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Controllability of Fuzzy Solutions for Neutral Impulsive Functional Differential Equations with Nonlocal Conditions

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Abstract: In this paper, the controllability of fuzzy solutions for first order nonlocal impulsive neutral functional differential equations is explored using the Banach fixed point theorem. We utilized the concepts of the fuzzy set theory, functional analysis, and the Hausdorff metric. In the conclusion, an illustration is given to bolster the hypothesis.

Keywords: controllability; fuzzy set; fuzzy number; neutral impulsive differential equation; fixed point; fuzzy solution; nonlocal conditions

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1. Introduction

The authors studied the controllability of fuzzy solutions for the following nonlocal functional differential equations with impulse

$$\left.\begin{aligned}\frac{d}{dt}[x(t)-h(t,x_t)] = Ax(t)+f(t,x_t)+u(t); t\in J=[0,T],\\ \Delta x(t_k)=I_k x(t_k^-), t\neq t_k, k=1,2,\ldots,m\\ x(t)+g\left(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_p}\right)(t)=\varphi(t), t\in[-r,0],\end{aligned}\right\} \quad (1)$$

where $A: J\to E^n$ is the fuzzy coefficient, E^n is the set of all upper semicontinuous, convex, and normal fuzzy numbers with bounded α-levels. The functions $f,h: J\times C([-r,0],E^n)\to E^n$ and $g:(C[-r,0],E^n)^p\to E^n$ are nonlinear regular fuzzy functions, $\varphi:[-r,0]\to E^n$. $u: J\to E^n$ is an admissible control function, and $I_k \epsilon C(E^n,E^n)$ are bounded functions.

$\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, represents the left and right limits of $x(t)$ at $t=t_k$, respectively, $k=1,2,\ldots,m$. $x(t_k^+)=\lim_{h\to 0^+} x(t_k+h)$ and $x(t_k^-)=\lim_{h\to 0^+} x(t_k-h)$. Moreover, $x_t(.)$ represents the history where $x_t(\theta)=x(t+\theta); \theta\epsilon[-r,0]$.

Let Ω be the space given by $\Omega=\{x|x: J\to E^n \text{ is continuous}\}$. In addition, there exists $x(t_k^+), x(t_k^-)$ where $k=1,2,\ldots,m$ with $(t_k^-)=x(t_k)$, $\Omega'=\Omega\cap \mathrm{C}(J,E^n)$.

The basis of fuzzy logic was put forward by Zadeh [1] in the year 1965. It is based on the principle that "everything in the world is unpredictable and unstable". This idea is further extended and used effectively in numerous fields of research, such as medicine, computer science, engineering, and economics, due to its outstanding problem-solving ability that was not solved through traditional logic. The use of fuzzy logic is applied to dynamic systems expressed in differential equations.

Dynamical systems in the real world are subject to all kinds of uncertainties, such as the growth of a population [2–5], contaminant migration in porous media [6], and the life cycle of a human [7]. To determine the current position of a particle from the history of its past movement, a dynamic system with a time delay can be applied effectively. In this paper, the main purpose was to investigate the fuzzy differential equation of a dynamic system constrained by time delay. The primary objective of this investigation was to establish the definitions and theorems on fuzzy control systems with time delay and to discover necessary conditions for the existence of a solution to this type of system by functional analysis. One of the most thoroughly studied classes of equations with distributed arguments is neutral differential equations. They occur naturally in applied problems that contain some recurrence property in their statement. The overview of fuzzy differential equations was made by Puri and Relescue (1986) [8] and further studied by Kaleva (1987) [9]. Impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given wide consideration, and were studied by Lakshminatham et al. (1989) [10]. Li and Kou (2009) proved the existence results for second-order impulsive neutral functional differential equations with nonlocal conditions using Sadovskii's fixed point theorem [11]. Impulsive functional differential equations of neutral type have been studied in [12,13]. However, in addition to impulsive effects, stochastic effects also exist in real systems.

The existence and uniqueness of a fuzzy solution for the nonlinear fuzzy neutral functional differential equation using the Banach fixed point theorem has been studied broadly by Balasubramaniam and Muralisankar (2001) [14]. Priyadharsini and Balasubramaniam (2020) proved the existence of the fuzzy fractional stochastic differential system with impulses using a granular derivative via the contraction principle [15]. Balachandran and Dauer (1997) showed the existence of solutions of perturbed fuzzy integral equations in Banach spaces using Darbo's fixed point theorem [16]. The existence and uniqueness for fuzzy impulsive functional differential equations have been studied by Vu H, Van Hao N (2016) under generalized Hukuhara differentiability, using contraction mapping [4]. Rivaz et al. (2017) introduced and defined a new metric on the space of fuzzy continuous functions in the fractional calculus. They established particular conditions that assured the existence and uniqueness of a solution to a nonlinear fuzzy fractional differential equation using the Banach fixed point theorem [17]. Some basic results on fuzzy differential equations can be found in [18–21].

Controllability has played a vital role in investigating and proposing control systems. It means the presence of a control function, which steers the solution of the system from its initial state to the desired concluding state. According to the best of our knowledge, there are only a few papers that deal with the controllability of fuzzy differential systems in the literature. Chalishajar, D. N., and Ramesh, R. (2019) developed controllability for impulsive fuzzy neutral functional integrodifferential equations [22]. Balachandran et al. (2000) proved the controllability of neutral functional integrodifferential systems in Banach spaces [23]. Narayanamoorthy et al. (2013) established the existence and controllability results for the nonlinear first order fuzzy neutral integrodifferential equations with nonlocal conditions [3]. Machado et al. (2013) investigated the controllability results for impulsive mixed-type functional integrodifferential evolution equations with nonlocal conditions using fixed point theory [13]. Radhakrishan et al. (2017) proved controllability results for nonlinear impulsive fuzzy neutral integrodifferential evolution systems [24]. Kumar et al. (2018) established the controllability of the second-order nonlinear differential equations with non-instantaneous impulses [25]. Arora et al. (2020) demonstrated the approximate controllability of semilinear impulsive functional differential systems with nonlocal conditions [26]. Motivated by the above papers, in this paper, we investigated the controllability results of fuzzy solutions of the nonlocal functional differential equations with impulse using the Banach fixed point theorem.

