IVTSFN and given by:*

$$\begin{aligned} &IVTSFHHG(\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \ldots \mathrm{T}_l) \\ &= \left(\begin{array}{c} \left[\begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(\dot{m}_{\sigma(j)}^{l}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-\dot{m}_{\sigma(j)}^{l}{}^{q})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(\dot{m}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ ,\frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}\left(\dot{m}_{\sigma(j)}^{u}\right)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-\dot{m}_{\sigma(j)}^{u}{}^{q})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(\dot{m}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{i}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\dot{i}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{i}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\dot{i}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ ,\sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{i}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\dot{i}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{i}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\dot{i}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array}\right] \\ \left[\begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{n}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\dot{n}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{n}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\dot{n}_{\sigma(j)}^{l}{}^{q}\right)^{w_j}}} \\ ,\sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{n}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-\dot{n}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)\dot{n}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-\dot{n}_{\sigma(j)}^{u}{}^{q}\right)^{w_j}}} \end{array}\right] \end{array}\right) \end{aligned} \tag{16}$$

**Proof.** Trivial. □

**Remark 7.** *Equation (11) is reduced to the IVTSFHWG operator for* $w_j = \left(\frac{1}{l},\ \frac{1}{l},\ \frac{1}{l},\ \ldots\ \frac{1}{l}\right)^T$. *Then, it is reduced into the IVTSFHOWG operator for* $\boldsymbol{w}_j = \left(\frac{1}{l},\ \frac{1}{l},\ \frac{1}{l},\ \ldots\ \frac{1}{l}\right)^T$.

## 6. Special Cases

It can be noticed that the AOs defined for some existing fuzzy sets, such as IVIFSs, IVPyFSs, IVq-ROPFSs, IVPFSs, and IVSFSs, can be reduced from the proposed operators. It means that the earlier defined HAOs become special cases of the proposed IVTSFHWA and IVTSFHWG operators. Proposed IVTSFHWA and IVTSFHWG operators are defined as following, respectively:

$$IVTSFHWA(\text{Ꚍ}_1, \text{Ꚍ}_2, \text{Ꚍ}_3, \ldots \text{Ꚍ}_l)$$
$$= \left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l} (1+(\gamma-1)m_j^{lq})^{w_j} - \prod_{j=1}^{l} (1-m_j^{lq})^{w_j}}{\prod_{j=1}^{l} (1+(\gamma-1)m_j^{lq})^{w_j} + (\gamma-1)\prod_{j=1}^{l} (1-m_j^{lq})^{w_j}}}, \\ \sqrt[q]{\frac{\prod_{j=1}^{l} (1+(\gamma-1)m_j^{uq})^{w_j} - \prod_{j=1}^{l} (1-m_j^{uq})^{w_j}}{\prod_{j=1}^{l} (1+(\gamma-1)m_j^{uq})^{w_j} + (\gamma-1)\prod_{j=1}^{l} (1-m_j^{uq})^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} (i_j^{l})^{w_j}}{\sqrt[q]{\prod_{j=1}^{l} (1+(\gamma-1)(1-i_j^{lq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l} (i_j^{lq})^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} (i_j^{u})^{w_j}}{\sqrt[q]{\prod_{j=1}^{l} (1+(\gamma-1)(1-i_j^{uq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l} (i_j^{uq})^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} (n_j^{l})^{w_j}}{\sqrt[q]{\prod_{j=1}^{l} (1+(\gamma-1)(1-n_j^{lq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l} (n_j^{lq})^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} (n_j^{u})^{w_j}}{\sqrt[q]{\prod_{j=1}^{l} (1+(\gamma-1)(1-n_j^{uq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l} (n_j^{uq})^{w_j}}} \end{array} \right] \end{array} \right)$$

$$IVTSFHWG(\text{Ꚍ}_1, \text{Ꚍ}_2, \text{Ꚍ}_3, \ldots \text{Ꚍ}_l)$$
$$= \left( \begin{array}{c} \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} (m_j^{l})^{w_j}}{\sqrt[q]{\prod_{j=1}^{l} (1+(\gamma-1)(1-m_j^{lq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l} (m_j^{lq})^{w_j}}}, \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} (m_j^{u})^{w_j}}{\sqrt[q]{\prod_{j=1}^{l} (1+(\gamma-1)(1-m_j^{uq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l} (m_j^{uq})^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l} (1+(\gamma-1)i_j^{lq})^{w_j} - \prod_{j=1}^{l} (1-i_j^{lq})^{w_j}}{\prod_{j=1}^{l} (1+(\gamma-1)i_j^{lq})^{w_j} + (\gamma-1)\prod_{j=1}^{l} (1-i_j^{lq})^{w_j}}}, \\ \sqrt[q]{\frac{\prod_{j=1}^{l} (1+(\gamma-1)i_j^{uq})^{w_j} - \prod_{j=1}^{l} (1-i_j^{uq})^{w_j}}{\prod_{j=1}^{l} (1+(\gamma-1)i_j^{uq})^{w_j} + (\gamma-1)\prod_{j=1}^{l} (1-i_j^{uq})^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l} (1+(\gamma-1)n_j^{lq})^{w_j} - \prod_{j=1}^{l} (1-n_j^{lq})^{w_j}}{\prod_{j=1}^{l} (1+(\gamma-1)n_j^{lq})^{w_j} + (\gamma-1)\prod_{j=1}^{l} (1-n_j^{lq})^{w_j}}}, \\ \sqrt[q]{\frac{\prod_{j=1}^{l} (1+(\gamma-1)n_j^{uq})^{w_j} - \prod_{j=1}^{l} (1-n_j^{uq})^{w_j}}{\prod_{j=1}^{l} (1+(\gamma-1)n_j^{uq})^{w_j} + (\gamma-1)\prod_{j=1}^{l} (1-n_j^{uq})^{w_j}}} \end{array} \right] \end{array} \right)$$

