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Regularization of the Ill-Posed Cauchy Problem for Matrix Factorizations of the Helmholtz Equation on the Plane

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Abstract: In this paper, we present an explicit formula for the approximate solution of the Cauchy problem for the matrix factorizations of the Helmholtz equation in a bounded domain on the plane. Our formula for an approximate solution also includes the construction of a family of fundamental solutions for the Helmholtz operator on the plane. This family is parameterized by function $K(w)$ which depends on the space dimension. In this paper, based on the results of previous works, the better results can be obtained by choosing the function $K(w)$.

Keywords: cauchy problem; regularization; factorization; regular solution; fundamental solution

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1. Introduction

The paper studies the construction of the exact and approximate solutions of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation. Such problems naturally arise in mathematical physics and in various fields of natural science such as electro-geological exploration, cardiology, electrodynamics and so on. In general, the theory of ill-posed problems for elliptic system of equations has been sufficiently formed by Tikhonov, Ivanov, Lavrent'ev and Tarkhanov in [1–5]. Among them, the most important and applicable topic is related to the conditionally well-posed problems, characterized by stability in the presence of additional information about the nature of the problem data. One of the most effective ways to study such problems is to construct the regularizing operators. For example, it can be done by the Carleman-type formulas (as in complex analysis) or iterative processes (the Kozlov–Maz'ya–Fomin algorithm, etc.).

The work is devoted to the main problem for partial differential equations, which is the Cauchy problem. The main aim of this study is to find the regularization formulas to find the solutions of the Cauchy problem for matrix factorizations of the Helmholtz equation. The question of the existence of a solution of the problem is not considered—it is assumed a priori. At the same time, it should be noted that any regularization formula leads to an approximate solution of the Cauchy problem for all data, even if there is no solution in the usual classic sense. Moreover, for explicit regularization formulas, the optimal solution can be obtained. In this sense, exact regularization formulas are very useful for real numerical calculations. There is good reason to hope that numerous practical applications of regularization formulas are still ahead. In [6–8] some applications of the Cauchy problem and the regularization technique for solving different kinds of integral equations have been presented.

The Cauchy problem for matrix factorizations of the Helmholtz equation is among ill-posed and unstable problems. It is known that the Cauchy problem for elliptic equations

is among unstable problems which by a small change in the data the problem will be incorrect [1,4,9–13]. Tarkhanov [14] has published a criterion for the solvability of a large class of boundary value problems for elliptic systems. In some cases of unstable problems, we should apply some operators for solving the problem. But the image of these operators are not closed, therefore, the solvability condition can not be written in terms of continuous linear functions. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique and the problem is solvable for everywhere dense a set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of Aizenberg, Kytmanov and Tarkhanov [5].

The uniqueness of the solution follows from Holmgren's general theorem [10]. The conditional stability of the problem follows from the work of Tikhonov [1], if we restrict the class of possible solutions to a compactum.

Formulas that allow finding a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carleman type formulas. In [13], Carleman established a formula giving a solution to the Cauchy–Riemann equations in a a special form of a domain. Developing his idea, Goluzin and Krylov [15] derived a formula to determine the values of analytic functions from known data. A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in [11]. The Carleman formula to find the solution of the differential operator with special properties can be found in [3,4]. Yarmukhamedov [16–19] applied this method to construct the Carleman functions for the Laplace and Helmholtz operators for special form and domain. In [5] an integral formula was proved for the first order elliptic type system of equations with constant coefficients in a bounded domain. In [20], Ikehata applied the presented methodologies in [16–19] to consider the probe method and Carleman functions for the Laplace and Helmholtz equations in the three-dimensional domain. In [21], a formula for solving the Helmholtz equation with a variable coefficient for regions in space where the unknown data are located on a section of the hypersurface $\{x \cdot s = t\}$ has been presented by Ikehata.

Carleman type formulas for various elliptic equations and systems were also obtained in [5,14–28] and [29–41]. In [22] the Cauchy problem for the Helmholtz equation in an arbitrary bounded plane domain with Cauchy data which is known only on the boundary region was discussed. The solvability criterion of the Cauchy problem for the Laplace equation in $\mathbb{R}^m$ was considered by Shlapunov [25]. In [42], the continuation of the Helmholtz equation was investigated and the results of the numerical experiments were presented.

