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Approximations of an Equilibrium Problem without Prior Knowledge of Lipschitz Constants in Hilbert Spaces with Applications

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Abstract: The objective of this paper is to introduce an iterative method with the addition of an inertial term to solve equilibrium problems in a real Hilbert space. The proposed iterative scheme is based on the Mann-type iterative scheme and the extragradient method. By imposing certain mild conditions on a bifunction, the corresponding theorem of strong convergence in real Hilbert space is well-established. The proposed method has the advantage of requiring no knowledge of Lipschitz-type constants. The applications of our results to solve particular classes of equilibrium problems is presented. Numerical results are established to validate the proposed method's efficiency and to compare it to other methods in the literature.

Keywords: equilibrium problem; pseudomonotone bifunction; Lipschitz-type conditions; strong convergence theorems; variational inequality problems; fixed-point problem

MSC: 47H05; 47H10; 65Y05; 65K15

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1. Introduction

Suppose that $\mathcal{C}$ is a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$. The inner product and induced norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ be a bifunction and $f(y, y) = 0$, for all $y \in \mathcal{C}$. The equilibrium problem (EP) [1,2] for a bifunction f on $\mathcal{C}$ is defined in the following way:

$$\text{Find } u^* \in \mathcal{C} \text{ such that } f(u^*, y) \geq 0, \ \forall y \in \mathcal{C}. \tag{EP}$$

The equilibrium problem is a general mathematical problem in the sense that it unifies various mathematical problems, i.e., fixed-point problems, vector and scalar minimization problems, problems of variational inequality, complementarity problems, Nash equilibrium problems in noncooperative games, saddle point problems, and inverse optimization problems [2–4]. The equilibrium problem is also known as the well-known Ky Fan inequality due to the result [1]. Many authors established and generalized several results on the existence and nature of the solution of the equilibrium problems (see for more detail [1,4,5]). Due to the importance of this problem (EP) in both pure and applied sciences, many researchers studied it in recent years [6–17] and other in [18–22].

Tran et al. in [23] introduced iterative sequence $\{u_n\}$ in the following way:

$$\begin{cases} u_0 \in \mathcal{C}, \\ y_n = \underset{z \in \mathcal{C}}{\arg\min}\{\chi f(u_n, z) + \frac{1}{2}\|u_n - z\|^2\}, \\ u_{n+1} = \underset{z \in \mathcal{C}}{\arg\min}\{\chi f(y_n, z) + \frac{1}{2}\|u_n - z\|^2\}, \end{cases} \tag{1}$$

where $0 < \chi < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. This method is also known as the extragradient method in [23] due to the previous contribution of Korpelevich [24] to solve the saddle-point problems. The iterative sequence generated by the above-mentioned method is weakly convergent to the solution with prior knowledge of Lipschitz-type constants. These Lipschitz-like constants are often not known or are difficult to compute. Recently, Hieu et al. [25] introduced an extension of the method (1) for solving the equilibrium problem. Let us consider that $[p]_+ := \max\{p, 0\}$ and choose $u_0 \in \mathcal{C}$, $\mu \in (0,1)$ with $\chi_0 > 0$ such that

$$\begin{cases} y_n = \underset{z \in \mathcal{C}}{\arg\min}\{\chi_n f(u_n, z) + \frac{1}{2}\|u_n - z\|^2\}, \\ u_{n+1} = \underset{z \in \mathcal{C}}{\arg\min}\{\chi_n f(y_n, z) + \frac{1}{2}\|u_n - z\|^2\}, \end{cases} \tag{2}$$

where $\{\chi_n\}$ is updated in the following manner:

$$\chi_{n+1} = \min\left\{\chi_n, \frac{\mu(\|u_n - y_n\|^2 + \|u_{n+1} - y_n\|^2)}{2[f(u_n, u_{n+1}) - f(u_n, y_n) - f(y_n, u_{n+1})]_+}\right\}.$$

Inertial-like methods are well-known two-step iterative methods in which the next iteration is derived from the previous two iterations (see [26,27] for more details). To speed up the iterative sequence convergence rate, an inertial extrapolation term is used. Numerical examples show that inertial effects improve numerical performance in terms of execution time and the expected number of iterations. Recently, many existing methods were established for the case of equilibrium problems (see [28–31] for more details).

In this paper, inspired by the methods in [23,25,26,32], we introduce a general inertial Mann-type subgradient extragradient method to evaluate the approximate solution of the equilibrium problems involving pseudomonotone bifunction. A strong convergence result corresponding to the proposed algorithm is well-established by assuming certain mild conditions. Some of the applications for our main results are considered to solve the fixed-point problems. Lastly, computational results show that the new method is more successful than existing ones [23,33,34].

2. Preliminaries

A *metric projection* $P_{\mathcal{C}}(u)$ of $u \in \mathcal{H}$ onto a closed and convex subset $\mathcal{C}$ of $\mathcal{H}$ is defined by

$$P_{\mathcal{C}}(u) = \underset{y \in \mathcal{C}}{\arg\min}\{\|y - u\|\}.$$

In this study, the equilibrium problem under the following conditions:

(c1). A bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is said to be *pseudomonotone* [3,35] on $\mathcal{C}$ if

$$f(y_1, y_2) \geq 0 \Longrightarrow f(y_2, y_1) \leq 0, \ \ \forall y_1, y_2 \in \mathcal{C}.$$

(c2). A bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ is said to be Lipschitz-type continuous [36] on $\mathcal{C}$ if there exist constants $c_1, c_2 > 0$ such that

$$f(y_1, y_3) \leq f(y_1, y_2) + f(y_2, y_3) + c_1\|y_1 - y_2\|^2 + c_2\|y_2 - y_3\|^2, \ \ \forall y_1, y_2, y_3 \in \mathcal{C}.$$

(c3). $\limsup_{n\to\infty} f(y_n, y) \leq f(q^*, y)$ for all $y \in \mathcal{C}$ and $\{y_n\} \subset \mathcal{C}$ satisfy $y_n \rightharpoonup q^*$.

(c4). $f(u, \cdot)$ is convex and subdifferentiable on $\mathcal{H}$ for each $u \in \mathcal{H}$.

A *cone on* $\mathcal{C}$ at $u \in \mathcal{C}$ is defined by

$$N_{\mathcal{C}}(u) = \{t \in \mathcal{H} : \langle t, y - u \rangle \leq 0, \forall y \in \mathcal{C}\}.$$

Let a convex function $\urcorner : \mathcal{C} \to \mathcal{R}$ and *subdifferential of* $\urcorner$ at $u \in \mathcal{C}$ is defined by

$$\partial \urcorner(u) = \{t \in \mathcal{H} : \urcorner(y) - \urcorner(u) \geq \langle t, y - u \rangle, \forall y \in \mathcal{C}\}.$$

Lemma 1. [37] *Let $\urcorner : \mathcal{C} \to \mathcal{R}$ be a subdifferentiable, lower semicontinuous, and convex function on $\mathcal{C}$. Then, $u \in \mathcal{C}$ is said to be a minimizer of $\urcorner$ if and only if $0 \in \partial \urcorner(u) + N_{\mathcal{C}}(u)$, where $\partial \urcorner(u)$ stands for the subdifferential of $\urcorner$ at $u \in \mathcal{C}$ and $N_{\mathcal{C}}(u)$ is a normal cone of $\mathcal{C}$ on u.*

Lemma 2. [38] *Assume that $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ be a metric projection such that*