This paper is an extension of work [21] and proves the controllability results of fuzzy solutions of the nonlocal functional differential equations with impulse using the Banach fixed point theorem. In this study, Section 2 summarizes the fundamental aspects. The controllability results of the nonlocal fuzzy differential equation with impulse are proved in Section 3. In Section 4, an example has been illustrated to validate the theorem, and we conclude the results in Section 5.

2. Preliminaries

Definition 1. *Fuzzy Set*
A fuzzy set $A \subseteq X \neq \phi$ *is characterized by its membership function* $A: X \to [0,1]$ *and* $A(x)$ *is interpreted as the degree of membership of element* x *in fuzzy set* A *for each* $x \epsilon X$.

Definition 2. *[27] Let* $CC(\Re^n)$ *denote the family of all nonempty, compact, and convex subsets of* $\Re^n$. *Define addition and scalar multiplication in* $CC(\Re^n)$ *by*

$$A + B = \{z: z = x + y, x\epsilon A, y\epsilon B, \forall A, B\epsilon CC(\Re^n)\}$$

and

$$\lambda A = \{z: z = \lambda x, x\epsilon A, \forall A\epsilon CC(\Re^n)\}$$

Let $J = [a, b] \subset \Re$ be a compact interval and denote

$$E^n = \{v: \Re^n \to [0,1] \text{ such that } v \text{ satisfy } (1) - (4) \text{ as below: }\}$$

1. v is normal, that is, there exists an $x_0 \epsilon \Re^n$ such that $v(x_0) = 1$.
2. v is fuzzy convex, that is, for $x, z\epsilon\Re^n$ *and* $0 < \lambda \leq 1$, $v(\lambda x + (1 - \lambda)z) \geq min\big(v(x), v(z)\big)$.
3. v is upper semicontinuous, that is, $\forall \delta > 0$ such that $v(x) - v(x_0) < \varepsilon, |x - x_0| < \delta$.
4. $[v]^0 = \overline{\{x\epsilon\Re^n: v(x) > 0\}}$ is compact.

For, $0 < \lambda \leq 1$ we denote $[v]^\alpha = \{x\epsilon\Re^n: v(x) \geq \alpha\}$.
Then from (1) – (4), it follows that the α- level sets $[v]^\alpha \epsilon CC(\Re^n)$.
If $g: \Re^n \times \Re^n \to \Re^n$ is a function, then by using Zadeh's extension principle, we can extend g to $E^n \times E^n \to E^n$ by the equation

$$[g(v,w)(z)] = \sup_{z=g(x,y)} min\{v(x), w(y)\}.$$

It is already known that $[g(v,w)]^\alpha = g([v]^\alpha, [w]^\alpha)\ \forall v, w\epsilon E^n, 0 \le \alpha \le 1$ and g is a continuous function. Further we have

$$[v+w]^\alpha = [v]^\alpha + [w]^\alpha, [kv]^\alpha = k[v]^\alpha$$

where

$$v, w\epsilon E^n, k\epsilon\Re, 0 \le \alpha < 1.$$

Let A and B be two nonempty bounded subsets of $\Re^n$. The distance between A and B is defined by the Hausdorff metric

$$H_d(A,B) = max\left\{\sup_{a\in A}\inf_{b\in B}\left\|a-b\right\|, \sup_{b\in B}\inf_{a\in A}\left\|a-b\right\|\right\},$$

where $\| . \|$ denotes the usual Euclidean norm in $\Re^n$. Then, $(CC(\Re^n), H_d)$ is a complete and separable metric space [7].

Definition 3. *[13] The complete metric d_∞ on E^n is defined by*

$$d_\infty(v,w) = \sup_{0<\alpha\le 1} H_d([v]^\alpha, [w]^\alpha) = \sup_{0<\alpha\le 1} |v_l^\alpha - w_l^\alpha, v_r^\alpha - w_r^\alpha|$$

for any $v, w, z\epsilon E^n$, which satisfies $H_d(v+z, w+z) = H_d(v,w)$. Hence, (E^n, d_∞) is a complete metric space.

Definition 4. *[28] The supremum metric H_1on $C(J, E^n)$ is defined by*

$$H_1(v,w) = \sup_{0\le t\le T} d_\infty(v(t), w(t))$$

Hence, $(C(J, E^n), H_1)$ is a complete metric space.

Definition 5. *[28] The derivative $x'(t)$ of a fuzzy process $x\epsilon E^n$ is defined by*

$$[x'(t)]^\alpha = [(x_l^\alpha)'(t), (x_r^\alpha)'(t)]$$

provided that the equation defines a fuzzy set $x'(t)\epsilon E^n$.

Definition 6. *[28] The fuzzy integral $\int_a^b x(t)dt, a, b\epsilon[0,T]$ is defined by*

$$\left[\int_a^b x(t)dt\right]^\alpha = \left[\int_a^b x_l^\alpha(t)dt, \int_a^b x_r^\alpha(t)dt\right]$$

provided that the Lebesgue integrals on the right-hand side exist.