The construction of the Carleman matrix for elliptic systems was carried out by Yarmukhamedov, Tarkhanov, Shlapunov, Niyozov, Juraev and others [5,14,16–19,23–41]. The system considered in this paper was introduced by Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain (see, for instance [5]).

In many well-posed problems of the system of equations of the first order elliptic type with constant coefficients that factorize the Helmholtz operator, calculating the values of the vector function on the entire boundary is not possible. Therefore, the problem of reconstructing the solution of system of equations of the first order elliptic type with constant coefficients and factorizing the Helmholtz operator [29–41] are among the more challenging problems in the theory of differential equations.

Additionally, the ill-posed problems of mathematical physics have been investigated by many researchers. The properties of solutions of the Cauchy problem for the Laplace equation were studied in [3,4,16–19] and subsequently developed in [5,14,15,20–41].

Let $\mathbb{R}^2$ be the two-dimensional real Euclidean space,

$$x = (x_1, x_2) \in \mathbb{R}^2, \ y = (y_1, y_2) \in \mathbb{R}^2.$$

$G \subset \mathbb{R}^2$ is a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_2 = 0$ and some smooth curve S lying in the half-space $y_2 > 0$, i.e., $\partial G = S \bigcup T$.

We introduce the following notation:

$$r = |y - x|, \alpha = |y_1 - x_1|, w = i\sqrt{u^2 + \alpha^2} + y_2,\ u \geq 0,$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^T, \frac{\partial}{\partial x} \to \xi^T, \xi^T = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ which is a transposed vector of } \xi,$$

$$U(x) = (U_1(x), ..., U_n(x))^T,\ u^0 = (1, ..., 1) \in \mathbb{R}^n,\ n = 2^m,\ m = 2,$$

$$E(z) = \left\| \begin{array}{c} z_1 ... 0 \\ ..... \\ 0 ... z_n \end{array} \right\| - \text{diagonal matrix}, z = (z_1, ..., z_n) \in \mathbb{R}^n.$$

Let $D(\xi^T)$ be a $(n \times n)-$dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane which is satisfied in the following condition

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$ and λ is a real number.

We consider the following system of differential equations

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{1}$$

in the region G where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote the class of vector functions in the domain G by $A(G)$ which is continuous on $\overline{G} = G \bigcup \partial G$ and satisfy in the system (1).

2. Construction of the Carleman Matrix and the Cauchy Problem

Formulation of the problem: suppose $U(y) \in A(G)$ and

$$U(y)|_S = f(y),\ y \in S, \tag{2}$$

where $f(y)$ is a given continuous vector-function on S. It is required to note that the vector function $U(y)$ is in the domain G, based on $f(y)$ on S.

If $U(y) \in A(G)$, then the following Cauchy type integral formula

$$U(x) = \int\limits_{\partial G} N(y, x; \lambda)U(y)ds_y,\ \ x \in G, \tag{3}$$

is valid and

$$N(y, x; \lambda) = \left(E\left(\varphi_2(\lambda r)u^0\right)D^*\left(\frac{\partial}{\partial x}\right)\right)D(t^T),$$

where $t = (t_1, t_2)$ shows the unit exterior normal which is drawn at point y on curve ∂G and $\varphi_2(\lambda r)$ denotes the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$, which is defined in the following form

$$\varphi_2(\lambda r) = -\frac{i}{4}H_0^{(1)}(\lambda r). \tag{4}$$

Here $H_0^{(1)}(\lambda r)$ is the the Hankel function of the first kind [12].

We introduce $K(w)$ as an entire function which takes real values as real part of w, ($w = u + iv$, $u, v-$real numbers) and satisfies in the following conditions:

$$K(u) \neq 0, \ \sup_{v \geq 1} \left| v^p K^{(p)}(w) \right| = B(u, p) < \infty, \tag{5}$$

$$-\infty < u < \infty, \ p = 0, 1, 2.$$

We define the function $\Phi(y, x; \lambda)$ at $y \neq x$ by the following equality

$$\Phi(y, x; \lambda) = -\frac{1}{2\pi K(x_2)} \int_0^\infty \mathrm{Im}\left[\frac{K(w)}{w - x_2}\right] \frac{u\, I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \tag{6}$$

where $I_0(\lambda u) = J_0(i\lambda u)-$is the zero order Bessel function of the first kind [10].