(i) $\|y_1 - P_{\mathcal{C}}(y_2)\|^2 + \|P_{\mathcal{C}}(y_2) - y_2\|^2 \leq \|y_2 - y_1\|^2, \ y_1 \in \mathcal{C}, y_2 \in \mathcal{H}.$

(ii) $y_3 = P_{\mathcal{C}}(y_1)$ *if and only if* $\langle y_1 - y_3, y_2 - y_3 \rangle \leq 0, \ \forall y_2 \in \mathcal{C}.$

(iii) $\|y_1 - P_{\mathcal{C}}(y_1)\| \leq \|y_1 - y_2\|, \ y_2 \in \mathcal{C}, y_1 \in \mathcal{H}.$

Lemma 3. [39] *Assume that $\{\urcorner_n\} \subset (0, +\infty)$ is a sequence satisfying, i.e., $\urcorner_{n+1} \leq (1 - \mho_n)\urcorner_n + \mho_n \eth_n$, for all $n \in \mathbb{N}$. Moreover, let $\{\mho_n\} \subset (0, 1)$ and $\{\eth_n\} \subset \mathcal{R}$ be two sequences, such that $\lim_{n \to \infty} \mho_n = 0$, $\sum_{n=1}^{\infty} \mho_n = +\infty$ and $\limsup_{n \to \infty} \eth_n \leq 0$. Then, $\lim_{n \to \infty} \urcorner_n = 0$.*

Lemma 4. [40] *Assume that $\{\urcorner_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\urcorner_{n_i} < \urcorner_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then, there is a nondecreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$, and the following conditions are fullfiled by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\urcorner_{m_k} \leq \urcorner_{m_{k+1}} \text{ and } \urcorner_k \leq \urcorner_{m_{k+1}}.$$

In fact, $m_k = \max\{j \leq k : \urcorner_j \leq \urcorner_{j+1}\}$.

Lemma 5. [41] *For all $y_1, y_2 \in \mathcal{H}$ and $\eth \in \mathcal{R}$, the following inequalities hold.*

(i) $\|\eth y_1 + (1 - \eth) y_2\|^2 = \eth \|y_1\|^2 + (1 - \eth)\|y_2\|^2 - \eth(1 - \eth)\|y_1 - y_2\|^2.$

(ii) $\|y_1 + y_2\|^2 \leq \|y_1\|^2 + 2\langle y_2, y_1 + y_2 \rangle.$

3. Main Results

We propose an iterative method for solving equilibrium problems involving a pseudomonotone that is based on Tran et al. in [23], and the Mann-type method [32] and the inertial scheme [26]. For clarity in the presentation, we use notation $[t]_+ = \max\{0, t\}$ and follow conventions $\frac{0}{0} = +\infty$ and $\frac{a}{0} = +\infty$ $(a \neq 0)$.

Lemma 6. *A sequence $\{\chi_n\}$ generated by (5) is monotonically decreasing, converges to $\chi > 0$, and has a lower bound $\min\left\{\frac{\mu}{2\max\{c_1, c_2\}}, \chi_0\right\}$.*

Proof. Assume that $f(t_n, z_n) - f(t_n, y_n) - f(y_n, z_n) > 0$ such that

$$\begin{aligned} \frac{\mu(\|t_n - y_n\|^2 + \|z_n - y_n\|^2)}{2[f(t_n, z_n) - f(t_n, y_n) - f(y_n, z_n)]} &\geq \frac{\mu(\|t_n - y_n\|^2 + \|z_n - y_n\|^2)}{2[c_1\|t_n - y_n\|^2 + c_2\|z_n - y_n\|^2]} \\ &\geq \frac{\mu}{2\max\{c_1, c_2\}}. \end{aligned} \tag{3}$$

This implies that $\{\chi_n\}$ has a lower bound $\min\left\{\frac{\mu}{2\max\{c_1, c_2\}}, \chi_0\right\}$. Moreover, there exists a fixed real number $\chi > 0$, such that $\lim_{n \to \infty} \chi_n = \chi$. $\square$

Lemma 7. *Suppose that Conditions* (c1)–(c4) *are satisfied. Then, sequence* $\{u_n\}$ *generated by the Algorithm 1 is a bounded sequence.*

Algorithm 1 (Explicit Accelerated Strong Convergence Iterative Scheme)

STEP 0: Choose $u_{-1}, u_0 \in \mathcal{C}$, $\phi > 0$, $\chi_0 > 0$, $\{\rho_n\} \subset (a, b) \subset (0, 1 - \varrho_n)$ and $\{\varrho_n\} \subset (0, 1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \varrho_n = 0 \text{ and } \sum_{n=1}^{+\infty} \varrho_n = +\infty.$$

STEP 1: Compute $t_n = u_n + \phi_n(u_n - u_{n-1})$ and choose ϕ_n such that

$$0 \leq \phi_n \leq \hat{\phi_n} \quad \text{and} \quad \hat{\phi_n} = \begin{cases} \min\left\{\frac{\phi}{2}, \frac{\varsigma_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \frac{\phi}{2} & \text{otherwise,} \end{cases} \tag{4}$$

where $\varsigma_n = \circ(\varrho_n)$, i.e., $\lim_{n\to\infty} \frac{\varsigma_n}{\varrho_n} = 0$.

STEP 2: Compute

$$y_n = \arg\min_{y \in \mathcal{C}} \{\chi_n f(t_n, y) + \frac{1}{2}\|t_n - y\|^2\}.$$

If $t_n = y_n$, then STOP the sequence. Else, go to STEP 3.

STEP 3: Construct a half-space $\mathcal{H}_n = \{z \in \mathcal{H} : \langle t_n - \chi_n \omega_n - y_n, z - y_n \rangle \leq 0\}$ where $\omega_n \in \partial_2 f(t_n, y_n)$ and compute

$$z_n = \arg\min_{y \in \mathcal{H}_n} \{\chi_n f(y_n, y) + \frac{1}{2}\|t_n - y\|^2\}.$$

STEP 4: Compute $u_{n+1} = (1 - \rho_n - \varrho_n)u_n + \rho_n z_n$.

STEP 5: Compute

$$\chi_{n+1} = \min\left\{\chi_n, \frac{\mu\|t_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2[f(t_n, z_n) - f(t_n, y_n) - f(y_n, z_n)]_+}\right\}. \tag{5}$$

Set $n := n + 1$ and go back to **Step 1**.

Proof. From the value of z_n, we have

$$0 \in \partial_2\Big\{\chi_n f(y_n, y) + \frac{1}{2}\|t_n - y\|^2\Big\}(z_n) + N_{\mathcal{H}_n}(z_n).$$

For $\omega \in \partial f(y_n, z_n)$ there exists $\overline{\omega} \in N_{\mathcal{H}_n}(z_n)$ such that

$$\chi_n \omega + z_n - t_n + \overline{\omega} = 0.$$

This implies that

$$\langle t_n - z_n, y - z_n \rangle = \chi_n \langle \omega, y - z_n \rangle + \langle \overline{\omega}, y - z_n \rangle, \ \forall y \in \mathcal{H}_n.$$

Due to $\overline{\omega} \in N_{\mathcal{H}_n}(z_n)$, it implies that $\langle \overline{\omega}, y - z_n \rangle \leq 0$ for each $y \in \mathcal{H}_n$. Thus, we have

$$\langle t_n - z_n, y - z_n \rangle \leq \chi_n \langle \omega, y - z_n \rangle, \ \forall y \in \mathcal{H}_n. \tag{6}$$

Moreover, $\omega \in \partial f(y_n, z_n)$ and owing to the subdifferential, we have

$$f(y_n, y) - f(y_n, z_n) \geq \langle \omega, y - z_n \rangle, \ \forall y \in \mathcal{H}. \tag{7}$$