Definition 7. *[29] A mapping $f: J \to E^n$ is strongly measurable if, the set valued map $f_\alpha: J \to CC(\Re^n)$defined by $f_\alpha(t) = [f(t)]^\alpha$ is Lebesgue measurable when $CC(\Re^n)$ has the topology induced by the Hausdorff metric.*

Definition 8. *[29] A mapping $f: J \times E^n \to E^n$ is called level wise continuous at a point $(t_0, x_0)\epsilon J \times E^n$ provided, for any fixed $\alpha\epsilon[0,1]$ and arbitrary $\varepsilon > 0$, there exists a $\delta(\varepsilon, \alpha) > 0$, such that $H_d([f(t,x)]^\alpha, [f(t_0,x_0)]^\alpha < \varepsilon)$ whenever $|t - t_0| < \delta(\varepsilon, \alpha)$ and $H_d([x]^\alpha, [x_0]^\alpha) < \delta(\varepsilon, \alpha), \forall t\epsilon J, x\epsilon E^n$.*

Definition 9. *[29] A mapping $f: J \to E^n$ is called level wise continuous at $t_0\epsilon J$ if the multivalued map $f_\alpha(t) = [f(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric for all $\alpha\epsilon[0,1]$.*

A map $f: J \rightarrow E^n$ is said to be integrably bounded if there is an integrable function $h(t)$, such that $\|x(t)\| \leq h(t)$ for every $x(t) \epsilon f_0(t)$.

Definition 10. *[29] A strongly measurable and integrably bounded map $f: J \rightarrow E^n$ is said to be integrable over J, if $\int_0^T f(t)dt \, \epsilon E^n$. If $f: J \rightarrow E^n$ is strongly measurable and integrably bounded, then f is integrable.*

3. Controllability Results

Assumptions: Assume the following hypothesis.

H1. *$S(t)$ is a fuzzy number, where*

$$[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)], S(0) = I \text{ and } S_j^\alpha(t)(j = l, r) \text{ is continuous with}$$

$$|S_j(t)| \leq M, M > 0, |AS(t)| \leq M_1 \ \forall t \epsilon J = [0, T].$$

H2. *The nonlinear function $h: J \times E^n \rightarrow E^n$ is continuous and there exists a constant $d_1 > 0$, satisfying the global Lipschitz condition, such that*

$$H_d([h(t,x)]^\alpha, [h(t,y)]^\alpha) \leq d_1 H_d([x_t(\theta)]^\alpha, [y_t(\theta)]^\alpha); \ \forall t \epsilon J \text{ and } x, y \epsilon E^n.$$

H3. *If g is continuous and there exists constants $G_k, k = 1, 2, \ldots, p$, such that*

$$H_d\left(\left[g\left(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_p}\right)(s)\right]^\alpha, \left[g\left(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_p}\right)(s)\right]^\alpha\right)$$

$$\leq \sum_{k=1}^{p} G_k \, H_d\left(\left[x_{\tau_k}(s)\right]^\alpha, \left[y_{\tau_k}(s)\right]^\alpha\right), \forall s \epsilon [-r, 0]$$

and all

$$x_{\tau_k}, y_{\tau_k} \epsilon C([-r, 0], E^n), k = 1, 2, \ldots, p.$$

H4. *There exists a non-negative d_k, such that*

$H_d([I_k(x(t_k^-))]^\alpha, [I_k(y(t_k^-))]^\alpha) \leq \ d_k H_d([x(t)]^\alpha, [y(t)]^\alpha), k = 1, 2, \ldots, m$

and for each $x, y \epsilon E^n$.

H5. *The nonlinear function $f: J \times E^n \rightarrow E^n$ is continuous and there exists a constant $d_2 > 0$, satisfying the global Lipschitz condition, such that*

$$H_d([f(t,x)]^\alpha, [f(t,y)]^\alpha) \leq d_2 H_d([x_t(\theta)]^\alpha, [y_t(\theta)]^\alpha) \forall t \epsilon J \text{ and } x, y \epsilon E^n.$$

H6.

$$d_1(1 + MM_1T + M) + d_2(MM_1T + M^2T) + (1 + MT)\sum_{k=1}^{p} G_k + M(M+1)d_k + (M^2 M_1 T) < 1.$$

Definition 11. *If $x(t)$ is an integral solution of the problem (1), then $x(t)$ is given by*

$$x(t) = S(t)\left[\varphi(0) - g\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0) - h(0,\varphi)\right] + h(t,x_t) + \int_0^t AS(t-S)h(s,x_s)ds + \int_0^t S(t-S)f(s,x_s)ds + \int_0^t S(t-S)u(s)ds + \sum_{0<t_k<t} S(t-t_k)\, I_k x(t_k^-) \quad (2)$$

where $t\epsilon J, t \neq t_k, k = 1,2,\dots,m$ *and* $S(t)$ *satisfies H1.*

Theorem 1. *[22] The nonlocal problem (1) is said to be controllable on the interval* J *if there exists a control* $u(t)$, *such that the fuzzy solution* $x(t)$ *for (2) is controllable and satisfies*

$$x[T] = x^1 \; ie., [x(T)]^\alpha = [x^1]^\alpha$$

where $x^1 \epsilon E^n$.

Proof: Now define the α −level set of fuzzy mapping $\tilde{G}: \tilde{P}(R) \to E^n$ by

$$\tilde{G}^\alpha(v) = \begin{cases} \int_0^T S^\alpha(T-S)\, v(s)ds; v \subset \widetilde{\Gamma_u} \\ 0 \quad ; otherwise \end{cases}$$

where $\widetilde{\Gamma_u}$ is the closure of support u. In [26], the support Γ_u of a fuzzy number u is defined as a special case of the level set by $\Gamma_u = \{x: \mu_u(x) > 0\}$.