In the Formula (6), choosing

$$K(w) = \exp(\sigma w^2), \ K(x_2) = \exp(\sigma x_2^2), \ \sigma > 0, \tag{7}$$

we get

$$\Phi_\sigma(y, x; \lambda) = -\frac{e^{-\sigma x_2^2}}{2\pi} \int_0^\infty \mathrm{Im}\left[\frac{\exp(\sigma w^2)}{w - x_2}\right] \frac{u\, I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du. \tag{8}$$

Substituting

$$\Phi_\sigma(y, x; \lambda) = \varphi_2(\lambda r) + g_\sigma(y, x; \lambda), \tag{9}$$

in Equation (3) instead of $\varphi_2(\lambda r)$, the formula will be correct where $g_\sigma(y, x)$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula can written in the follwoing form:

$$U(x) = \int_{\partial G} N_\sigma(y, x; \lambda) U(y) ds_y, \ \ x \in G, \tag{10}$$

where

$$N_\sigma(y, x; \lambda) = \left(E\left(\Phi_\sigma(y, x; \lambda) u^0\right) D^*\left(\frac{\partial}{\partial x}\right)\right) D(t^T).$$

3. The Continuation Formula and Regularization According to M.M. Lavrent'ev's

Theorem 1. *Let $U(y) \in A(G)$ satisfy in the following inequality*

$$|U(y)| \leq M, \ y \in T. \tag{11}$$

If

$$U_\sigma(x) = \int_S N_\sigma(y, x; \lambda) U(y) ds_y, \ \ x \in G, \tag{12}$$

then the following estimations are correct:

$$|U(x) - U_\sigma(x)| \leq C(\lambda, x) \sigma M e^{-\sigma x_2^2}, \ \sigma > 1, \ x \in G, \tag{13}$$

$$\left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j} \right| \leq C(\lambda, x) \sigma M e^{-\sigma x_2^2}, \ \sigma > 1, \ x \in G, \ j = 1, 2, \tag{14}$$

where $C(\lambda, x)$ shows the bounded functions on compact subsets of the domain G.

Proof. Let us first estimate inequality (13). Using the integral Formula (10) and the equality (12), we obtain

$$U(x) = \int_S N_\sigma(y, x; \lambda) U(y) ds_y + \int_T N_\sigma(y, x; \lambda) U(y) ds_y$$

$$= U_\sigma(x) + \int_T N_\sigma(y, x; \lambda) U(y) ds_y, \ x \in G.$$

Taking into account the inequality (11), we estimate the following

$$|U(x) - U_\sigma(x)| \leq \left| \int_T N_\sigma(y, x; \lambda) U(y) ds_y \right|$$

$$\leq \int_T |N_\sigma(y, x; \lambda)||U(y)| ds_y \leq M \int_T |N_\sigma(y, x; \lambda)| ds_y, \ x \in G. \tag{15}$$

For this aim, we estimate the integrals $\int_T |\Phi_\sigma(y, x; \lambda)| ds_y$, $\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_1} \right| ds_y$, and $\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_2} \right| ds_y$ on the part T of the plane $y_2 = 0$.

Separating the imaginary part of (8), we obtain

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{\sigma(y_2^2 - x_2^2)}}{2\pi} \left[\int_0^\infty \frac{e^{-\sigma(u^2+\alpha^2)} \cos \sigma \sqrt{u^2 + \alpha^2}}{u^2 + r^2} \, u I_0(\lambda u) \, du \right.$$

$$\left. - \int_0^\infty \frac{e^{-\sigma(u^2+\alpha^2)} (y_2 - x_2) \sin \sigma \sqrt{u^2 + \alpha^2}}{u^2 + r^2} \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du \right], \ x_2 > 0. \tag{17}$$

Given (16) and the inequality

$$I_0(\lambda u) \leq \sqrt{\frac{2}{\lambda \pi u}}, \tag{17}$$

we have

$$\int_T |\Phi_\sigma(y, x; \lambda)| ds_y \leq C(\lambda, x) \sigma e^{-\sigma x_2^2}, \ \sigma > 1, \ x \in G. \tag{18}$$