1. If $m^l = m^u = m$, $i^l = i^u = i$ and $n^l = n^u = n$, then the IVTSFHWA and IVTSFHWG operators are converted into TSFHWA and TSFHWG, given as follows:

$$TSFHWA(\text{Ꚍ}_1, \text{Ꚍ}_2, \text{Ꚍ}_3, \ldots \text{Ꚍ}_l)$$
$$= \left( \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l} (1+(\gamma-1)m_j^{q})^{w_j} - \prod_{j=1}^{l} (1-m_j^{q})^{w_j}}{\prod_{j=1}^{l} (1+(\gamma-1)m_j^{q})^{w_j} + (\gamma-1)\prod_{j=1}^{l} (1-m_j^{q})^{w_j}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} i_j^{w_j}}{\sqrt[q]{\prod_{j=1}^{l} (1+(\gamma-1)(1-i_j^{q}))^{w_j} + (\gamma-1)\prod_{j=1}^{l} (i_j^{q})^{w_j}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} n_j^{w_j}}{\sqrt[q]{\prod_{j=1}^{l} (1+(\gamma-1)(1-n_j^{q}))^{w_j} + (\gamma-1)\prod_{j=1}^{l} (n_j^{q})^{w_j}}} \end{array} \right) \tag{17}$$

$$TSFHWG(\text{Ţ}_1,\text{Ţ}_2,\text{Ţ}_3,\ldots\text{Ţ}_l) = \begin{pmatrix} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}(m_j^{w_j})}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-m_j^q)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(m_j^q)^{w_j}}} \\ \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)i_j^q\right)^{w_j}-\prod_{j=1}^{l}\left(1-i_j^q\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)i_j^q\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-i_j^q\right)^{w_j}}} \\ \sqrt[q]{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^q\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^q\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^q\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^q\right)^{w_j}}} \end{pmatrix} \tag{18}$$

2. If $q = 2$, then aggregated operators (AOs) of the IVTSFHWA and IVTSFHWG are converted to IVSFHWA and IVSFWG, given as follows:

$$IVSFHWA(\text{Ţ}_1,\text{Ţ}_2,\text{Ţ}_3,\ldots\text{Ţ}_l) = \begin{pmatrix} \begin{bmatrix} \sqrt{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{l2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j^{l2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{l2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{l2}\right)^{w_j}}} \\ ,\sqrt{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{u2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j^{u2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{u2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{u2}\right)^{w_j}}} \end{bmatrix} \\ \begin{bmatrix} \frac{\sqrt{\gamma}\prod_{j=1}^{l}(i_j^l)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-i_j^{l2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(i_j^{l2})^{w_j}}} \\ ,\frac{\sqrt{\gamma}\prod_{j=1}^{l}(i_j^u)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-i_j^{u2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(i_j^{u2})^{w_j}}} \end{bmatrix} \\ \begin{bmatrix} \frac{\sqrt{\gamma}\prod_{j=1}^{l}(n_j^l)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-n_j^{l2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(n_j^{l2})^{w_j}}} \\ ,\frac{\sqrt{\gamma}\prod_{j=1}^{l}(n_j^u)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-n_j^{u2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(n_j^{u2})^{w_j}}} \end{bmatrix} \end{pmatrix} \tag{19}$$

$$IVSFHWG(\text{Ţ}_1,\text{Ţ}_2,\text{Ţ}_3,\ldots\text{Ţ}_l) = \begin{pmatrix} \begin{bmatrix} \frac{\sqrt{\gamma}\prod_{j=1}^{l}(m_j^l)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-m_j^{l2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(m_j^{l2})^{w_j}}} \\ ,\frac{\sqrt{\gamma}\prod_{j=1}^{l}(m_j^u)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-m_j^{u2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(m_j^{u2})^{w_j}}} \end{bmatrix} \\ \begin{bmatrix} \sqrt{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{l2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j^{l2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{l2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{l2}\right)^{w_j}}}, \\ \sqrt{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{u2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j^{u2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{u2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{u2}\right)^{w_j}}} \end{bmatrix} \\ \begin{bmatrix} \sqrt{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{l2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^{l2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{l2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^{l2}\right)^{w_j}}} \\ ,\sqrt{\frac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{u2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^{u2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{u2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^{u2}\right)^{w_j}}} \end{bmatrix} \end{pmatrix} \tag{20}$$

3. If $q = 2$, $m^l = m^u = m$, $i^l = i^u = i$ and $n^l = n^u = n$, then the IVTSFHWA and IVTSFHWG operators are converted into a spherical fuzzy environment.

$$SFHWA(\mathrm{Ƭ}_1, \mathrm{Ƭ}_2, \mathrm{Ƭ}_3, \ldots \mathrm{Ƭ}_l) = \begin{pmatrix} \sqrt{\frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j^2)^{w_j}-\prod_{j=1}^{l}(1-m_j^2)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j^2)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-m_j^2)^{w_j}}} \\ \frac{\sqrt{\gamma}\prod_{j=1}^{l} i_j^{w_j}}{\sqrt{\prod_{j=1}^{l}(1+(\gamma-1)(1-i_j^2))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(i_j^2)^{w_j}}} \\ \frac{\sqrt{\gamma}\prod_{j=1}^{l} n_j^{w_j}}{\sqrt{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j^2))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(n_j^2)^{w_j}}} \end{pmatrix} \tag{21}$$

$$SFHWG(\mathrm{Ƭ}_1, \mathrm{Ƭ}_2, \mathrm{Ƭ}_3, \ldots \mathrm{Ƭ}_l) = \begin{pmatrix} \frac{\sqrt{\gamma}\prod_{j=1}^{l}(m_j^{w_j})}{\sqrt{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j^2))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(m_j^2)^{w_j}}} \\ \sqrt{\frac{\prod_{j=1}^{l}(1+(\gamma-1)i_j^2)^{w_j}-\prod_{j=1}^{l}(1-i_j^2)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)i_j^2)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-i_j^2)^{w_j}}} \\ \sqrt{\frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j^2)^{w_j}-\prod_{j=1}^{l}(1-n_j^2)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j^2)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-n_j^2)^{w_j}}} \end{pmatrix} \tag{22}$$

4. If $q = 1$, then A the IVTSFHWA and IVTSFHWG are converted into interval-valued picture fuzzy settings and can be defined as:

$$IVPFHWA(\mathrm{Ƭ}_1, \mathrm{Ƭ}_2, \mathrm{Ƭ}_3, \ldots \mathrm{Ƭ}_l) = \begin{pmatrix} \begin{bmatrix} \frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j^l)^{w_j}-\prod_{j=1}^{l}(1-m_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j^l)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-m_j^l)^{w_j}}, \\ \frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j^u)^{w_j}-\prod_{j=1}^{l}(1-m_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j^u)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-m_j^u)^{w_j}} \end{bmatrix} \\ \begin{bmatrix} \frac{\gamma\prod_{j=1}^{l}(i_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-i_j^l))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(i_j^l)^{w_j}} \\ , \frac{\gamma\prod_{j=1}^{l}(i_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-i_j^u))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(i_j^u)^{w_j}} \end{bmatrix} \\ \begin{bmatrix} \frac{\gamma\prod_{j=1}^{l}(n_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j^l))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(n_j^l)^{w_j}} \\ , \frac{\gamma\prod_{j=1}^{l}(n_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j^u))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(n_j^u)^{w_j}} \end{bmatrix} \end{pmatrix} \tag{23}$$