Then, there exists $\tilde{G_j}^\alpha (j = l, r)$, such that

$$\widetilde{G_l^\alpha}(v_l) = \int_0^T S_l^\alpha(T-S)v_l(s)ds, v_l(s)\epsilon[u_l^\alpha(s), u^1(s)],$$

$$\widetilde{G_r^\alpha}(v_r) = \int_0^T S_r^\alpha(T-S)v_r(s)ds, v_r(s)\epsilon[u^1(s), u_r^\alpha(s)],$$

Consider that $\widetilde{G_l^\alpha}, \widetilde{G_r^\alpha}$ are bijective mapping. Introduce an α −level set of *u(s)* for (2) and we get

$$[u(s)]^\alpha = [u_l^\alpha(s), u_r^\alpha(s)]$$

$$\left(\widetilde{G_l^\alpha}\right)^{-1}\Bigg((x^1)_l^\alpha - S_l^\alpha(T)\left[\varphi_l^\alpha(0) - g_l^\alpha\left(x_{\tau_1}, x_{\tau_2}, \dots\dots, x_{\tau_p}\right)(0) - h_l^\alpha(0,\varphi)\right] - h_l^\alpha\left(T, x_{Tl}{}^\alpha\right) - \int_0^T A_l^\alpha S_l^\alpha(T-S)h_l^\alpha\left(s, x_{sl}{}^\alpha\right)ds - \int_0^T S_l^\alpha(T-S)f_l^\alpha\left(s, x_{sl}{}^\alpha\right)ds - \sum_{0\prec t_k \prec T} S_l^\alpha(T-t_k)I_{kl}\left(x_l^\alpha(t_k^-)\right)\Bigg),$$

$$\left(\widetilde{G_r^\alpha}\right)^{-1}\Bigg((x^1)_r^\alpha - S_r^\alpha(T)\left[\varphi_r^\alpha(0) - g_r^\alpha\left(x_{\tau_1}, x_{\tau_2}, \dots\dots, x_{\tau_p}\right)(0) - h_r^\alpha(0,\varphi)\right] - h_r^\alpha\left(T, x_{Tr}{}^\alpha\right) - \int_0^T A_r^\alpha S_r^\alpha(T-S)h_r^\alpha\left(s, x_{sr}{}^\alpha\right)ds - \int_0^T S_r^\alpha(T-S)f_r^\alpha\left(s, x_{sr}{}^\alpha\right)ds - \sum_{0\prec t_k \prec T} S_r^\alpha(T-t_k)I_{kr}\left(x_r^\alpha(t_k^-)\right)\Bigg)$$

Substituting this in equation (2), we get an α −level set of $x(T)$ as

$$[x(T)]^\alpha =$$

$\Big[S_l^\alpha(T)\left[\varphi_l^\alpha(0) - g_l^\alpha\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0) - h_l^\alpha(0,\varphi)\right] + h_l^\alpha\left(T, x_{Tl}{}^\alpha\right) + \int_0^T A_l^\alpha S_l^\alpha(T-$
$S)h_l^\alpha\left(s, x_{sl}{}^\alpha\right)ds + \int_0^T S_l^\alpha(T-S)f_l^\alpha\left(s, x_{sl}{}^\alpha\right)ds + \sum_{0<t_k<T} S_l^\alpha(T-t_k)\, I_{kl}\left(x_l^\alpha(t_k^-)\right) +$

$$\int_0^T S_l^\alpha(T-S)\left(\widetilde{G_l^\alpha}\right)^{-1}\left((x^1)_l^\alpha - S_l^\alpha(T)\left[\varphi_l^\alpha(0) - g_l^\alpha\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0) - h_l^\alpha(0,\varphi)\right] - h_l^\alpha\left(T, x_{Tl}{}^\alpha\right) - \int_0^T A_l^\alpha S_l^\alpha(T-S) h_l^\alpha\left(s, x_{sl}{}^\alpha\right) ds - \int_0^T S_l^\alpha(T-S) f_l{}^\alpha\left(s, x_{sl}{}^\alpha\right) ds - \sum_{0<t_k<T} S_l^\alpha(T-t_k)\, I_{kl}\left(x_l^\alpha(t_k^-)\right)\right)\Big]\, ds,$$

$$\Bigg[S_r^\alpha(T)\left[\varphi_r^\alpha(0) - g_r^\alpha\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0) - h_r^\alpha(0,\varphi)\right] + h_r^\alpha\left(T, x_{Tr}{}^\alpha\right) + \int_0^T A_r^\alpha S_r^\alpha(T-S) h_r^\alpha\left(s, x_{sr}{}^\alpha\right) ds$$
$$+ \int_0^T S_r^\alpha(T-S) f_r^\alpha\left(s, x_{sr}{}^\alpha\right) ds + \sum_{0<t_k<T} S_r^\alpha(T-t_k)\, I_{kr}\left(x_r^\alpha(t_k^-)\right)$$
$$+ \int_0^T S_r^\alpha(T-S)\left(\widetilde{G_r^\alpha}\right)^{-1}\Bigg((x^1)_r^\alpha - S_r^\alpha(T)\left[\varphi_r^\alpha(0) - g_r^\alpha\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0) - h_r^\alpha(0,\varphi)\right]$$
$$- h_r^\alpha\left(T, x_{Tr}{}^\alpha\right) - \int_0^T A_r^\alpha S_r^\alpha(T-S) h_r^\alpha\left(s, x_{sr}{}^\alpha\right) ds - \int_0^T S_r^\alpha(T-S) f_r^\alpha\left(s, x_{sr}{}^\alpha\right) ds$$
$$- \sum_{0<t_k<T} S_r^\alpha(T-t_k)\, I_{kr}\left(x_r^\alpha(t_k^-)\right)\Bigg)\Bigg] ds$$

$$= \left[(x^1)_l^\alpha, (x^1)_r^\alpha\right] = [x^1]^\alpha.$$

Hence, the fuzzy solution $x(t)$ for equation (2) satisfies $[x(T)]^\alpha = [x^1]^\alpha$. □

Define

$$\Phi\left(x(t)\right) = S(t)\left[\varphi(0) - g\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0) - h(0,\varphi)\right] + h(t, x_t) + \int_0^t AS(t-S) h(s, x_s) ds + \int_0^t S(t-S) f(s, x_s) ds + \sum_{0<t_k<t} S(t-t_k)\, I_k\left(x(t_k^-)\right) + \int_0^t S(t-S)\left(\tilde{G}\right)^{-1}\left((x^1) - S(T)\left[\varphi(0) - g\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0) - h(0,\varphi)\right] - h(T, x_T) - \int_0^T AS(T-S) h(s, x_s) ds - \int_0^T S(T-S) f(s, x_s) ds - \sum_{0<t_k<T} S(T-t_k)\, I_k\left(x(t_k^-)\right)\right) ds, \quad (3)$$

where $\left(\tilde{G}\right)^{-1}$ satisfies the previous statements.