To estimate the second integral, we use the equality

$$\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_1} = \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s} \frac{\partial s}{\partial y_1} = 2(y_1 - x_1) \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s},$$

$$s = \alpha^2. \tag{19}$$

Given equality (16), inequality (17) and equality (19), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_1} \right| ds_y \leq C(\lambda, x) \sigma e^{-\sigma x_2^2}, \ \sigma > 1, \ x \in G, \tag{20}$$

Now, we estimate the integral $\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_2} \right| ds_y$.

Taking into account equality (16) and inequality (17), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_2} \right| ds_y \le C(\lambda, x) \sigma e^{-\sigma x_2^2}, \ \sigma > 1, \ x \in G, \tag{21}$$

From inequalities (17), (20) and (21), bearing in mind (15), we obtain an estimate (13).

Now the inequality (14) can be proved. To do this, we take the derivatives from equalities (10) and (12) with respect to x_j, $(j = 1, 2)$ then we get:

$$\begin{gathered} \frac{\partial U(x)}{\partial x_j} = \int_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y + \int_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y, \\ \frac{\partial U_\sigma(x)}{\partial x_j} = \int_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y, \quad x \in G, \ j = 1, 2. \end{gathered} \tag{22}$$

Taking into account the (22) and inequality (11), we estimate the following

$$\begin{gathered} \left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial_\sigma U(x)}{\partial x_j} \right| \le \left| \int_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y \right| \\ \le \int_T \left| \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} \right| |U(y)| ds_y \le M \int_T \left| \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} \right| ds_y, \\ x \in G, \ j = 1, 2. \end{gathered} \tag{23}$$

To do this, we estimate the integrals $\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_1} \right| ds_y$ and $\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_2} \right| ds_y$ on the part T of the plane $y_2 = 0$.

To estimate the first integrals, we use the equality

$$\begin{gathered} \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_1} = \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s} \frac{\partial s}{\partial x_1} = -2(y_1 - x_1) \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s}, \\ s = \alpha^2. \end{gathered} \tag{24}$$

Applying equality (16), inequality (17) and equality (24), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_1} \right| ds_y \le C(\lambda, x) \sigma e^{-\sigma x_2^2}, \ \sigma > 1, \ x \in G. \tag{25}$$

Now, we estimate the integral $\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_2} \right| ds_y$.

Taking into account equality (16) and inequality (17), we obtain

$$\int_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_2} \right| ds_y \le C(\lambda, x) \sigma e^{-\sigma x_2^2}, \ \sigma > 1, \ x \in G. \tag{26}$$

From inequalities (25) and (26), bearing in mind (23), we obtain an estimate of (14). Theorem 1 is proved. □

Corollary 1. *For each $x \in G$, the equalities are true*

$$\lim_{\sigma \to \infty} U_\sigma(x) = U(x), \ \lim_{\sigma \to \infty} \frac{\partial U_\sigma(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \ j = 1, 2.$$

We define $\overline{G}_\varepsilon$ as

$$\overline{G}_\varepsilon = \left\{ (x_1, x_2) \in G,\ a > x_2 \geq \varepsilon,\ a = \max_T \psi(x_1),\ 0 < \varepsilon < a \right\}.$$

Here $\psi(x_1)$-is a curve. It is easy to see that the set $\overline{G}_\varepsilon \subset G$ is compact.

Corollary 2. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_\sigma(x)\}$ and $\left\{ \dfrac{\partial U_\sigma(x)}{\partial x_j} \right\}$ converge uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x),\ \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j},\ j = 1, 2.$$

We should note that the set $E_\varepsilon = G \backslash \overline{G}_\varepsilon$ serves as a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

4. Estimation of the Stability of the Solution to the Cauchy Problem

Suppose that the curve S is given by the equation

$$y_2 = \psi(y_1),\ y_1 \in \mathbb{R},$$

where $\psi(y_1)$ is a single-valued function satisfying the Lyapunov conditions.