$$IVPSFHWG(\mathrm{Ƭ}_1, \mathrm{Ƭ}_2, \mathrm{Ƭ}_3, \ldots \mathrm{Ƭ}_l) = \begin{pmatrix} \begin{bmatrix} \frac{\gamma\prod_{j=1}^{l}(m_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j^l))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(m_j^l)^{w_j}} \\ , \frac{\gamma\prod_{j=1}^{l}(m_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j^u))^{w_j}+(\gamma-1)\prod_{j=1}^{l}(m_j^u)^{w_j}} \end{bmatrix} \\ \begin{bmatrix} \frac{\prod_{j=1}^{l}(1+(\gamma-1)i_j^l)^{w_j}-\prod_{j=1}^{l}(1-i_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)i_j^l)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-i_j^l)^{w_j}}, \\ \frac{\prod_{j=1}^{l}(1+(\gamma-1)i_j^u)^{w_j}-\prod_{j=1}^{l}(1-i_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)i_j^u)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-i_j^u)^{w_j}} \end{bmatrix} \\ \begin{bmatrix} \frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j^l)^{w_j}-\prod_{j=1}^{l}(1-n_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j^l)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-n_j^l)^{w_j}}, \\ \frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j^u)^{w_j}-\prod_{j=1}^{l}(1-n_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j^u)^{w_j}+(\gamma-1)\prod_{j=1}^{l}(1-n_j^u)^{w_j}} \end{bmatrix} \end{pmatrix} \tag{24}$$

5. If $q = 1$ and $m^l = m^u = m$, $i^l = i^u = i$, $n^l = n^u = n$, then IVTSFHWA and IVTSFHWG are converted into picture fuzzy settings, given as follows:

$$PFHWA(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j)^{w_j} - \prod_{j=1}^{l}(1-m_j)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-m_j)^{w_j}} \\ \frac{\gamma \prod_{j=1}^{l} i_j^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-i_j))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(i_j)^{w_j}} \\ \frac{\gamma \prod_{j=1}^{l} n_j^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(n_j)^{w_j}} \end{array} \right) \tag{25}$$

$$PFHWG(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \frac{\gamma \prod_{j=1}^{l}(m_j^{w_j})}{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(m_j)^{w_j}} \\ \frac{\prod_{j=1}^{l}(1+(\gamma-1)i_j)^{w_j} - \prod_{j=1}^{l}(1-i_j)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)i_j)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-i_j)^{w_j}} \\ \frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j)^{w_j} - \prod_{j=1}^{l}(1-n_j)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-n_j)^{w_j}} \end{array} \right) \tag{26}$$

6. If $i^l = i^u = i = 0$, then IVTSFHWA and IVTSFHWG are converted into interval-valued q-ROPFSs, given as follows:

$$IVq-ROPFHWA(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j^{lq})^{w_j} - \prod_{j=1}^{l}(1-m_j^{lq})^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j^{lq})^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-m_j^{lq})^{w_j}}}, \\ \sqrt[q]{\frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j^{uq})^{w_j} - \prod_{j=1}^{l}(1-m_j^{uq})^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j^{uq})^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-m_j^{uq})^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}(n_j^l)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j^{lq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(n_j^{lq})^{w_j}}} \\ , \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}(n_j^u)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j^{uq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(n_j^{uq})^{w_j}}} \end{array} \right] \end{array} \right) \tag{27}$$

$$IVq-ROPFHWG(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \left[ \begin{array}{c} \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}(m_j^l)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j^{lq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(m_j^{lq})^{w_j}}} \\ , \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l}(m_j^u)^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j^{uq}))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(n_j^{uq})^{w_j}}} \end{array} \right] \\ \left[ \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j^{lq})^{w_j} - \prod_{j=1}^{l}(1-n_j^{lq})^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j^{lq})^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-n_j^{lq})^{w_j}}}, \\ \sqrt[q]{\frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j^{uq})^{w_j} - \prod_{j=1}^{l}(1-n_j^{uq})^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j^{uq})^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-n_j^{uq})^{w_j}}} \end{array} \right] \end{array} \right) \tag{28}$$

7. If $m^l = m^u = m$, $i^l = i^u = i = 0$ and $n^l = n^u = n$, then the IVTSFHWA and IVTSFHWG operators are converted into q-ring orthpair fuzzy layouts, given as follows:

$$q-ROPFHWA(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \sqrt[q]{\frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j^{q})^{w_j} - \prod_{j=1}^{l}(1-m_j^{q})^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j^{q})^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-m_j^{q})^{w_j}}} \\ \frac{\sqrt[q]{\gamma}\prod_{j=1}^{l} n_j^{w_j}}{\sqrt[q]{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j^{q}))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(n_j^{q})^{w_j}}} \end{array} \right) \tag{29}$$

$$
q-ROPFHWG(\text{Ƭ}_1,\text{Ƭ}_2,\text{Ƭ}_3,\ldots\text{Ƭ}_l)=
\begin{pmatrix}
\dfrac{\sqrt[q]{\gamma}\prod_{j=1}^{l}(m_j^{w_j})}{\sqrt[q]{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-m_j^{q})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(m_j^{q}\right)^{w_j}}} \\
\sqrt[q]{\dfrac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{q}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^{q}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^{q}\right)^{w_j}}}
\end{pmatrix}
\tag{30}
$$

8. If $q=2$ and $i^l=i^u=i=0$, then IVTSFHWA and IVTSFHWG are converted into interval-valued Pythagorean fuzzy layouts, given as follows:

$$
IVPyFHWA(\text{Ƭ}_1,\text{Ƭ}_2,\text{Ƭ}_3,\ldots\text{Ƭ}_l)=
\begin{pmatrix}
\begin{bmatrix}
\sqrt{\dfrac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{l2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j^{l2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{l2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{l2}\right)^{w_j}}}, \\
\sqrt{\dfrac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{u2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j^{u2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{u2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{u2}\right)^{w_j}}}
\end{bmatrix} \\
\begin{bmatrix}
\dfrac{\sqrt{\gamma}\prod_{j=1}^{l}\left(n_j^{l}\right)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-n_j^{l2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(n_j^{l2}\right)^{w_j}}} \\
,\dfrac{\sqrt{\gamma}\prod_{j=1}^{l}\left(n_j^{u}\right)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-n_j^{u2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(n_j^{u2}\right)^{w_j}}}
\end{bmatrix}
\end{pmatrix}
\tag{31}
$$

$$
IVPyFHWG(\text{Ƭ}_1,\text{Ƭ}_2,\text{Ƭ}_3,\ldots\text{Ƭ}_l)=
\begin{pmatrix}
\begin{bmatrix}
\dfrac{\sqrt{\gamma}\prod_{j=1}^{l}\left(m_j^{l}\right)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-m_j^{l2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(m_j^{l2}\right)^{w_j}}} \\
,\dfrac{\sqrt{\gamma}\prod_{j=1}^{l}\left(m_j^{u}\right)^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-m_j^{u2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(m_j^{u2}\right)^{w_j}}}
\end{bmatrix} \\
\begin{bmatrix}
\sqrt{\dfrac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{l2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^{l2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{l2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^{l2}\right)^{w_j}}} \\
,\sqrt{\dfrac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{u2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^{u2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{u2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^{u2}\right)^{w_j}}}
\end{bmatrix}
\end{pmatrix}
\tag{32}
$$