From Expressions (6) and (7), we obtain

$$\chi_n f(y_n, y) - \chi_n f(y_n, z_n) \geq \langle t_n - z_n, y - z_n \rangle, \ \forall y \in \mathcal{H}_n. \tag{8}$$

Due to the definition of $\mathcal{H}_n$, we have

$$\chi_n \langle \omega_n, z_n - y_n \rangle \geq \langle t_n - y_n, z_n - y_n \rangle. \tag{9}$$

Now, using $\omega_n \in \partial f(t_n, y_n)$, we obtain

$$f(t_n, y) - f(t_n, y_n) \geq \langle \omega_n, y - y_n \rangle, \ \forall y \in \mathcal{H}.$$

By letting $y = z_n$, we obtain

$$f(t_n, z_n) - f(t_n, y_n) \geq \langle \omega_n, z_n - y_n \rangle, \ \forall y \in \mathcal{H}. \tag{10}$$

Combining Expressions (9) and (10), we obtain

$$\chi_n \{ f(t_n, z_n) - f(t_n, y_n) \} \geq \langle t_n - y_n, z_n - y_n \rangle. \tag{11}$$

By substituting $y = u^*$ in Expression (8), we obtain

$$\chi_n f(y_n, u^*) - \chi_n f(y_n, z_n) \geq \langle t_n - z_n, u^* - z_n \rangle. \tag{12}$$

Since $u^* \in Ep(f, \mathcal{C})$, we have $f(u^*, y_n) \geq 0$. From the pseudomonotonicity of bifunction f, we achieve $f(y_n, u^*) \leq 0$. It follows from Expression (12) that

$$\langle t_n - z_n, z_n - u^* \rangle \geq \chi_n f(y_n, z_n). \tag{13}$$

From the description of χ_{n+1}, we obtain

$$f(t_n, z_n) - f(t_n, y_n) - f(y_n, z_n) \leq \frac{\mu \|t_n - y_n\|^2 + \mu \|z_n - y_n\|^2}{2\chi_{n+1}} \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned} \langle t_n - z_n, z_n - u^* \rangle \geq\ & \chi_n \{ f(t_n, z_n) - f(t_n, y_n) \} \\ & - \frac{\mu \chi_n}{2\chi_{n+1}} \|t_n - y_n\|^2 - \frac{\mu \chi_n}{2\chi_{n+1}} \|z_n - y_n\|^2. \end{aligned} \tag{15}$$

Combining Expressions (11) and (15), we have

$$\begin{aligned} \langle t_n - z_n, z_n - u^* \rangle \geq\ & \langle t_n - y_n, z_n - y_n \rangle \\ & - \frac{\mu \chi_n}{2\chi_{n+1}} \|t_n - y_n\|^2 - \frac{\mu \chi_n}{2\chi_{n+1}} \|z_n - y_n\|^2. \end{aligned} \tag{16}$$

We have the given formula in place:

$$-2\langle t_n - z_n, z_n - u^* \rangle = -\|t_n - u^*\|^2 + \|z_n - t_n\|^2 + \|z_n - u^*\|^2. \tag{17}$$

$$2\langle y_n - t_n, y_n - z_n \rangle = \|t_n - y_n\|^2 + \|z_n - y_n\|^2 - \|t_n - z_n\|^2. \tag{18}$$

Combining (16)–(18), we obtain

$$\|z_n - u^*\|^2 \leq \|t_n - u^*\|^2 - \left(1 - \frac{\mu \chi_n}{\chi_{n+1}}\right) \|t_n - y_n\|^2 - \left(1 - \frac{\mu \chi_n}{\chi_{n+1}}\right) \|z_n - y_n\|^2. \tag{19}$$

Since $\chi_n \to \chi$, then there is number $\Im \in (0, 1-\mu)$ that

$$\lim_{n\to\infty}\left(1-\frac{\mu\chi_n}{\chi_{n+1}}\right)=1-\mu>\Im>0.$$

Thus, there exists a finite number $n_1 \in \mathbb{N}$, such that

$$\left(1-\frac{\mu\chi_n}{\chi_{n+1}}\right)>\Im>0,\ \forall n\geq n_1. \tag{20}$$

From Expression (19), we obtain

$$\|u_{n+1}-u^*\|^2 \leq \|t_n-u^*\|^2,\ \forall n\geq n_1. \tag{21}$$

From Expression (4), we have $\phi_n\|u_n-u_{n-1}\| \leq \varsigma_n$, for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\left(\frac{\varsigma_n}{\varrho_n}\right)=0$ implies that

$$\lim_{n\to\infty}\frac{\phi_n}{\varrho_n}\|u_n-u_{n-1}\| \leq \lim_{n\to\infty}\frac{\varsigma_n}{\varrho_n}=0. \tag{22}$$

From Expression (21) and $\{t_n\}$, we have

$$\begin{aligned}\|z_n-u^*\| \leq \|t_n-u^*\| &= \|u_n+\phi_n(u_n-u_{n-1})-u^*\|\\ &\leq \|u_n-u^*\|+\phi_n\|u_n-u_{n-1}\|\\ &\leq \|u_n-u^*\|+\varrho_n\frac{\phi_n}{\varrho_n}\|u_n-u_{n-1}\|\\ &\leq \|u_n-u^*\|+\varrho_n\beth_1,\end{aligned} \tag{23}$$

where for some fixed $\beth_1>0$ and

$$\frac{\phi_n}{\varrho_n}\|u_n-u_{n-1}\| \leq \beth_1,\ \forall n\geq 1. \tag{24}$$

It is given that $u^*\in Ep(f,\mathcal{C})$ and by definition of $\{u_{n+1}\}$, we have

$$\begin{aligned}\|u_{n+1}-u^*\| &= \|(1-\rho_n-\varrho_n)u_n+\rho_n z_n-u^*\|\\ &= \|(1-\rho_n-\varrho_n)(u_n-u^*)+\rho_n(z_n-u^*)-\varrho_n u^*\|\\ &\leq \|(1-\rho_n-\varrho_n)(u_n-u^*)+\rho_n(z_n-u^*)\|+\varrho_n\|u^*\|.\end{aligned} \tag{25}$$

Next, we compute

$$\begin{aligned}&\|(1-\rho_n-\varrho_n)(u_n-u^*)+\rho_n(z_n-u^*)\|^2\\ &= (1-\rho_n-\varrho_n)^2\|u_n-u^*\|^2+\rho_n^2\|z_n-u^*\|^2+2\langle(1-\rho_n-\varrho_n)(u_n-u^*),\rho_n(z_n-u^*)\rangle\\ &\leq (1-\rho_n-\varrho_n)^2\|u_n-u^*\|^2+\rho_n^2\|z_n-u^*\|^2+2\rho_n(1-\rho_n-\varrho_n)\|u_n-u^*\|\|z_n-u^*\|\\ &\leq (1-\rho_n-\varrho_n)^2\|u_n-u^*\|^2+\rho_n^2\|z_n-u^*\|^2\\ &\quad+\rho_n(1-\rho_n-\varrho_n)\|u_n-u^*\|^2+\rho_n(1-\rho_n-\varrho_n)\|z_n-u^*\|^2\\ &\leq (1-\rho_n-\varrho_n)(1-\varrho_n)\|u_n-u^*\|^2+\rho_n(1-\varrho_n)\|z_n-u^*\|^2 \qquad (26)\\ &\leq (1-\rho_n-\varrho_n)(1-\varrho_n)\|u_n-u^*\|^2+\rho_n(1-\varrho_n)(\|u_n-u^*\|+\varrho_n\beth_1)^2\\ &\leq (1-\varrho_n)^2\|u_n-u^*\|^2+\varrho_n^2\beth_1^2+2\varrho_n\beth_1(1-\varrho_n)\|u_n-u^*\|^2. \qquad (27)\end{aligned}$$