We put

$$a = \max_T \psi(y_1),\ b = \max_T \sqrt{1 + \psi'^2(y_1)}.$$

Theorem 2. *Let $U(y) \in A(G)$ satisfies in the condition (20), and on a smooth curve S the inequality*

$$|U(y)| \leq \delta,\ 0 < \delta \leq Me^{-\sigma a^2}. \tag{27}$$

Then the following relations are true

$$|U(x)| \leq C(\lambda, x)\sigma M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}},\ \sigma > 1,\ x \in G. \tag{28}$$

$$\left| \frac{\partial U(x)}{\partial x_j} \right| \leq C(\lambda, x)\sigma M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}},\ \sigma > 1,\ x \in G,\quad j = 1, 2. \tag{29}$$

Proof. Let us first estimate inequality (28). Using the integral formula (10), we have

$$U(x) = \int_S N_\sigma(y, x; \lambda) U(y) ds_y + \int_T N_\sigma(y, x; \lambda)) U(y) ds_y,\ x \in G. \tag{30}$$

We estimate the following

$$|U(x)| \leq \left| \int_S N_\sigma(y, x; \lambda) U(y) ds_y \right| + \left| \int_T N_\sigma(y, x; \lambda) U(y) ds_y \right|,\ x \in G. \tag{31}$$

Given inequality (27), we estimate the first integral of inequality (31).

$$\left|\int_S N_\sigma(y,\,x;\lambda)U(y)ds_y\right| \le \int_S |N_\sigma(y,\,x;\lambda)||U(y)|ds_y \le \delta\int_S |N_\sigma(y,\,x;\lambda)|ds_y,\ x\in G. \tag{32}$$

To do this, we estimate the integrals $\int_S |\Phi_\sigma(y,\,x;\lambda)|ds_y$, $\int_S \left|\frac{\partial\Phi_\sigma(y,\,x;\lambda)}{\partial y_1}\right|ds_y$ and $\int_S \left|\frac{\partial\Phi_\sigma(y,\,x;\lambda)}{\partial y_2}\right|ds_y$ on a smooth curve S.

Given equality (16) and the inequality (17), we have

$$\int_S |\Phi_\sigma(y,\,x;\lambda)|ds_y \le C(\lambda,x)\sigma e^{\sigma(a^2-x_2^2)},\ \sigma>1,\ x\in G. \tag{33}$$

To estimate the second integral, using equalities (16) and (19) as well as inequality (17), we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,\,x;\lambda)}{\partial y_1}\right|ds_y \le C(\lambda,x)\sigma e^{\sigma(a^2-x_2^2)},\ \sigma>1,\ x\in G. \tag{34}$$

To find the integral $\int_S \left|\frac{\partial\Phi_\sigma(y,\,x;\lambda)}{\partial y_2}\right|ds_y$, using equality (16) and inequality (17), we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,\,x;\lambda)}{\partial y_2}\right|ds_y \le C(\lambda,x)\sigma e^{\sigma(a^2-x_2^2)},\ \sigma>1,\ x\in G. \tag{35}$$

From (33)–(35) and applying (32), we obtain

$$\left|\int_S N_\sigma(y,\,x;\lambda)U(y)ds_y\right| \le C(\lambda,x)\sigma\delta\, e^{\sigma(a^2-x_2^2)},\ \sigma>1,\ x\in G. \tag{36}$$

The following is known

$$\left|\int_T N_\sigma(y,\,x;\lambda)U(y)ds_y\right| \le C(\lambda,x)\sigma M e^{-\sigma x_2^2},\ \sigma>1,\ x\in G. \tag{37}$$

Now taking into account (36)–(37) and using (31), we have

$$|U(x)| \le \frac{C(\lambda,x)\sigma}{2}(\delta e^{\sigma a^2}+M)e^{-\sigma x_2^2},\ \sigma>1,\ x\in G. \tag{38}$$

Choosing σ from the equality

$$\sigma = \frac{1}{a^2}\ln\frac{M}{\delta}, \tag{39}$$

we obtain an estimate (28).