9. If $q=2$ and $m^l=m^u=m$, $i^l=i^u=0$, $n^l=n^u=n$, then IVTSFHWA and IVTSFHWG are converted into PyFSs, given as follows:

$$
PyFHWA(\text{Ƭ}_1,\text{Ƭ}_2,\text{Ƭ}_3,\ldots\text{Ƭ}_l)=
\begin{pmatrix}
\sqrt{\dfrac{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j^{2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)m_j^{2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-m_j^{2}\right)^{w_j}}} \\
\dfrac{\sqrt{\gamma}\prod_{j=1}^{l}n_j^{w_j}}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-n_j^{2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(n_j^{2}\right)^{w_j}}}
\end{pmatrix}
\tag{33}
$$

$$
PyFHWG(\text{Ƭ}_1,\text{Ƭ}_2,\text{Ƭ}_3,\ldots\text{Ƭ}_l)=
\begin{pmatrix}
\dfrac{\sqrt{\gamma}\prod_{j=1}^{l}(m_j^{w_j})}{\sqrt{\prod_{j=1}^{l}\left(1+(\gamma-1)(1-m_j^{2})\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(m_j^{2}\right)^{w_j}}} \\
\sqrt{\dfrac{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{2}\right)^{w_j}-\prod_{j=1}^{l}\left(1-n_j^{2}\right)^{w_j}}{\prod_{j=1}^{l}\left(1+(\gamma-1)n_j^{2}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{l}\left(1-n_j^{2}\right)^{w_j}}}
\end{pmatrix}
\tag{34}
$$

10. If $q = 1$ and, $i^l = i^u = i = 0$, then IVTSFHWA and IVTSFHWG are converted to interval-valued IFSs, given as follows:

$$IVIFHWA(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \left[ \begin{array}{c} \frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j^l)^{w_j} - \prod_{j=1}^{l}(1-m_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j^l)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-m_j^l)^{w_j}}, \\ \frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j^u)^{w_j} - \prod_{j=1}^{l}(1-m_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j^u)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-m_j^u)^{w_j}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\gamma\prod_{j=1}^{l}(n_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j^l))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(n_j^l)^{w_j}}, \\ \frac{\gamma\prod_{j=1}^{l}(n_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j^u))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(n_j^u)^{w_j}} \end{array} \right] \end{array} \right) \tag{35}$$

$$IVIFHWG(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \left[ \begin{array}{c} \frac{\gamma\prod_{j=1}^{l}(m_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j^l))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(m_j^l)^{w_j}} \\ , \frac{\gamma\prod_{j=1}^{l}(m_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j^u))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(m_j^u)^{w_j}} \end{array} \right] \\ \left[ \begin{array}{c} \frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j^l)^{w_j} - \prod_{j=1}^{l}(1-n_j^l)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j^l)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-n_j^l)^{w_j}}, \\ \frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j^u)^{w_j} - \prod_{j=1}^{l}(1-n_j^u)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j^u)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-n_j^u)^{w_j}} \end{array} \right] \end{array} \right) \tag{36}$$

11. If $q = 1$ and $m^l = m^u = m$, $i^l = i^u = 0$, $n^l = n^u = n$, then IVTSFHWA and IVTSFHWG are converted into intuitionistic fuzzy settings, given as follows.

$$IFHWA(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \frac{\prod_{j=1}^{l}(1+(\gamma-1)m_j)^{w_j} - \prod_{j=1}^{l}(1-m_j)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)m_j)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-m_j)^{w_j}} \\ \frac{\gamma\prod_{j=1}^{l}n_j^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)(1-n_j))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(n_j)^{w_j}} \end{array} \right) \tag{37}$$

$$IFHWG(\text{Ƭ}_1, \text{Ƭ}_2, \text{Ƭ}_3, \ldots \text{Ƭ}_l) = \left( \begin{array}{c} \frac{\gamma\prod_{j=1}^{l}(m_j^{w_j})}{\prod_{j=1}^{l}(1+(\gamma-1)(1-m_j))^{w_j} + (\gamma-1)\prod_{j=1}^{l}(m_j)^{w_j}} \\ \frac{\prod_{j=1}^{l}(1+(\gamma-1)n_j)^{w_j} - \prod_{j=1}^{l}(1-n_j)^{w_j}}{\prod_{j=1}^{l}(1+(\gamma-1)n_j)^{w_j} + (\gamma-1)\prod_{j=1}^{l}(1-n_j)^{w_j}} \end{array} \right) \tag{38}$$

## 7. Multi-Attribute Decision Making

The aim of this section is to utilize the HAOs of TSFSs defined in Sections 4 and 5 in a MADM problem. We propose an algorithm for MADM based on HAOs of IVTSFSs. A numerical example to demonstrate the applicability of the HAOs and the effect of parameters $\gamma$ and $q$ is also studied.

In MADM problems, the aim is to choose the most preferred alternative from a set of alternatives by AOs using score functions. The information used in this process is based on the human opinion that can be represented by IVTSFNs. IVTSFNs are enabled to discuss four aspects of evaluations provided by experts, including MD, NMD, AD, and RD, of an uncertain environment. Let $A = \{A_1, A_2, \ldots, A_k\}$ be a set of alternatives and a set of attributes $G = \{G_1, G_2, \ldots, G_j\}$ where $j$ is finite, with weight vector $w_j$. $D_{k\times j} = (T)_{k\times j} = \left(\left[m^l, m^u\right], \left[i^l, i^u\right], \left[n^l, n^u\right]\right)$ represents the decision matrix containing information on the alternatives concerning the attributes in the form of IVTSFNs. The algorithm of the MADM process based on HAOs is given below, followed by a flowchart in Figure 1.