Observe $\Phi\left(x(t)\right) = [x^1]$, which represents that the control $u(t)$ steers condition (3) from the arbitrary stage to x^1 in time T, given that there must exist a fixed point of the nonlinear operator Φ.

Similarly,

$$\Phi\big(y(t)\big) = S(t)\left[\varphi(0) - g\left(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_p}\right)(0) - h(0,\varphi)\right] + h(t, y_t)$$
$$+ \int_0^t AS(t-S)h(s, y_s)ds$$
$$+ \int_0^t S(t-S)f(s, y_s)ds + \sum_{0<t_k<t} S(t-t_k)\, I_k(y(t_k^-))$$
$$+ \int_0^t S(t-S)\left(\tilde{G}\right)^{-1}\left((y^1) - S(T)\left[\varphi(0) - g\left(y_{\tau_1,}y_{\tau_2,} \dots \dots, y_{\tau_p}\right)(0)\right.\right.$$
$$\left.-h(0,\varphi)\right] - h(T, y_T) - \int_0^T AS(T-S)h(s, y_s)ds - \int_0^T S(T-S)f(s, y_s)ds$$
$$\left. - \sum_{0<t_k<T} S(T-t_k)\, I_k\big(y(t_k^-)\big)\right) ds.$$

The controllability of fuzzy solutions for the neutral impulsive functional differential equation with nonlocal conditions is discussed in the following theorem.

Theorem 2. *[22] Equation (3) is controllable if the hypothesis (H1–H6) is satisfied.*

Proof: For $x, y\epsilon\Omega'$

$$\leq H_d([h(t,x_t)]^\alpha, [h(t,y_t)]^\alpha) + H_d\left(\left[g\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0)\right]^\alpha, \left[g\left(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_p}\right)(0)\right]^\alpha\right)$$
$$+ H_d\left(\left[\int_0^t AS(t-S)h(s, x_s)ds\right]^\alpha, \left[\int_0^t AS(t-S)h(s, y_s)ds\right]^\alpha\right)$$
$$+ H_d\left(\left[\int_0^t S(t-S)f(s, x_s)ds\right]^\alpha, \left[\int_0^t S(t-S)f(s, y_s)ds\right]^\alpha\right)$$
$$+ H_d\left(\left[\int_0^t S(t-S)\left(\tilde{G}\right)^{-1}\left((x^1) - S(T)\left[\varphi(0) - g\left(x_{\tau_1}, x_{\tau_2}, \dots, x_{\tau_p}\right)(0) - h(0,\varphi)\right] - h(T, x_T)\right.\right.\right.$$
$$\left.\left. - \int_0^T AS(T-S)h(s, x_s)ds - \int_0^T S(T-S)f(s, x_s)ds - \sum_{0<t_k<T} S(T-t_k)\, I_k\big(x(t_k^-)\big)\right) ds\right]^\alpha, \left[\int_0^t S(t\right.$$
$$- S)\left(\tilde{G}\right)^{-1}\left((y^1) - S(T)\left[\varphi(0) - g\left(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_p}\right)(0) - h(0,\varphi)\right] - h(T, y_T)\right.$$
$$\left.\left.\left. - \int_0^T AS(T-S)h(s, y_s)ds - \int_0^T S(T-S)f(s, y_s)ds - \sum_{0<t_k<T} S(T-t_k)\, I_k\big(y(t_k^-)\big)\right) ds\right]^\alpha\right)$$

$$\leq d_1 H_d([x(t+\theta)]^\alpha,[y(t+\theta)]^\alpha)+\sum_{k=1}^{p} G_k\, H_d\left(\left[x_{\tau_k}(0)\right]^\alpha,\left[y_{\tau_k}(0)\right]^\alpha\right)$$

$$+\int_0^t MM_1\, d_1 H_d([x(s+\theta)]^\alpha,[y(s+\theta)]^\alpha)ds+\int_0^t MM_1\, d_2 H_d([x(t+\theta)]^\alpha,[y(t+\theta)]^\alpha)ds$$

$$+ M d_k H_d([x(t)]^\alpha,[y(t)]^\alpha)$$

$$+ M\int_0^t\left\{\left(\sum_{k=1}^{p} G_k\, H_d\left(\left[x_{\tau_k}(0)\right]^\alpha,\left[y_{\tau_k}(0)\right]^\alpha\right)\right)+d_1 H_d([x(T+\theta)]^\alpha,[y(T+\theta)]^\alpha)\right.$$

$$+\int_0^T MM_1\, d_1 H_d([x(s+\theta)]^\alpha,[y(s+\theta)]^\alpha)ds+\int_0^T M\, d_2 H_d([x(T+\theta)]^\alpha,[y(T+\theta)]^\alpha)ds$$

$$\left.+ M d_k H_d([x(s)]^\alpha,[y(s)]^\alpha)\right\}ds$$

Therefore,

$$d_\infty\left(\Phi x(t),\Phi y(t)\right)=\sup_{0<\alpha\leq 1} H_d([\Phi x(t)]^\alpha,[\Phi y(t)]^\alpha)$$

$$\leq d_1 \sup_{0<\alpha\leq 1} H_d([x(t+\theta)]^\alpha, [y(t+\theta)]^\alpha) + \sum_{k=1}^{p} G_k \sup_{0<\alpha\leq 1} H_d\left(\left[x_{\tau_k}(0)\right]^\alpha, \left[y_{\tau_k}(0)\right]^\alpha\right)$$

$$+ \int_0^t MM_1\, d_1 \sup_{0<\alpha\leq 1} H_d([x(s+\theta)]^\alpha, [y(s+\theta)^\alpha])ds$$

$$+ \int_0^t MM_1\, d_2 \sup_{0<\alpha\leq 1} H_d([x(t+\theta)]^\alpha, [y(t+\theta)]^\alpha)ds$$