Now let us prove inequality (29). To do this, we find the partial derivative from the integral formula (10) with respect to the variable $x_j,\ j = 1, 2$:

$$\frac{\partial U(x)}{\partial x_j} = \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y + \int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y$$
$$= \frac{\partial U_\sigma(x)}{\partial x_j} + \int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y,\ x \in G,\ j = 1, 2. \tag{40}$$

Here

$$\frac{\partial U_\sigma(x)}{\partial x_j} = \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y. \tag{41}$$

We estimate the following

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \le \left|\int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y\right|$$
$$+\left|\int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y\right| \le \left|\frac{\partial U_\sigma(x)}{\partial x_j}\right| \tag{42}$$
$$+\left|\int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y\right|,\ x \in G,\ j = 1, 2.$$

Given inequality (27), we estimate the first integral of inequality (42).

$$\left|\int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y\right| \le \int\limits_S \left|\frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}\right| |U(y)| ds_y$$
$$\le \delta \int\limits_S \left|\frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}\right| ds_y,\ x \in G,\ j = 1, 2. \tag{43}$$

To do this, we estimate the integrals $\int\limits_S \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_1}\right| ds_y$, and $\int\limits_S \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_2}\right| ds_y$ on a smooth curve S.

Given equality (16), inequality (17) and equality (24), we obtain

$$\int\limits_S \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_1}\right| ds_y \le C(\lambda, x) \sigma e^{\sigma(a^2 - x_2^2)},\ \sigma > 1,\ x \in G, \tag{44}$$

Now, we estimate the integral $\int\limits_S \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_2}\right| ds_y$.

Taking into account equality (16) and inequality (17), we obtain

$$\int\limits_S \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_2}\right| ds_y \le C(\lambda, x) \sigma e^{\sigma(a^2 - x_2^2)},\ \sigma > 1,\ x \in G, \tag{45}$$

From (44) and (45), bearing in mind (43), we obtain

$$\left|\int_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right| \le C(\lambda,x)\sigma\delta e^{-\sigma x_2^2},\ \sigma>1,\ x\in G, \quad j=1,2. \tag{46}$$

The following is known

$$\left|\int_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right| \le C(\lambda,x)\sigma M e^{-\sigma x_2^2},\ \sigma>1,\ x\in G, \quad j=1,2. \tag{47}$$

Now taking into account (46)–(47), bearing in mind (42), we have

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \le \frac{C(\lambda,x)\sigma}{2}(\delta e^{\sigma a^2}+M)e^{-\sigma x_2^2},\ \sigma>1,\ x\in G, \quad j=1,2. \tag{48}$$

Choosing σ from the equality (39) we obtain an estimate (29). Theorem 2 is proved. □

Assume that $U(y)\in A(G)$ is defined on S and $f_\delta(y)$ is its approximation with an error $0<\delta\le Me^{-\sigma a^2}$ then

$$\max_S|U(y)-f_\delta(y)|\le\delta. \tag{49}$$

We put

$$U_{\sigma(\delta)}(x)=\int_S N_\sigma(y,x;\lambda)f_\delta(y)ds_y,\ x\in G. \tag{50}$$

Theorem 3. *Let $U(y)\in A(G)$ on the part of the plane $y_2=0$ satisfies in the condition (11). Then the following estimates is true*

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \le C(\lambda,x)\sigma M^{1-\frac{x_2^2}{a^2}}\delta^{\frac{x_2^2}{a^2}},\ \sigma>1,\ x\in G. \tag{51}$$

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \le C(\lambda,x)\sigma M^{1-\frac{x_2^2}{a^2}}\delta^{\frac{x_2^2}{a^2}},\ \sigma>1,\ x\in G, \quad j=1,2. \tag{52}$$

Proof. From the integral formulas (10) and (50), we have

$$U(x)-U_{\sigma(\delta)}(x)=\int_{\partial G}N_\sigma(y,x;\lambda)U(y)ds_y$$

$$-\int_S N_\sigma(y,x;\lambda)f_\delta(y)ds_y=\int_S N_\sigma(y,x;\lambda)U(y)ds_y$$

$$+\int_T N_\sigma(y,x;\lambda)U(y)ds_y-\int_S N_\sigma(y,x;\lambda)f_\delta(y)ds_y$$

$$=\int_S N_\sigma(y,x;\lambda)\{U(y)-f_\delta(y)\}ds_y+\int_T N_\sigma(y,x;\lambda)U(y)ds_y.$$