The above expression implies that

$$\|(1-\rho_n-\varrho_n)(u_n-u^*)+\rho_n(z_n-u^*)\| \leq (1-\varrho_n)\|u_n-u^*\|+\varrho_n\beth_1. \tag{28}$$

Combining Expressions (25) and (28), we obtain

$$\begin{aligned}\|u_{n+1} - u^*\| &\leq (1-\varrho_n)\|u_n - u^*\| + \varrho_n \gimel_1 + \varrho_n\|u^*\| \\ &\leq \max\left\{\|u_n - u^*\|, \gimel_1 + \|u^*\|\right\} \\ &\leq \vdots \\ &\leq \max\left\{\|u_0 - u^*\|, \gimel_1 + \|u^*\|\right\}. \end{aligned} \tag{29}$$

Therefore, we conclude that $\{u_n\}$ is bounded sequence. □

Theorem 1. *Let $\{u_n\}$ be a sequence generated by Algorithm 1, and Conditions* (c1)–(c4) *are satisfied. Then, $\{u_n\}$ strongly converges to $u^* = P_{Ep(f,\mathcal{C})}(0)$.*

Proof. By using definition of $\{u_{n+1}\}$, we have

$$\begin{aligned}\|u_{n+1} - u^*\|^2 &= \|(1-\rho_n-\varrho_n)u_n + \rho_n z_n - u^*\|^2 \\ &= \|(1-\rho_n-\varrho_n)(u_n - u^*) + \rho_n(z_n - u^*) - \varrho_n u^*\|^2 \\ &= \|(1-\rho_n-\varrho_n)(u_n - u^*) + \rho_n(z_n - u^*)\|^2 + \varrho_n^2\|u^*\|^2 \\ &\quad - 2\langle(1-\rho_n-\varrho_n)(u_n - u^*) + \rho_n(z_n - u^*),\, \varrho_n u^*\rangle. \end{aligned} \tag{30}$$

From Expression (26), we have

$$\begin{aligned}&\|(1-\rho_n-\varrho_n)(u_n - u^*) + \rho_n(z_n - u^*)\|^2 \\ &\leq (1-\rho_n-\varrho_n)(1-\varrho_n)\|u_n - u^*\|^2 + \rho_n(1-\varrho_n)\|z_n - u^*\|^2. \end{aligned} \tag{31}$$

Combining Expressions (30) and (31) (for some $\gimel_2 > 0$), we obtain

$$\begin{aligned}&\|u_{n+1} - u^*\|^2 \\ &\leq (1-\rho_n-\varrho_n)(1-\varrho_n)\|u_n - u^*\|^2 + \rho_n(1-\varrho_n)\|z_n - u^*\|^2 + \varrho_n\gimel_2 \\ &\leq (1-\rho_n-\varrho_n)(1-\varrho_n)\|u_n - u^*\|^2 + \varrho_n\gimel_2 \\ &\quad + \rho_n(1-\varrho_n)\left[\|t_n - u^*\|^2 - \left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\|t_n - y_n\|^2 - \left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\|z_n - y_n\|^2\right]. \end{aligned} \tag{32}$$

From Expression (23), we have

$$\|t_n - u^*\|^2 \leq \|u_n - u^*\|^2 + \varrho_n\gimel_3, \tag{33}$$

for some $\gimel_3 > 0$. Substituting (33) into (32), we obtain

$$\begin{aligned}&\|u_{n+1} - u^*\|^2 \\ &\leq (1-\rho_n-\varrho_n)(1-\varrho_n)\|u_n - u^*\|^2 + \varrho_n\gimel_2 \\ &\quad +\rho_n(1-\varrho_n)\left[\|u_n - u^*\|^2 + \varrho_n\gimel_3 - \left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\|t_n - y_n\|^2 - \left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\|z_n - y_n\|^2\right] \\ &= (1-\varrho_n)^2\|u_n - u^*\|^2 + \varrho_n\gimel_2 + \rho_n(1-\varrho_n)\varrho_n\gimel_3 \\ &\quad -\rho_n(1-\varrho_n)\left[\left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\|t_n - y_n\|^2 + \left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\|z_n - y_n\|^2\right] \\ &\leq \|u_n - u^*\|^2 + \varrho_n\gimel_4 - \rho_n(1-\varrho_n)\left[\left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\|t_n - y_n\|^2 + \left(1 - \frac{\mu\chi_n}{\chi_{n+1}}\right)\|z_n - y_n\|^2\right], \end{aligned} \tag{34}$$

for some $\gimel_4 > 0$. It is given that $u^* = P_{Ep(f,\mathcal{C})}(0)$ and by using Lemma 2 (ii) ($Ep(f,\mathcal{C})$ is a convex and closed set ([23,34])), we obtain

$$\langle u^*, u^* - y\rangle \leq 0,\ \forall\, y \in Ep(f,\mathcal{C}). \tag{35}$$

The remainder of the proof shall be taken into account in the following two parts:

Case 1: Assume that there is a fixed number $n_2 \in \mathbb{N}$ $(n_2 \geq n_1)$ such as

$$\|u_{n+1} - u^*\| \leq \|u_n - u^*\|, \ \forall\, n \geq n_2. \tag{36}$$

It implies that $\lim_{n\to\infty} \|u_n - u^*\|$ exists, and due to (34), we obtain

$$\begin{aligned}
&\rho_n(1-\varrho_n)\Big[\Big(1 - \frac{\mu\chi_n}{\chi_{n+1}}\Big)\|t_n - y_n\|^2 + \Big(1 - \frac{\mu\chi_n}{\chi_{n+1}}\Big)\|z_n - y_n\|^2\Big] \\
&\leq \|u_n - u^*\|^2 + \varrho_n \daleth_4 - \|u_{n+1} - u^*\|^2.
\end{aligned} \tag{37}$$

Due to the existence of $\lim_{n\to\infty} \|u_n - u^*\|$, $\varrho_n \to 0$ and $\chi_n \to \chi$, we infer that

$$\lim_{n\to\infty} \|y_n - t_n\| = \lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{38}$$

We can calculate that

$$\lim_{n\to\infty} \|z_n - t_n\| \leq \lim_{n\to\infty} \|t_n - y_n\| + \lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{39}$$

It follows that

$$\begin{aligned}
\big\|u_{n+1} - u_n\big\| &= \big\|(1 - \rho_n - \varrho_n)u_n + \rho_n z_n - u_n\big\| \\
&= \big\|u_n - \varrho_n u_n + \rho_n z_n - \rho_n u_n - u_n\big\| \\
&\leq \rho_n\big\|z_n - u_n\big\| + \varrho_n\big\|u_n\big\|.
\end{aligned} \tag{40}$$

The term is referred to above that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{41}$$