$$+ Md_k \sup_{0<\alpha\leq 1} H_d([x(t)]^\alpha, [y(t)]^\alpha)$$

$$+ M\int_0^T \left\{ \left(\sum_{k=1}^{p} G_k \sup_{0<\alpha\leq 1} H_d\left(\left[x_{\tau_k}(0)\right]^\alpha, \left[y_{\tau_k}(0)\right]^\alpha\right) \right) \right.$$

$$+ d_1 \sup_{0<\alpha\leq 1} H_d([x(T+\theta)]^\alpha, [y(T+\theta)]^\alpha)$$

$$+ \int_0^T MM_1\, d_1 \sup_{0<\alpha\leq 1} H_d([x(s+\theta)]^\alpha, [y(s+\theta)]^\alpha)ds$$

$$+ \int_0^T M\, d_2 \sup_{0<\alpha\leq 1} H_d([x(T+\theta)]^\alpha, [y(T+\theta)]^\alpha)ds$$

$$\left. + Md_k \sup_{0<\alpha\leq 1} H_d([x(s)]^\alpha, [y(s)]^\alpha) \right\} ds$$

$$\leq d_1 d_\infty([x(t+\theta)], [y(t+\theta)]) + \sum_{k=1}^{p} G_k d_\infty\left(x_{\tau_k}, y_{\tau_k}\right) + MM_1 \int_0^t d_1\, d_\infty([x(s+\theta)], [y(s+\theta)])ds$$

$$+ \int_0^t MM_1\, d_2 d_\infty([x(t+\theta)], [y(t+\theta)])ds + Md_k d_\infty([x(t)], [y(t)])$$

$$+ M\int_0^T \left\{ \left(\sum_{k=1}^{p} G_k d_\infty\left([x(\tau_k)], [y(\tau_k)]\right) \right) ds + d_1 d_\infty([x(T+\theta)], [y(T+\theta)]) \right.$$

$$+ \int_0^T MM_1\, d_1 d_\infty([x(s+\theta)], [y(s+\theta)])ds + \int_0^T M\, d_2 d_\infty([x(T+\theta)], [y(T+\theta)])ds$$

$$\left. + Md_k d_\infty([x(s)], [y(s)]) \right\} ds$$

Hence, $H_1\big(\Phi(x),\Phi(y)\big) = \sup_{0\le t\le T} d_\infty(\Phi(x(t)),(\Phi y(t)))$

$$\le d_1 \sup_{0\le t\le T} d_\infty([x(t+\theta)],[y(t+\theta)]) + \sum_{k=1}^{p} G_k \sup_{0\le t\le T} d_\infty\left(x_{\tau_k}, y_{\tau_k}\right) + MM_1\int_0^t d_1 \sup_{0\le t\le T} d_\infty([x(s+\theta)],[y(s+\theta)])ds + MM_1\int_0^t d_2 \sup_{0\le t\le T} d_\infty([x(t+\theta)],[y(t+\theta)])ds + Md_k \sup_{0\le t\le T} d_\infty([x(t)],[y(t)]) + M\int_0^T\left\{\left(\sum_{k=1}^{p} G_k \sup_{0\le t\le T} d_\infty([x(\tau_k)],[y(\tau_k)])\right) + d_1 \sup_{0\le t\le T} d_\infty([x(T+\theta)],[y(T+\theta)]) + \int_0^T MM_1\, d_1\left(\sup_{0\le t\le T} d_\infty[x(s+\theta)],[y(s+\theta)]\right)ds + \int_0^T M\, d_2 \sup_{0\le t\le T} d_\infty([x(T+\theta)],[y(T+\theta)])ds + Md_k \sup_{0\le t\le T} d_\infty([x(s)],[y(s)])\right\} ds$$

$$\le d_1H_1(x,y) + \sum_{k=1}^{p} G_kH_1(x,y) + MM_1\int_0^t d_1\, H_1(x,y)ds + MM_1\int_0^t d_2\, H_1(x,y)ds + Md_kH_1(x,y) + M\int_0^T\left\{\left(\sum_{k=1}^{p} G_kH_1(x,y)\right)ds + d_1H_1(x,y) + \int_0^T MM_1\, d_1H_1(x,y)ds + \int_0^T M\, d_2H_1(x,y)ds + Md_kH_1(x,y)\right\} ds$$

$$\le \Big(d_1(1+MM_1T+M) + d_2(MM_1T+M^2T) + (1+MT)\sum_{k=1}^{p} G_k + M(M+1)d_k + (\,M^2M_1T)\Big)H_1(x,y).$$

Hence, Φ is a contraction mapping. By applying the Banach fixed point theorem, Equation (3) has a unique fixed point $x\epsilon\Omega'$. ☐

4. Examples

Example 1. *In the study, the fuzzy solution of nonlinear fuzzy neutral impulsive functional differential equations with nonlocal conditions of the form [24]*

$$\frac{d}{dt}[x(t) - 2tx(t+h)^2] = 2[x(t)] + 3tx(t+h)^2 + u(t)$$
$$x(0) + \sum_{k=1}^{p} c_kx(t_k) = 0\epsilon E^n$$
$$I_k\big(x(t_k^-)\big) = \frac{1}{1+x(t_k)},$$

where x^1 *is the target set, and the* α*-level set of fuzzy numbers 0, 2, 3 are given by*

$$[0]^\alpha = [\alpha-1, 1-\alpha], [2]^\alpha = [\alpha+1, 3-\alpha], [3]^\alpha = [\alpha+2, 4-\alpha], \text{for } \alpha\epsilon[0,1]$$
$$f(t,x_t) = 3tx(t+h)^2, h(t,x_t) = 2tx(t+h)^2$$

then an $\alpha - level\ set\ of\ g(x) = \sum_{k=1}^{p} c_kx(t_k)$ *is*

$$[g(x)]^{\alpha} = \left[\sum_{k=1}^{p} c_k x(t_k)\right]^{\alpha} = \left[\sum_{k=1}^{p} c_k x_l^{\alpha}(t_k)\ ,\sum_{k=1}^{p} c_k x_r^{\alpha}(t_k)\right]\ and$$

$$H_d([g(x)]^{\alpha},[g(y)]^{\alpha}) = H_d\left(\left[\sum_{k=1}^{p} c_k x(t_k)\right]^{\alpha},\left[\sum_{k=1}^{p} c_k y(t_k)\right]^{\alpha}\right) \le \sum_{k=1}^{p} G_k H_d\left(\left[x_{t_k}(s)\right]^{\alpha},\left[y_{t_k}(s)\right]^{\alpha}\right)$$