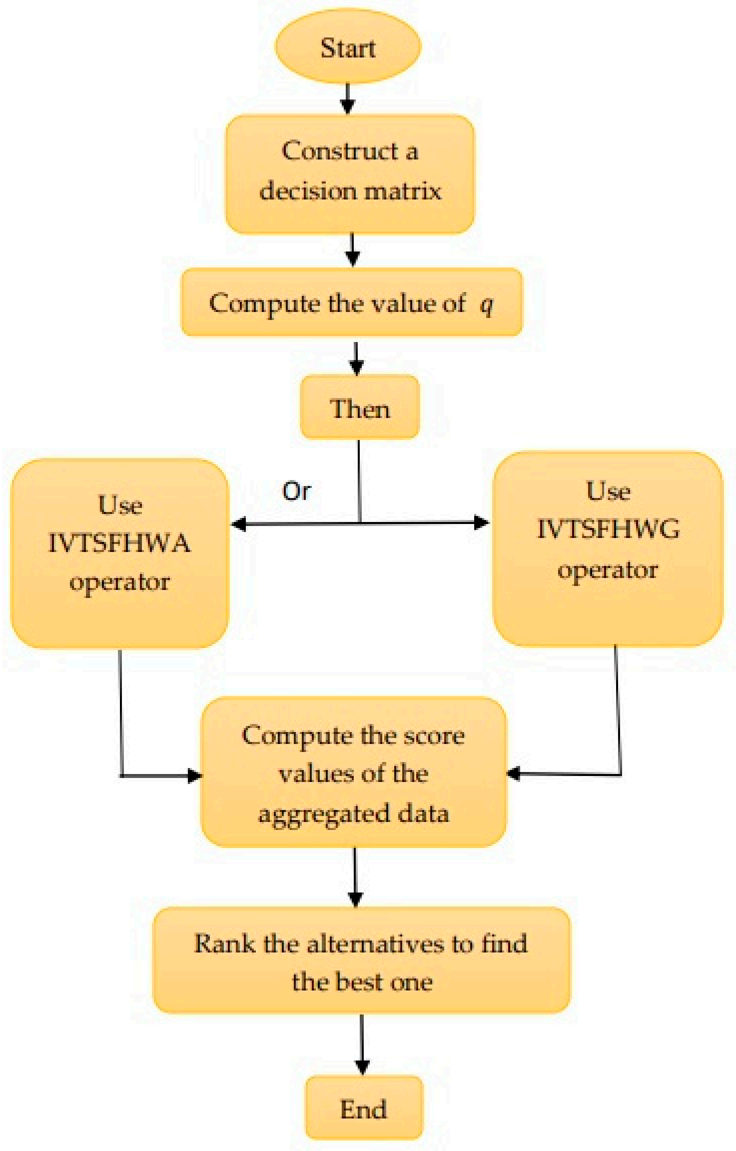


**Figure 1.** Flowchart of the MADM algorithm.

**Step 1.** A decision matrix is formed from the data gathered from the decision-makers about alternatives based on attributes. The $q$-values are also considered IVTSFNs.

**Step 2.** The decision matrix is aggregated using the IVTSFHWA and IVTSFHWG operators.

**Step 3.** The score values of aggregated IVTSFNs are calculated using

$$SC(I) = \frac{\left(m^l\right)^q\left(1-\left(i^l\right)^q-\left(n^l\right)^q\right)+(m^u)^q(1-(i^u)^q-(n^u)^q)}{3},\ SC(I) \in [-1,1]$$

**Step 4.** The score values of the alternative are examined to find the optimum one.

## 7.1. Numerical Example

This subsection aims to take a practical example for utilizing MADM based on the HAOs of IVTSFNs. We adapt an example from [33], where the selection of optimum enterprise is carried out using the MADM algorithm.

*In this example, we take the problem of evaluating enterprise financial performance, where we analyze some enterprises under some attributes to get the most optimum enterprise using the HAOs based on IVTSF information. The four possible enterprises denoted by $A_j$ $(1 \leq j \leq 4)$, according to four attributes, are denoted by $G_i$ $(1 \leq i \leq 4)$, where $G_1$ is the debt-paying ability, $G_2$ is the operation capability, $G_3$ is the earning capacity, and $G_4$ is the development capability. The four possible enterprises $(A_1,\ A_2,\ A_3,\ A_4)$ are to be evaluated using the IVTSFHWA and IVTSFHWG operators by the decision-maker under the four attributes with weights* $w_j = (0.2,\ 0.5,\ 0.25,\ 0.05)^T$.

The decision matrix is formed by IVTSFNs.

**Step 1.** The evaluations about enterprises are provided by the experts in Table 1. Note that in this problem, evaluations are represented by IVTSFNs for $q = 5$.

**Table 1.** Decision matrix.

| | $G_1$ | $G_2$ | $G_3$ | $G_4$ |
|---|---|---|---|---|
| $A_1$ | $\begin{pmatrix}[0.6,0.8],\\ [0.3,0.5],\\ [0.3,0.6]\end{pmatrix}$ | $\begin{pmatrix}[0.4,0.5],\\ [0.1,0.6],\\ [0.5,0.9]\end{pmatrix}$ | $\begin{pmatrix}[0.5,0.7],\\ [0.4,0.8],\\ [0.1,0.3]\end{pmatrix}$ | $\begin{pmatrix}[0.3,0.6],\\ [0.1,0.6],\\ [0.4,0.7]\end{pmatrix}$ |
| $A_2$ | $\begin{pmatrix}[0.7,0.9],\\ [0.1,0.8],\\ [0.3,0.4]\end{pmatrix}$ | $\begin{pmatrix}[0.3,0.6],\\ [0.3,0.5],\\ [0.4,0.8]\end{pmatrix}$ | $\begin{pmatrix}[0.1,0.6],\\ [0.1,0.5],\\ [0.1,0.9]\end{pmatrix}$ | $\begin{pmatrix}[0.4,0.6],\\ [0.2,0.5],\\ [0.5,0.6]\end{pmatrix}$ |
| $A_3$ | $\begin{pmatrix}[0.3,0.5],\\ [0.2,0.3],\\ [0.2,0.8]\end{pmatrix}$ | $\begin{pmatrix}[0.2,0.5],\\ [0.6,0.7],\\ [0.2,0.9]\end{pmatrix}$ | $\begin{pmatrix}[0.2,0.4],\\ [0.3,0.4],\\ [0.3,0.6]\end{pmatrix}$ | $\begin{pmatrix}[0.2,0.7],\\ [0.4,0.6],\\ [0.1,0.4]\end{pmatrix}$ |
| $A_4$ | $\begin{pmatrix}[0.2,0.4],\\ [0.1,0.3],\\ [0.7,0.9]\end{pmatrix}$ | $\begin{pmatrix}[0.7,0.8],\\ [0.3,0.5],\\ [0.1,0.2]\end{pmatrix}$ | $\begin{pmatrix}[0.5,0.8],\\ [0.4,0.9],\\ [0.2,0.4]\end{pmatrix}$ | $\begin{pmatrix}[0.4,0.7],\\ [0.5,0.6],\\ [0.3,0.7]\end{pmatrix}$ |

**Step 2.** The IVTSFHWA and IVTSFHWG operators are applied to get aggregated information, given as in Table 2. Note that while using IVTSFHWA and IVTSFHWG operators, we take $q = 5, \gamma = 2$ and $w = (0.2,\ 0.5,\ 0.25,\ 0.05)^T$.