Thus, this implies that $\{y_n\}$ and $\{z_n\}$ are bounded. The reflexivity of $\mathcal{H}$ and the boundedness of $\{u_n\}$ guarantee that there is a subsequence $\{u_{n_k}\}$, such that $\{u_{n_k}\} \rightharpoonup \hat{x} \in \mathcal{H}$ as $k \to \infty$. Next, our aim to prove that $\hat{x} \in Ep(f, \mathcal{C})$. Using (8), due to χ_{n+1} and (11), we write

$$\begin{aligned}
\chi_{n_k} f(y_{n_k}, y) &\geq \chi_{n_k} f(y_{n_k}, z_{n_k}) + \langle t_{n_k} - z_{n_k}, y - z_{n_k}\rangle \\
&\geq \chi_{n_k} f(t_{n_k}, z_{n_k}) - \chi_{n_k} f(t_{n_k}, y_{n_k}) - \tfrac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|t_{n_k} - y_{n_k}\|^2 \\
&\quad - \tfrac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|y_{n_k} - z_{n_k}\|^2 + \langle t_{n_k} - z_{n_k}, y - z_{n_k}\rangle \\
&\geq \langle t_{n_k} - y_{n_k}, z_{n_k} - y_{n_k}\rangle - \tfrac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|t_{n_k} - y_{n_k}\|^2 \\
&\quad - \tfrac{\mu\chi_{n_k}}{2\chi_{n_k+1}}\|y_{n_k} - z_{n_k}\|^2 + \langle t_{n_k} - z_{n_k}, y - z_{n_k}\rangle,
\end{aligned} \tag{42}$$

while y is an any arbitrary member in $\mathcal{H}_n$. It continues from (38) and (39) that the right-hand side approaches to zero. From $\chi > 0$, Condition (c3) and $y_{n_k} \rightharpoonup \hat{x}$, we have

$$0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(\hat{x}, y), \ \forall\, y \in \mathcal{H}_n. \tag{43}$$

The following is that $f(\hat{x}, y) \geq 0$, $\forall y \in \mathcal{C}$; thus $\hat{x} \in Ep(f, \mathcal{C})$. It continues from that

$$\limsup_{n\to\infty} \langle u^*, u^* - u_n\rangle = \limsup_{k\to\infty} \langle u^*, u^* - u_{n_k}\rangle = \langle u^*, u^* - \hat{x}\rangle \leq 0. \tag{44}$$

Due to $\lim_{n\to\infty} \big\|u_{n+1} - u_n\big\| = 0.$, we can deduce that

$$\limsup_{n\to\infty} \langle u^*, u^* - u_{n+1}\rangle \leq \limsup_{k\to\infty} \langle u^*, u^* - u_n\rangle + \limsup_{k\to\infty} \langle u^*, u_n - u_{n+1}\rangle \leq 0. \tag{45}$$

Next, consider the following value

$$\begin{aligned}
\|t_n - u^*\|^2 &= \|u_n + \phi_n(u_n - u_{n-1}) - u^*\|^2 \\
&= \|u_n - u^* + \phi_n(u_n - u_{n-1})\|^2 \\
&= \|u_n - u^*\|^2 + \phi_n^2\|u_n - u_{n-1}\|^2 + 2\langle u_n - u^*, \phi_n(u_n - u_{n-1})\rangle \\
&\le \|u_n - u^*\|^2 + \phi_n^2\|u_n - u_{n-1}\|^2 + 2\phi_n\|u_n - u^*\|\|u_n - u_{n-1}\| \\
&= \|u_n - u^*\|^2 + \phi_n\|u_n - u_{n-1}\|[2\|u_n - u^*\| + \phi_n\|u_n - u_{n-1}\|] \\
&\le \|u_n - u^*\|^2 + \phi_n\|u_n - u_{n-1}\|\beth_5,
\end{aligned} \tag{46}$$

Substituting $q_n = (1-\rho_n)u_n + \rho_n z_n$, we have

$$u_{n+1} = q_n - \varrho_n u_n = (1-\varrho_n)q_n - \varrho_n(u_n - q_n) = (1-\varrho_n)q_n - \varrho_n\rho_n(u_n - z_n). \tag{47}$$

where $u_n - q_n = u_n - (1-\rho_n)u_n - \rho_n z_n = \rho_n(u_n - z_n)$. Consider that

$$\begin{aligned}
&\|q_n - u^*\|^2 \\
&= \|(1-\rho_n)u_n + \rho_n z_n - u^*\|^2 \\
&= \|(1-\rho_n)(u_n - u^*) + \rho_n(z_n - u^*)\|^2 \\
&= (1-\rho_n)^2\|u_n - u^*\|^2 + \rho_n^2\|z_n - u^*\|^2 + 2\langle(1-\rho_n)(u_n - u^*), \rho_n(z_n - u^*)\rangle \\
&\le (1-\rho_n)^2\|u_n - u^*\|^2 + \rho_n^2\|z_n - u^*\|^2 + 2\rho_n(1-\rho_n)\|u_n - u^*\|\|z_n - u^*\| \\
&\le (1-\rho_n)^2\|u_n - u^*\|^2 + \rho_n^2\|z_n - u^*\|^2 + \rho_n(1-\rho_n)\|u_n - u^*\|^2 + \rho_n(1-\rho_n)\|z_n - u^*\|^2 \\
&= (1-\rho_n)\|u_n - u^*\|^2 + \rho_n\|z_n - u^*\|^2 \\
&\le (1-\rho_n)\|u_n - u^*\|^2 + \rho_n\|t_n - u^*\|^2 \\
&\le (1-\rho_n)\|u_n - u^*\|^2 + \rho_n[\|u_n - u^*\|^2 + \phi_n\|u_n - u_{n-1}\|\beth_5] \\
&\le \|u_n - u^*\|^2 + \phi_n\|u_n - u_{n-1}\|\beth_5.
\end{aligned} \tag{48}$$

Next, consider that

$$\begin{aligned}
&\|u_{n+1} - u^*\|^2 \\
&= \|(1-\varrho_n)q_n + \rho_n\varrho_n(z_n - u_n) - u^*\|^2 \\
&= \|(1-\varrho_n)(q_n - u^*) + [\rho_n\varrho_n(z_n - u_n) - \varrho_n u^*]\|^2 \\
&\le (1-\varrho_n)^2\|q_n - u^*\|^2 + 2\langle\rho_n\varrho_n(z_n - u_n) - \varrho_n u^*,\ (1-\varrho_n)(q_n - u^*) + \rho_n\varrho_n(z_n - u_n) - \varrho_n u^*\rangle \\
&= (1-\varrho_n)^2\|q_n - u^*\|^2 + 2\langle\rho_n\varrho_n(z_n - u_n) - \varrho_n u^*,\ q_n - \varrho_n q_n - \varrho_n(u_n - q_n) - u^*\rangle \\
&= (1-\varrho_n)\|q_n - u^*\|^2 + 2\rho_n\varrho_n\langle z_n - u_n,\ u_{n+1} - u^*\rangle + 2\varrho_n\langle u^*,\ u^* - u_{n+1}\rangle \\
&\le (1-\varrho_n)\|q_n - u^*\|^2 + 2\rho_n\varrho_n\|z_n - u_n\|\|u_{n+1} - u^*\| + 2\varrho_n\langle u^*,\ u^* - u_{n+1}\rangle
\end{aligned} \tag{49}$$

for some $\beth_5 > 0$. Combining Expressions (46), (48), and (49), we obtain

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &\le (1-\varrho_n)\|u_n - u^*\|^2 + (1-\varrho_n)\phi_n\|u_n - u_{n-1}\|\beth_5 \\
&\quad + 2\rho_n\varrho_n\|z_n - u_n\|\|u_{n+1} - u^*\| + 2\varrho_n\langle u^*,\ u^* - u_{n+1}\rangle \\
&\le (1-\varrho_n)\|u_n - u^*\|^2 + \varrho_n\bigg[\frac{\phi_n}{\varrho_n}(1-\varrho_n)\|u_n - u_{n-1}\|\beth_5 \\
&\quad + 2\rho_n\|z_n - u_n\|\|u_{n+1} - u^*\| + 2\langle u^*,\ u^* - u_{n+1}\rangle\bigg].
\end{aligned} \tag{50}$$

Due to (45), (50), and the implemented Lemma 3, we conclude that $\|u_n - u^*\| \to 0$ as $n \to \infty$.