Similarly an, $\alpha -$ level set of $f(t,x_t)$, $h(t,x_t)$ and $I_k\left(x(\mathrm{t_k})\right)$ is

$$[f(t,x_t)]^{\alpha} = [3tx(t+h)^2]^{\alpha} = t[(\alpha+2)(x_l^{\alpha}(t+h)^2,(4-\alpha)(x_r^{\alpha}(t+h)^2]$$
$$[h(t,x_t)]^{\alpha} = [2tx(t+h)^2]^{\alpha} = t[(\alpha+1)(x_l^{\alpha}(t+h)^2,(3-\alpha)(x_r^{\alpha}(t+h)^2]$$

$$\left[I_k\left(x(\mathrm{t_k})\right)\right]^{\alpha} = \left[\frac{1}{1+x(t_k)}\right]^{\alpha} = \left[\frac{1}{1+x_l^{\alpha}(t_k)},\frac{1}{1+x_r^{\alpha}(t_k)}\right]$$

and $H_d([f(t,x_t)]^{\alpha},[f(t,y_t)]^{\alpha}) \le d_2 H_d([x(t+h)]^{\alpha},[y(t+h)]^{\alpha})$,
where $[x(t+h)]^{\alpha} = [x_l^{\alpha}(t+h),x_r^{\alpha}(t+h)]$ and $d_2 = 4b|x_r^{\alpha}(t+h),y_r^{\alpha}(t+h)|$.

Similarly, $H_d([h(t,x_t)]^{\alpha},[h(t,y_t)]^{\alpha}) \le d_1 H_d([x(t+h)]^{\alpha},[y(t+h)]^{\alpha})$,

$$H_d\left(\left[I_k\left(x(\mathrm{t_k})\right)\right]^{\alpha},\left[I_k\left(y(\mathrm{t_k})\right)\right]^{\alpha}\right) \le d_k H_d([x(t)]^{\alpha},[y(t)]^{\alpha})$$

where $d_1 = 3b|x_r^{\alpha}(t+h),y_r^{\alpha}(t+h)|, d_k = \frac{1}{((1+|x_r^{\alpha}(t_k)|)(1+|y_r^{\alpha}(t_k)|))}$.

Hence, the unique fuzzy solution is obtained by choosing $b \to 0$.

Assuming $x^1 = 2$, we prove the nonlocal controllability;

$$[u(s)]^{\alpha} = [u_l^{\alpha}(s),u_r^{\alpha}(s)]$$

$$= \Bigg[\left(\widetilde{G_l^{\alpha}}\right)^{-1}\Bigg((x^1)_l^{\alpha} - S_l^{\alpha}(T)\left[\varphi_l^{\alpha}(0) - g_l^{\alpha}\left(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_p}\right)(0) - h_l^{\alpha}(0,\varphi)\right] - h_l^{\alpha}\left(T,x_{Tl}{}^{\alpha}\right) - \int_0^T A_l^{\alpha} S_l^{\alpha}(T-S) h_l^{\alpha}\left(s,x_{sl}{}^{\alpha}\right)ds - \int_0^T S_l^{\alpha}(T-S) f_l^{\alpha}\left(s,x_{sl}{}^{\alpha}\right)ds - \sum_{0\prec t_k\prec T} S_l^{\alpha}(T-t_k) I_{kl}\left(x_l^{\alpha}(t_k^-)\right)\Bigg),$$

$$\left(\widetilde{G_r^{\alpha}}\right)^{-1}\Bigg((x^1)_r^{\alpha} - S_r^{\alpha}(T)\left[\varphi_r^{\alpha}(0) - g_r^{\alpha}\left(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_p}\right)(0) - h_r^{\alpha}(0,\varphi)\right] - h_r^{\alpha}\left(T,x_{Tr}{}^{\alpha}\right) - \int_0^T A_r^{\alpha} S_r^{\alpha}(T-\ S) h_r^{\alpha}\left(s,x_{sr}{}^{\alpha}\right)ds - \int_0^T S_r^{\alpha}(T-S) f_r^{\alpha}\left(s,x_{sr}{}^{\alpha}\right)ds - \sum_{0\prec t_k\prec T} S_r^{\alpha}(T-t_k) I_{kr}\left(x_r^{\alpha}(t_k^-)\right)\Bigg)\Bigg]$$

Substituting the above derived values into the integral system with respect to (1) yields an α −level set of $x(T)$ as $[x(T)]^{\alpha} = [2]^{\alpha} = [x^1]$. So, the system (1) is controllable on [0, T].