**Table 2.** Aggregated values of IVTSFHWA and IVTSFHWG operators.

| | IVTSFHWA Operator | IVTSFHWG Operator |
|---|---|---|
| $A_1$ | $\begin{pmatrix}[0.491328,\ 0.664195],\\ [0.1762,\ 0.6243],\\ [0.2990,\ 0.63395]\end{pmatrix}$ | $\begin{pmatrix}[0.4526,\ 0.6061],\\ [0.005313,\ 0.230905],\\ [0.02895,\ 0.586139]\end{pmatrix}$ |
| $A_2$ | $\begin{pmatrix}[0.513349,\ 0.721466],\\ [0.179335,\ 0.5518],\\ [0.270219,\ 0.7183]\end{pmatrix}$ | $\begin{pmatrix}[0.274969,\ 0.6558],\\ [0.0022,\ 0.160593],\\ [0.012486,\ 0.568143]\end{pmatrix}$ |
| $A_3$ | $\begin{pmatrix}[0.236636\ ,\ 0.505216],\\ [0.3981\ ,\ 0.5128],\\ [0.213815\ ,\ 0.7738]\end{pmatrix}$ | $\begin{pmatrix}[0.21691,\ 0.4812],\\ [0.069814,\ 0.158957],\\ [0.001449,\ 0.684677]\end{pmatrix}$ |
| $A_4$ | $\begin{pmatrix}[0.621815\ ,\ 0.763604],\\ [0.265618,\ 0.5357],\\ [0.186066\ ,\ 0.3471]\end{pmatrix}$ | $\begin{pmatrix}[0.4904,\ 0.6991],\\ [0.009297,\ 0.316949],\\ [0.059314,\ 0.247336]\end{pmatrix}$ |

**Step 3.** The scores of the aggregated value of data in Table 2 are computed. The score values for each alternative are computed and shown in Table 3.

**Table 3.** Score values of IVTSFHWA and IVTSFHWG operators.

| | IVTSFHWA Operator | IVTSFHWG Operator |
|---|---|---|
| $A_1$ | 0.0439 | 0.03169 |
| $A_2$ | 0.0612 | 0.03856 |
| $A_3$ | | 0.0075 |
| $A_4$ | 0.1132 | 0.06489 |

**Step 4.** Alternatives are ranked using score values in decreasing order. The ranking is handled as $A_4 > A_2 > A_1 > A_3$. Thus, the best enterprise from different companies is $A_4$. The comparison between the results using the IVTSFHWA and IVTSFHWG operators is discussed in Table 3.

### *7.2. Effect of "$\gamma$" on Ranking of Alternatives*

Ullah et al. [41] observed a significant change in the ranking results while dealing with TSF HAOs for various values of $q$ and $\gamma$. Therefore, we examined the consequence of deviations in $\gamma$ on the ranking result. Hence, we solved the MADM problem discussed in Section 7.2 for different $\gamma$ values, and the effect on the ranking of the alternatives is given in Table 4.

**Table 4.** Impact of $\gamma$ on the ranking results.

| $\gamma$ | Operators | Score Values of IVTSFHWA Operator and IVTSFHWG Operator | Resulting Pattern |
|---|---|---|---|
| 2 | IVTSFHWA | $S_1 = 0.0439,\ S_2 = 0.0612,\ S_3 = 0.0078, S_4 = 0.1132$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0317, S_2 = 0.0386,\ S_3 = 0.0075,\ S_4 = 0.0649$ | $A_4 > A_2 > A_1 > A_3$ |
| 4 | IVTSFHWA | $S_1 = 0.0342,\ S_2 = 0.0456,\ S_3 = 0.0074,\ S_4 = 0.0824$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0336,\ S_2 = 0.0415,\ S_3 = 0.0084,\ S_4 = 0.0669$ | $A_4 > A_2 > A_1 > A_3$ |
| 5 | IVTSFHWA | $S_1 = 0.0334,\ S_2 = 0.04398,\ S_3 = 0.0073,\ S_4 = 0.0809$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0338,\ S_2 = 0.0418,\ S_3 = 0.0085,\ S_4 = 0.0673$ | $A_4 > A_2 > A_1 > A_3$ |
| 6 | IVTSFHWA | $S_1 = 0.0328,\ S_2 = 0.0428,\ S_3 = 0.0073,\ S_4 = 0.0798$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0339,\ S_2 = 0.0421,\ S_3 = 0.0085,\ S_4 = 0.0676$ | $A_4 > A_2 > A_1 > A_3$ |
| 8 | IVTSFHWA | $S_1 = 0.0319,\ S_2 = 0.0412,\ S_3 = 0.0072,\ S_4 = 0.0783$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0341,\ S_2 = 0.0424,\ S_3 = 0.0086,\ S_4 = 0.0679$ | $A_4 > A_2 > A_1 > A_3$ |
| 9 | IVTSFHWA | $S_1 = 0.0316,\ S_2 = 0.0406,\ S_3 = 0.0071,\ S_4 = 0.0777$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0341, S_2 = 0.0425, S_3 = 0.0086,\ S_4 = 0.0680$ | $A_1 > A_4 > A_3 > A_2$ |
| 12 | IVTSFHWA | $S_1 = 0.0306,\ S_2 = 0.0393,\ S_3 = 0.0070,\ S_4 = 0.0760$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.03424,\ S_2 = 0.0427,\ S_3 = 0.0086, S_4 = 0.0683$ | $A_1 > A_3 > A_4 > A_2$ |

From Table 4, we noticed that there is no significant change in the ranking results in the case of the IVTSFHWA operator and the IVTSFHWG operator for various values of $\gamma$. This whole scenario can be observed from Figure 2.