Case 2: Assume there is a subsequence $\{n_i\}$ of $\{n\}$ that

$$\|u_{n_i} - u^*\| \le \|u_{n_{i+1}} - u^*\|,\ \forall i \in \mathbb{N}.$$

Using Lemma 4, there is a $\{m_k\} \subset \mathbb{N}$ sequence, such as $\{m_k\} \to \infty$,

$$\|u_{m_k} - u^*\| \le \|u_{m_{k+1}} - u^*\| \quad \text{and} \quad \|u_k - u^*\| \le \|u_{m_{k+1}} - u^*\|, \text{ for all } k \in \mathbb{N}. \tag{51}$$

Similar to Case 1, Relation (37) gives that

$$\begin{aligned}&\rho_{m_k}(1-\varrho_{m_k})\Big[\Big(1-\frac{\mu\chi_{m_k}}{\chi_{m_k+1}}\Big)\|t_{m_k}-y_{m_k}\|^2+\Big(1-\frac{\mu\chi_{m_k}}{\chi_{m_k+1}}\Big)\|z_{m_k}-y_{m_k}\|^2\Big]\\&\leq\|u_{m_k}-u^*\|^2+\varrho_{m_k}\gimel_4-\|u_{m_k+1}-u^*\|^2.\end{aligned}\tag{52}$$

Due to $\varrho_{m_k}\to 0$ and $\chi_{m_k}\to\chi$, we deduce the following:

$$\lim_{n\to\infty}\|t_{m_k}-y_{m_k}\|=\lim_{n\to\infty}\|z_{m_k}-y_{m_k}\|=0.\tag{53}$$

It continues on from that

$$\begin{aligned}\|u_{m_k+1}-u_{m_k}\| &= \|(1-\rho_{m_k}-\varrho_{m_k})u_{m_k}+\rho_{m_k}z_{m_k}-u_{m_k}\|\\&= \|u_{m_k}-\varrho_{m_k}u_{m_k}+\rho_{m_k}z_{m_k}-\rho_{m_k}u_{m_k}-u_{m_k}\|\\&\leq \rho_{m_k}\|z_{m_k}-u_{m_k}\|+\varrho_{m_k}\|u_{m_k}\|\longrightarrow 0.\end{aligned}\tag{54}$$

We use the same reasoning as that in Case 1:

$$\limsup_{k\to\infty}\langle u^*,u^*-u_{m_k+1}\rangle\leq 0.\tag{55}$$

Now, using Expressions (50) and (51), we have

$$\begin{aligned}&\|u_{m_k+1}-u^*\|^2\\&\leq(1-\varrho_{m_k})\|u_{m_k}-u^*\|^2+\varrho_{m_k}\Big[\frac{\phi_{m_k}}{\varrho_{m_k}}(1-\varrho_{m_k})\|u_{m_k}-u_{m_k-1}\|\gimel_5\\&\quad+2\rho_{m_k}\|z_{m_k}-u_{m_k}\|\|u_{m_k+1}-u^*\|+2\langle u^*,\,u^*-u_{m_k+1}\rangle\Big].\\&\leq(1-\varrho_{m_k})\|u_{m_k+1}-u^*\|^2+\varrho_{m_k}\Big[\frac{\phi_{m_k}}{\varrho_{m_k}}(1-\varrho_{m_k})\|u_{m_k}-u_{m_k-1}\|\gimel_5\\&\quad+2\rho_{m_k}\|z_{m_k}-u_{m_k}\|\|u_{m_k+1}-u^*\|+2\langle u^*,\,u^*-u_{m_k+1}\rangle\Big].\end{aligned}\tag{56}$$

It implies that

$$\begin{aligned}\|u_{m_k+1}-u^*\|^2 \leq &\Big[\frac{\phi_{m_k}}{\varrho_{m_k}}(1-\varrho_{m_k})\|u_{m_k}-u_{m_k-1}\|\gimel_5\\&+2\rho_{m_k}\|z_{m_k}-u_{m_k}\|\|u_{m_k+1}-u^*\|+2\langle u^*,\,u^*-u_{m_k+1}\rangle\Big].\end{aligned}\tag{57}$$

Since $\varrho_{m_k}\to 0$, and $\|u_{m_k}-u^*\|$ is bounded. Thus, with Expressions (55) and (57), we have

$$\|u_{m_k+1}-u^*\|^2\to 0,\ \text{as } k\to\infty.\tag{58}$$

The above implies that

$$\lim_{n\to\infty}\|u_k-u^*\|^2\leq\lim_{n\to\infty}\|u_{m_k+1}-u^*\|^2\leq 0.\tag{59}$$

As a result, $u_n\to u^*$. This completes the proof of the theorem. □

By letting $\phi_n=0$, we obtain a strong convergence of the result in [25].

Corollary 1. *Let $f:\mathcal{C}\times\mathcal{C}\to\mathcal{R}$ be a bifunction satisfying Conditions* (c1)–(c4)*. Choosing $u_0\in\mathcal{C}$, $\chi_0>0$, $\{\rho_n\}\subset(a,b)\subset(0,1-\varrho_n)$ and $\{\varrho_n\}\subset(0,1)$ satisfies the following conditions:*

$$\lim_{n\to\infty}\varrho_n=0 \text{ and } \sum_{n}^{\infty}\varrho_n=+\infty.$$

Let $\{u_n\}$ be a sequence that is generated in the following manner:

$$\begin{cases} y_n = \arg\min_{y \in \mathcal{C}} \{\chi_n f(u_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \\ z_n = \arg\min_{y \in \mathcal{H}_n} \{\chi_n f(y_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \\ u_{n+1} = (1 - \rho_n - \varrho_n) u_n + \rho_n z_n, \end{cases} \tag{60}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle u_n - \chi_n \omega_n - y_n, z - y_n \rangle \leq 0\}$ and $\omega_n \in \partial_2 f(u_n, y_n)$. The step size is updated in the following way:

$$\chi_{n+1} = \min\left\{\chi_n, \frac{\mu\|u_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2[f(u_n, z_n) - f(u_n, y_n) - f(y_n, z_n)]_+}\right\}.$$

Then, sequence $\{u_n\}$ converges strongly to $u^* \in Ep(f, \mathcal{C})$.