Example 2. *Consider the following system:*

$$\frac{d}{dt}[x(t) - tx(t+h)^2] = [x(t)] + 2tx(t+h)^2 + u(t)$$
$$x(0) + \sum_{k=1}^{p} c_k x(t_k) = 0 \epsilon E^n$$
$$I_k\big(x(t_k^-)\big) = \frac{1}{1 + x(t_k)}$$

where x^1 *is the target set, and the* α *-level set of fuzzy numbers 0, 1, 2 are given by*

$$[0]^\alpha = [\alpha - 1, 1 - \alpha], [1]^\alpha = [\alpha, 2 - \alpha], [2]^\alpha = [\alpha + 1, 3 - \alpha], \text{for } \alpha \epsilon [0,1].$$
$$f(t, x_t) = 2tx(t+h)^2, h(t, x_t) = tx(t+h)^2$$

Then, an $\alpha -$ level set of $g(x) = \sum_{k=1}^{p} c_k x(t_k)$ is

$$[g(x)]^\alpha = \left[\sum_{k=1}^{p} c_k x(t_k)\right]^\alpha = \left[\sum_{k=1}^{p} c_k x_l^\alpha(t_k), \sum_{k=1}^{p} c_k x_r^\alpha(t_k)\right]$$

and

$$H_d([g(x)]^\alpha, [g(y)]^\alpha) = H_d\left(\left[\sum_{k=1}^{p} c_k x(t_k)\right]^\alpha, \left[\sum_{k=1}^{p} c_k y(t_k)\right]^\alpha\right)$$
$$\leq \sum_{k=1}^{p} G_k H_d\left(\left[x_{t_k}(s)\right]^\alpha, \left[y_{t_k}(s)\right]^\alpha\right)$$

Similarly, an $\alpha -$ level set of $f(t, x_t), h(t, x_t)$ and $I_k\big(x(t_k)\big)$ is

$$[f(t, x_t)]^\alpha = [2tx(t+h)^2]^\alpha = t[(\alpha+1)(x_l^\alpha(t+h)^2, (3-\alpha)(x_r^\alpha(t+h)^2]$$
$$[h(t, x_t)]^\alpha = [tx(t+h)^2]^\alpha = t[(\alpha)(x_l^\alpha(t+h)^2, (2-\alpha)(x_r^\alpha(t+h)^2]$$
$$\left[I_k\big(x(t_k)\big)\right]^\alpha = \left[\frac{1}{1 + x(t_k)}\right]^\alpha = \left[\frac{1}{1 + x_l^\alpha(t_k)}, \frac{1}{1 + x_r^\alpha(t_k)}\right]$$

and

$$H_d([f(t, x_t)]^\alpha, [f(t, y_t)]^\alpha) \leq d_2 H_d([x(t+h)]^\alpha, [y(t+h)]^\alpha)$$

where

$$[x(t+h)]^\alpha = [x_l^\alpha(t+h), x_r^\alpha(t+h)]$$

and

$$d_2 = 3b|x_r^\alpha(t+h), y_r^\alpha(t+h)|$$

Similarly,

$$H_d([h(t, x_t)]^\alpha, [h(t, y_t)]^\alpha) \leq d_1 H_d([x(t+h)]^\alpha, [y(t+h)]^\alpha),$$
$$H_d\left(\left[I_k\big(x(t_k)\big)\right]^\alpha, \left[I_k\big(y(t_k)\big)\right]^\alpha\right) \leq d_k H_d([x(t)]^\alpha, [y(t)]^\alpha)$$

where

$$d_1 = 2b|x_r^\alpha(t+h), y_r^\alpha(t+h)|, d_k = \frac{1}{\big((1 + |x_r^\alpha(t_k)|)(1 + |y_r^\alpha(t_k)|)\big)}$$

Hence, the unique fuzzy solution is obtained by choosing $b \to 0$.
Assuming, $x^1 = 2$, we prove the nonlocal controllability;

$$[u(s)]^\alpha = [u_l^\alpha(s), u_r^\alpha(s)]$$

$$= \left[\left(\widetilde{G_l^\alpha}\right)^{-1} \left((x^1)_l^\alpha - S_l^\alpha(T)\left[\varphi_l^\alpha(0) - g_l^\alpha\left(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_p}\right)(0) - h_l^\alpha(0,\varphi)\right] - h_l^\alpha\left(T, x_{Tl}{}^\alpha\right) - \int_0^T A_l^\alpha S_l^\alpha(T-S) h_l^\alpha\left(s, x_{sl}{}^\alpha\right) ds - \int_0^T S_l^\alpha(T-S) f_l^\alpha\left(s, x_{sl}{}^\alpha\right) ds - \sum_{0 \prec t_k \prec T} S_l^\alpha(T-t_k) I_{kl}\left(x_l^\alpha(t_k^-)\right) \right), \right.$$

$$\left. \left(\widetilde{G_r^\alpha}\right)^{-1} \left((x^1)_r^\alpha - S_r^\alpha(T)\left[\varphi_r^\alpha(0) - g_r^\alpha\left(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_p}\right)(0) - h_r^\alpha(0,\varphi)\right] - h_r^\alpha\left(T, x_{Tr}{}^\alpha\right) - \int_0^T A_r^\alpha S_r^\alpha(T- S) h_r^\alpha\left(s, x_{sr}{}^\alpha\right) ds - \int_0^T S_r^\alpha(T-S) f_r^\alpha\left(s, x_{sr}{}^\alpha\right) ds - \sum_{0 \prec t_k \prec T} S_r^\alpha(T-t_k) I_{kr}\left(x_r^\alpha(t_k^-)\right) \right) \right]$$

Substituting the above derived values into the integral system with respect to (1) yields an $\alpha -$ level set of $x(T)$ as $[x(T)]^\alpha = [2]^\alpha = [x^1]$. So, the system (1) is controllable on [0, T].

5. Conclusions

In this paper, we have proved the controllability results of the fuzzy solutions for the first order impulsive neutral functional differential equations by applying the contraction mapping principle. Further, we can extend the controllability results for the fuzzy inclusions. The same concept can be generalized to study the controllability of second order systems/inclusions using sine and cosine operators (2012) [30]. One can also study the controllability of the fuzzy solution for neutral impulsive functional fractional order systems for time and state delay systems/inclusions (2013) [31]. Numerical aspects of the same would be quite interesting for the further study. Moreover, we can extend the results to real phenomena, considering a pendulum problem, and develop a fuzzy and impulsive controller design in $\mathfrak{R}^n$, applying a simulation to the proposed adaptive fuzzy and impulsive controllers to control the inverted pendulum using MATLAB. By stability analysis, we can make sure that all signals involved are uniformly bounded. At this stage, all the design parameters have specific numerical values that generate the graph of the adaptive control input signal and the graph of the state and its desired value.

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