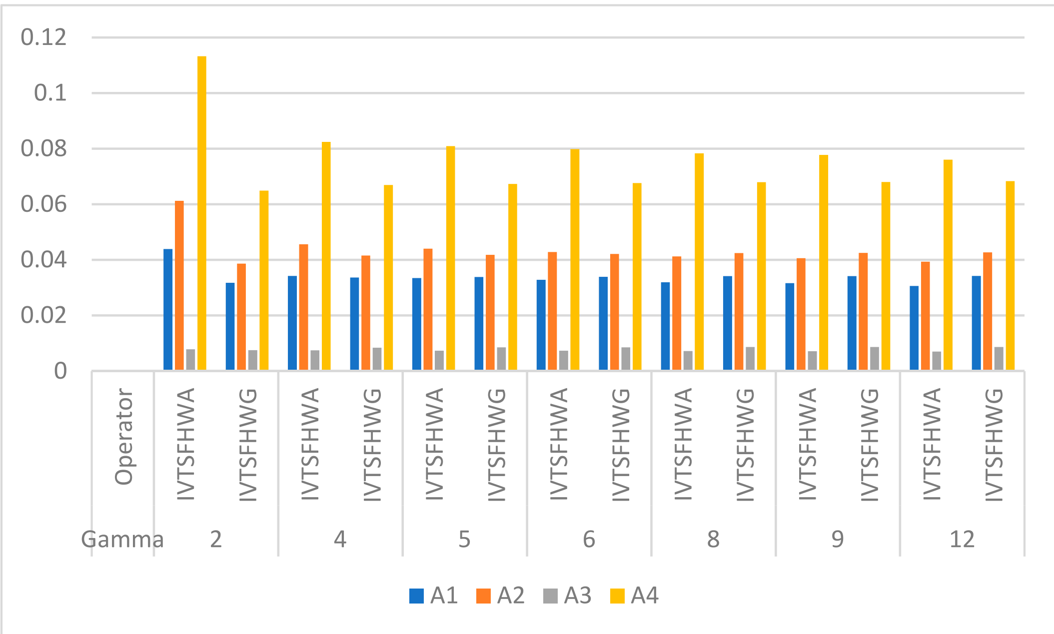


**Figure 2.** Graphical view of Table 4.

### 7.3. *Effect of Variations in "q" on Ranking Results*

As we have observed the impact of variations in "$\gamma$" on the ranking result in Section 7.3, here, we aim to analyze the effect of variations in "$q$" using IVTSFHWA and IVTSFHWG operators on ranking results. In the measured problem studied in Section 7.2, if we differentiate the values of "$q$" from 5 onward, then the variable ranking order of the given alternative is as displayed in Table 5 below.

**Table 5.** Ranking result for various values of "$q$" when $\gamma = 2$.

| $q$ | Operators | Score Values of IVTSFHWA and IVTSFHWG | Resulting Pattern |
|---|---|---|---|
| 5 | IVTSFHWA | $S_1 = 0.0439,\ S_2 = 0.0612,\ S_3 = 0.0078,\ S_4 = 0.1132$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0317,\ S_2 = 0.0386,\ S_3 = 0.0075,\ S_4 = 0.0649$ | $A_4 > A_2 > A_1 > A_3$ |
| 7 | IVTSFHWA | $S_1 = 0.0238,\ S_2 = 0.0429,\ S_3 = 0.0028,\ S_4 = 0.0679$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0111,\ S_2 = 0.0171,\ S_3 = 0.0020,\ S_4 = 0.0287$ | $A_4 > A_2 > A_1 > A_3$ |
| 9 | IVTSFHWA | $S_1 = 0.0134,\ S_2 = 0.0310,\ S_3 = 0.0010, S_4 = 0.0411$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0038,\ S_2 = 0.0072, S_3 = 0.00046,\ S_4 = 0.0130$ | $A_4 > A_2 > A_1 > A_3$ |
| 11 | IVTSFHWA | $S_1 = 0.0077,\ S_2 = 0.0233,\ S_3 = 0.0004,\ S_4 = 0.0251$ | $A_4 > A_2 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0013, S_2 = 0.0030, S_3 = 0.00011,\ S_4 = 0.0061$ | $A_4 > A_2 > A_1 > A_3$ |
| 12 | IVTSFHWA | $S_1 = 0.0059, S_2 = 0.0205, S_3 = 0.00028,\ S_4 = 0.0197$ | $A_2 > A_4 > A_1 > A_3$ |
| | IVTSFHWG | $S_1 = 0.0041, S_2 = 0.00198,\ S_3 = 0.0008,\ S_4 = 5.1E-05$ | $A_4 > A_2 > A_1 > A_3$ |

It can be noted that the ranking array is changed at $q = 12$ in the case of the IVTSFHWA operator, but the final ranking order does not change in the case of the IVTSFHWG operator. However, the ranking array does not change after $q = 12$ in both cases. This demonstrates the consistency in ranking consequences at $q = 12$. The whole scenario can be seen at a glance in Figure 3 below.

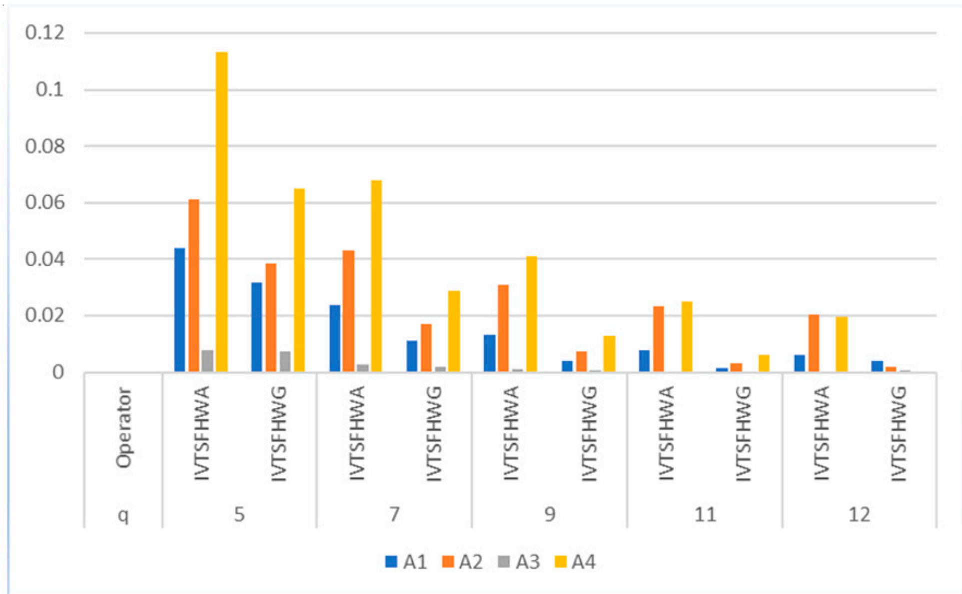


**Figure 3.** Graphical view of Table 5.

## 8. A Comparison of the Result Obtained Using Proposed and Existing Methods

The goal of this section is to set up a comparative analysis of the IVTSFHWA and IVTSFHWG operators with existing HAOs.