4. Applications to Solve Fixed-Point Problems

We propose our results to focus on fixed-point problems regarding κ-strict pseudocontraction mapping. The fixed-point problem (FPP) for $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is defined in the following manner:

$$\text{Find } u^* \in \mathcal{C} \text{ such that } \mathcal{S}(u^*) = u^*. \tag{FPP}$$

We assume that the following conditions were met:

(c1*) A mapping $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ is said to be *κ-strict pseudocontraction* [42] on $\mathcal{C}$ if

$$\|Ty_1 - Ty_2\|^2 \leq \|y_1 - y_2\|^2 + \kappa\|(y_1 - Ty_1) - (y_2 - Ty_2)\|^2, \ \forall y_1, y_2 \in \mathcal{C};$$

(c2*) A mapping that is weakly sequentially continuous on $\mathcal{C}$ if

$$\mathcal{S}(y_n) \rightharpoonup \mathcal{S}(q^*) \text{ for any sequence in } \mathcal{C} \text{ satisfying } y_n \rightharpoonup q^*.$$

If we consider that mapping $\mathcal{S}$ is weakly continuous and a κ-strict pseudocontraction, then $f(u, y) = \langle u - \mathcal{S}u, y - u \rangle$ satisfies the conditions (c1)–(c4) (see [43]) and $2c_1 = 2c_2 = \frac{3-2\kappa}{1-\kappa}$. The values of y_n and z_n in Algorithm 1 can be written as follows:

$$\begin{cases} y_n = \arg\min_{y \in \mathcal{C}} \{\chi_n f(t_n, y) + \frac{1}{2}\|t_n - y\|^2\} = P_{\mathcal{C}}[t_n - \chi_n(t_n - \mathcal{S}(t_n))], \\ z_n = \arg\min_{y \in \mathcal{H}_n} \{\chi_n f(y_n, y) + \frac{1}{2}\|t_n - y\|^2\} = P_{\mathcal{H}_n}[t_n - \chi_n(y_n - \mathcal{S}(y_n))]. \end{cases} \tag{61}$$

Corollary 2. *Suppose $\mathcal{C}$ is a nonempty, convex, and closed subset of a Hilbert space $\mathcal{H}$ and $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ is weakly continuous and κ-strict pseudocontraction with solution set $Fix(\mathcal{S}) \neq \emptyset$. Let $u_{-1}, u_0 \in \mathcal{C}$, $\phi > 0$, $\chi_0 > 0$, $\{\rho_n\} \subset (a, b) \subset (0, 1 - \varrho_n)$ and $\{\varrho_n\} \subset (0, 1)$ fulfill the items, i.e., $\lim_{n\to\infty} \varrho_n = 0$ and $\sum_{n=1}^{\infty} \varrho_n = +\infty$. Moreover, choose ϕ_n satisfying $0 \leq \phi_n \leq \hat{\phi_n}$ such that*

$$\hat{\phi_n} = \begin{cases} \min\left\{\frac{\phi}{2}, \frac{\varsigma_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \frac{\phi}{2} & \text{else}, \end{cases} \tag{62}$$

where $\varsigma_n = \circ(\varrho_n)$, i.e., $\lim_{n\to\infty} \frac{\varsigma_n}{\varrho_n} = 0$. Assume that $\{u_n\}$ is the sequence generated in the following manner:

$$\begin{cases} t_n = u_n + \phi_n(u_n - u_{n-1}), \\ y_n = P_{\mathcal{C}}[t_n - \chi_n(t_n - \mathcal{S}(t_n))], \\ z_n = P_{\mathcal{H}_n}[t_n - \chi_n(y_n - \mathcal{S}(y_n))], \\ u_{n+1} = (1 - \rho_n - \varrho_n) u_n + \rho_n z_n, \end{cases}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle (1-\chi_n)t_n + \chi_n \mathcal{S}(t_n) - y_n, z - y_n \rangle \le 0\}$. *Compute*

$$\chi_{n+1} = \min\left\{\chi_n, \frac{\mu\|t_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2\big[\langle (t_n - y_n) - [T(t_n) - T(y_n)], z_n - y_n\rangle\big]_+}\right\}$$

Then, $\{u_n\}$ *strongly converges to* $u^* \in Fix(\mathcal{S}, \mathcal{C})$.

Corollary 3. *Suppose* $\mathcal{C}$ *to be a convex and closed subset of a Hilbert space* $\mathcal{H}$ *and* $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ *is weakly continuous and* κ*-strict pseudocontraction with solution set* $Fix(\mathcal{S}) \neq \emptyset$. *Let* $u_0 \in \mathcal{C}$, $\chi_0 > 0$, $\{\rho_n\} \subset (a, b) \subset (0, 1 - \varrho_n)$ *and* $\{\varrho_n\} \subset (0, 1)$ *fulfills the requirement, i.e.,* $\lim_{n\to\infty} \varrho_n = 0$ *and* $\sum_{n=1}^{\infty} \varrho_n = +\infty$. *Assume that* $\{u_n\}$ *is the sequence formed as follows:*

$$\begin{cases} y_n = P_{\mathcal{C}}\big[u_n - \chi_n(u_n - \mathcal{S}(u_n))\big], \\ z_n = P_{\mathcal{H}_n}\big[u_n - \chi_n(y_n - \mathcal{S}(y_n))\big], \\ u_{n+1} = (1 - \rho_n - \varrho_n)u_n + \rho_n z_n, \end{cases}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle (1-\chi_n)u_n + \chi_n \mathcal{S}(u_n) - y_n, z - y_n \rangle \le 0\}$. *Compute*

$$\chi_{n+1} = \min\left\{\chi_n, \frac{\mu\|u_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2\big[\langle (u_n - y_n) - [T(u_n) - T(y_n)], z_n - y_n\rangle\big]_+}\right\}$$

Then, sequence $\{u_n\}$ *converges strongly to* $u^* \in Fix(\mathcal{S}, \mathcal{C})$.

5. Applications to Solve Variational-Inequality Problems

Next, we consider the application of our results in the problem of classical variational inequalities [44,45]. The variational-inequality problem (VIP) for an operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is stated in the following manner:

$$\text{Find } u^* \in \mathcal{C} \text{ such that } \big\langle \mathcal{L}(u^*), y - u^* \big\rangle \ge 0, \ \forall\, y \in \mathcal{C}. \tag{VIP}$$

We assume that the following conditions were met:

($\mathcal{L}$1) The solution set of problem (VIP) denoted by $VI(\mathcal{L}, \mathcal{C})$ is nonempty.

($\mathcal{L}$2) An operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is said to be pseudomonotone if

$$\big\langle \mathcal{L}(y_1), y_2 - y_1 \big\rangle \ge 0 \Longrightarrow \big\langle \mathcal{L}(y_2), y_1 - y_2 \big\rangle \le 0, \ \forall\, y_1, y_2 \in \mathcal{C}.$$

($\mathcal{L}$3) An operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ is said to be Lipschitz continuous through $L > 0$, such that

$$\|\mathcal{L}(y_1) - \mathcal{L}(y_2)\| \le L\|y_1 - y_2\|, \ \forall\, y_1, y_2 \in \mathcal{C};$$

($\mathcal{L}$4) $\limsup_{n\to\infty} \big\langle \mathcal{L}(y_n), y - y_n \big\rangle \le \big\langle \mathcal{L}(q^*), y - q^* \big\rangle$ for all $y \in \mathcal{C}$ and $\{y_n\} \subset \mathcal{C}$ satisfy $y_n \rightharpoonup q^*$.