and

$$\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \int\limits_{\partial G} \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y$$

$$- \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} f_\delta(y) ds_y = \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y$$

$$+ \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y - \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} f_\delta(y) ds_y$$

$$= \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \{U(y) - f_\delta(y)\} ds_y + \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y,$$

$$j = 1, 2.$$

Using conditions (11) and (49), we estimate the following:

$$\left| U(x) - U_{\sigma(\delta)}(x) \right| = \left| \int\limits_S N_\sigma(y,x;\lambda) \{U(y) - f_\delta(y)\} ds_y \right|$$

$$+ \left| \int\limits_T N_\sigma(y,x;\lambda) U(y) ds_y \right| \le \int\limits_S |N_\sigma(y,x;\lambda)| |\{U(y) - f_\delta(y)\}| ds_y$$

$$+ \int\limits_T |N_\sigma(y,x;\lambda)| |U(y)| ds_y \le \delta \int_S |N_\sigma(y,x;\lambda)| ds_y$$

$$+ M \int\limits_T |N_\sigma(y,x;\lambda)| ds_y.$$

and

$$\left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \right| = \left| \int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \{U(y) - f_\delta(y)\} ds_y \right|$$

$$+ \left| \int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y \right| \le \int\limits_S \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| |\{U(y) - f_\delta(y)\}| ds_y$$

$$+ \int\limits_T \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| |U(y)| ds_y \le \delta \int_S \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| ds_y$$

$$+ M \int\limits_T \left| \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} \right| ds_y, \; j = 1, 2.$$

Now, repeating the proof of Theorems 1 and 2, we obtain

$$\left| U(x) - U_{\sigma(\delta)}(x) \right| \le \frac{C(\lambda,x)\sigma}{2} (\delta e^{\sigma a^2} + M) e^{-\sigma x_2^2}.$$

$$\left| \frac{\partial U(x)}{\partial x_j} - \frac{U_{\sigma(\delta)}(x)}{\partial x_j} \right| \le \frac{C(\lambda,x)\sigma}{2} (\delta e^{\sigma a^2} + M) e^{-\sigma x_2^2}, \; j = 1, 2.$$

From here, choosing σ from equality (39), we obtain an estimates (51) and (52). Thus Theorem 3 is proved. □

Corollary 3. *For each $x \in G$, the following equalities are true*

$$\lim_{\delta \to 0} U_{\sigma(\delta)}(x) = U(x), \ \lim_{\delta \to 0} \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \ j = 1, 2.$$

Corollary 4. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\left\{ U_{\sigma(\delta)}(x) \right\}$ and $\left\{ \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \right\}$ are convergent uniformly for $\delta \to 0$, i.e.:*

$$U_{\sigma(\delta)}(x) \rightrightarrows U(x), \ \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j}, \ j = 1, 2.$$

5. Conclusions

The article obtained the following results:

Using the Carleman function, a formula can be obtained for the continuation of the solution of linear elliptic systems of the first order with constant coefficients in a spatial bounded domain $\mathbb{R}^2$. The resulting formula is an analogue of the classical formula of Riemann, Voltaire and Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation was presented. This problem can be considered when, instead of the exact data of the Cauchy problem we have their approximations with a given deviation in the uniform metric and under the assumption that the solution of the Cauchy problem is bounded on part T, of the boundary of the domain G.

We note that for solving applicable problems, the approximate values of $U(x)$ and $\frac{\partial U(x)}{\partial x_j}$, $x \in G$, $j = 1, 2$ should be found.

In this paper, we have built a family of vector-functions $U(x, f_\delta) = U_{\sigma(\delta)}(x)$ and $\frac{\partial U(x, f_\delta)}{\partial x_j} = \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$, $(j = 1, 2)$ depend on a parameter σ. Also, we prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ are convergent to a solution $U(x)$ and its derivative $\frac{\partial U(x)}{\partial x_j}$, $x \in G$ at point $x \in G$.

According to [1], a family of vector-functions $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ is called a regularized solution of the problem. A regularized solution determines a stable method to find the approximate solution of the problem.

Thus, functionals $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ determine the regularization of the solution of problems (1) and (2).

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