To examine the reliability and effectiveness of the presented approaches, we choose the information of the example studied in Section 7.1 and solve it by using some previously defined operators of IVTSFSs by Ullah et al. [25]. We further show the HAOs established in the environment of IFSs, PyFSs, q-ROPFSs, PFSs, and TSFSs cannot be applied to the problem where information is in the form of IVTSFNs. The relative analysis of the presented approaches with some current approaches is discussed in Table 6 below.

**Table 6.** Ranking of alternatives using existing and proposed methods.

| Method | Reference | Score Values | Ranking |
|---|---|---|---|
| **IVTSFWA** | Ullah et al. [25] | $S_1 = 0.0463,\ S_2 = 0.0683, S_3 = 0.0081, S_4 = 0.1164$ | $A_4 > A_2 > A_1 > A_3$ |
| **IVTSFWG** | Ullah et al. [25] | $S_1 = 0.0203, S_2 = 0.0239, S_3 = 0.0048,\ S_4 = 0.0503$ | $A_4 > A_2 > A_1 > A_3$ |
| **Proposed work WA** | This paper | $S_1 = 0.0439,\ S_2 = 0.0612,\ S_3 = 0.0078,\ S_4 = 0.1132$ | $A_4 > A_2 > A_1 > A_3$ |
| **Proposed work WG** | This paper | $S_1 = 0.0317,\ S_2 = 0.0386,\ S_3 = 0.0075,\ S_4 = 0.0649$ | $A_4 > A_2 > A_1 > A_3$ |
| **HAOs of IFSs** | Huang [33] | Failed | Cannot be specified |
| **HAOs of IVIFSs** | Liu [35] | Failed | Cannot be specified |
| **HAOs of PyFSs** | Gao [36] | Failed | Cannot be specified |
| **HAOs of IVPyFSs** | Peng and Yang [38] | Failed | Cannot be specified |
| **HAOs of q-ROPFSs** | Darko and Liang [39] | Failed | Cannot be specified |
| **HAOS of PFSs** | Jana & Pal [40] | Failed | Cannot be specified |
| **HAOs of TSFSs** | Ullah et al. [41] | Failed | Cannot be specified |

From the above analysis, we observed that the results obtained in this paper are compatible with that of previous work. Moreover, it is shown that the HAOs developed in Huang et al. [33], Liu [35], Gao [36], Peng and Yang [38], Darko and Liang [39], Jana and Pal [40], and Ullah et al. [41] cannot be applied to the problem. The comparison results are portrayed in Figure 4.

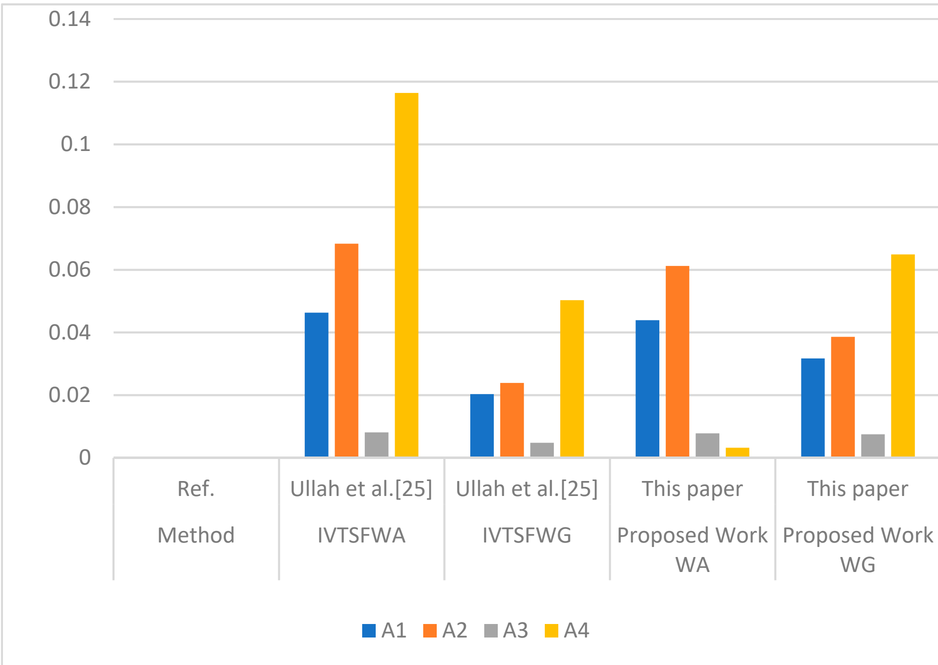


**Figure 4.** Interpretation of the information in Table 6.

## 9. Conclusions

In this paper, first, we discussed the importance of working in the setting of interval-valued fuzzy frameworks, as such fuzzy frameworks, which reduces the loss of information and ensures the effective modeling of human opinion. Based on this fact, we developed the notion of IVTSFHWA and IVTSFHWG operators that can aggregate the information given in the form of IVTSFNs. We exemplified each newly developed operator and studied its monotonicity, boundedness, and idempotency properties for newly defined HAOs. Some further study based on the HAOs of IVTSFNs is as follows:

1. To meet the situations where the ordered position and weights of the information matters, we proposed the IVTSFHOWA, IVTSFHHA, IVTSFHOWG, and IVTSFHHG operators.
2. We comprehensively studied the special cases of the newly developed HAOs.
3. A MADM algorithm based on the HAOs of IVTSFNs was produced and applied to the problem of the evaluation of the performance of enterprises.
4. The impact of parameters $q$ and $\gamma$ on the ranking pattern was analyzed and geometrically portrayed, where it was observed that severe fluctuations may occur by varying the values of $\gamma$ and $q$.
5. A comparative study of the newly developed HAOs and previously established HAOs was set up, where the advantage of using the proposed HAOs became prominent as all the existing HAOs failed to handle some situations without information loss.

The key advantage of the proposed HAOs is that they reduce information loss due to their ability to describe the information in terms of the closed subintervals of $[0, 1]$. Another advantage of the HAOs of IVTSFSs is that they describe the AD and RD of the information along with the MD and NMD, unlike the HAOs of IFSs, PyFSs, and QROFSs. However, these HAOs can be further generalized to the frame of complex TSFSs, so their ability to handle uncertain information would be increased.

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