If we define $f(u, y) := \big\langle \mathcal{L}(u), y - u \big\rangle$ for all $u, y \in \mathcal{C}$. Then, problem (EP) becomes the problem of variational inequalities described above where $L = 2c_1 = 2c_2$. From the above value of the bifunction f, we have

$$\begin{cases} y_n = \arg\min_{y\in\mathcal{C}} \{\chi_n f(t_n, y) + \frac{1}{2}\|t_n - y\|^2\} = P_{\mathcal{C}}(t_n - \chi_n \mathcal{L}(t_n)), \\ z_n = \arg\min_{y\in\mathcal{H}_n} \{\chi_n f(y_n, y) + \frac{1}{2}\|t_n - y\|^2\} = P_{\mathcal{H}_n}(t_n - \chi_n \mathcal{L}(y_n)). \end{cases} \tag{63}$$

Corollary 4. *Suppose that* $\mathcal{L} : \mathcal{C} \to \mathcal{H}$ *is a function satisfying the assumptions* ($\mathcal{L}$1)–($\mathcal{L}$4). *Let* $u_{-1}, u_0 \in \mathcal{C}$, $\phi > 0$, $\chi_0 > 0$, $\{\rho_n\} \subset (a, b) \subset (0, 1 - \varrho_n)$ *and* $\{\varrho_n\} \subset (0, 1)$ *satisfies the* *items, i.e.,* $\lim_{n\to\infty} \varrho_n = 0$ *and* $\sum_{n=1}^{\infty} \varrho_n = +\infty$*. Moreover, choose* ϕ_n *satisfying* $0 \le \phi_n \le \hat{\phi_n}$*, such that*

$$\hat{\phi_n} = \begin{cases} \min\left\{\frac{\phi}{2}, \frac{\varsigma_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \frac{\phi}{2} & \text{else}, \end{cases} \tag{64}$$

where $\varsigma_n = \circ(\varrho_n)$*, i.e.,* $\lim_{n\to\infty} \frac{\varsigma_n}{\varrho_n} = 0$*. Assume that* $\{u_n\}$ *is the sequence generated in the following manner:*

$$\begin{cases} t_n = u_n + \phi_n(u_n - u_{n-1}), \\ y_n = P_{\mathcal{C}}(t_n - \chi_n \mathcal{L}(t_n)), \\ z_n = P_{\mathcal{H}_n}(t_n - \chi_n \mathcal{L}(y_n)), \\ u_{n+1} = (1 - \rho_n - \varrho_n)u_n + \rho_n z_n, \end{cases}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle t_n - \chi_n \mathcal{L}(t_n) - y_n, z - y_n\rangle \le 0\}$*. Compute*

$$\chi_{n+1} = \min\left\{\chi_n, \frac{\mu\|t_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2\left[\langle \mathcal{L}(t_n) - \mathcal{L}(y_n), z_n - y_n\rangle\right]_+}\right\}.$$

Then, sequences $\{u_n\}$ *converge strongly to* $u^* \in VI(\mathcal{L}, \mathcal{C})$*.*

Corollary 5. *Suppose that* $\mathcal{L} : \mathcal{C} \to \mathcal{H}$ *is a function meeting conditions* ($\mathcal{L}$1)–($\mathcal{L}$4)*. Let* $u_0 \in \mathcal{C}$*,* $\chi_0 > 0$*,* $\{\rho_n\} \subset (a, b) \subset (0, 1 - \varrho_n)$ *and* $\{\varrho_n\} \subset (0, 1)$ *satisfies the conditions, i.e.,* $\lim_{n\to\infty} \varrho_n = 0$ *and* $\sum_{n=1}^{\infty} \varrho_n = +\infty$*. Assume that* $\{u_n\}$ *is the sequence generated in the following manner:*

$$\begin{cases} y_n = P_{\mathcal{C}}(u_n - \chi_n \mathcal{L}(u_n)), \\ z_n = P_{\mathcal{H}_n}(u_n - \chi_n \mathcal{L}(y_n)), \\ u_{n+1} = (1 - \rho_n - \varrho_n)u_n + \rho_n z_n, \end{cases}$$

where $\mathcal{H}_n = \{z \in \mathcal{H} : \langle u_n - \chi_n \mathcal{L}(u_n) - y_n, z - y_n\rangle \le 0\}$*.*

Compute

$$\chi_{n+1} = \min\left\{\chi_n, \frac{\mu\|u_n - y_n\|^2 + \mu\|z_n - y_n\|^2}{2\left[\langle \mathcal{L}(u_n) - \mathcal{L}(y_n), z_n - y_n\rangle\right]_+}\right\}.$$

Then, sequences $\{u_n\}$ *converge strongly to* $u^* \in VI(\mathcal{L}, \mathcal{C})$*.*

Remark 1. *Condition* ($\mathcal{L}$4) *could be exempted when* $\mathcal{L}$ *is monotone. Indeed, this condition, which is a particular case of Condition* (c3)*, is only used to prove* (43)*. Without Condition* ($\mathcal{L}$4)*, inequality* (42) *can be obtained by imposing monotonocity on* $\mathcal{L}$*. In that case,*

$$\langle \mathcal{L}(y), y - y_n\rangle \ge \langle \mathcal{L}(y_n), y - y_n\rangle, \quad \forall y \in \mathcal{C}. \tag{65}$$

By allowing $f(u, y) = \langle \mathcal{L}(u), y - u\rangle$ *in* (42)*, we have*

$$\limsup_{k\to\infty} \langle \mathcal{L}(y_{n_k}), y - y_{n_k}\rangle \ge 0, \ \forall y \in \mathcal{H}_n. \tag{66}$$

Combining (65) *with* (66)*, we conclude that*

$$\limsup_{k\to\infty} \langle \mathcal{L}(y), y - y_{n_k}\rangle \ge 0, \ \ \forall y \in \mathcal{C}. \tag{67}$$

Let $y_t = (1 - t)z + ty$*, for every* $t \in [0, 1]$*. By using the convexity of set* $\mathcal{C}$*,* $y_t \in \mathcal{C}$ *for every* $t \in (0, 1)$*. Since* $y_{n_k} \rightharpoonup z \in \mathcal{C}$ *and* $\langle \mathcal{L}(y), y - z\rangle \ge 0$ *for every* $y \in \mathcal{C}$*, we have*

$$0 \le \langle \mathcal{L}(y_t), y_t - z\rangle = t\langle \mathcal{L}(y_t), y - z\rangle. \tag{68}$$

Therefore, $\langle \mathcal{L}(y_t), y - z\rangle \geq 0$, $t \in (0,1)$. Since $y_t \to z$ as $t \to 0$ and due to $\mathcal{L}$ continuity, we have $\langle \mathcal{L}(z), y - z\rangle \geq 0$, for each $y \in \mathcal{C}$, which provides $z \in VI(\mathcal{L}, \mathcal{C})$.

Remark 2. *From Remark 1, it can be concluded that Corollaries 4 and 5 still hold, even if we remove Condition ($\mathcal{L}4$) in the case of monotone operators.*

6. Numerical Illustrations

Numerical results are presented in this section to demonstrate the efficiency of our proposed method. The MATLAB codes were run in MATLAB version 9.5 (R2018b) on an Intel(R) Core(TM)i5-6200 CPU PC @ 2.30 GHz 2.40 GHz, RAM 4.00 GB.

Example 1. *Let there be m companies that manufacture the same product. Assume vector u of each item u_i represents the quantity of the material produced by a company i. We consider that cost function P to be a declining affine function that relies on $\mu = \sum_{i=1}^{m} u_i$, i.e., $P_i(\mu) = \phi_i - \psi_i S$, where $\phi_i > 0$, $\psi_i > 0$. The formula for profit of every company i is taken as $F_i(u) = P_i(S)u_i - q_i(u_i)$, where $q_i(u_i)$ is the tax value and cost for developing item u_i. Moreover, consider that $\mathcal{C}_i = [u_i^{\min}, u_i^{\max}]$ is the set of actions related to each company i, and the plan to figure out the model as $\mathcal{C} := \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_m$. In addition, each member wants to achieve its peak turnover by a good level of production on the basis that the performance of other firms is an input parameter. The commonly used modelling methodology is based on the famous Nash equilibrium principle. A point $u^* \in \mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_m$ is the level of equilibrium of the model if*

$$F_i(u^*) \geq F_i(u^*[u_i]), \ \forall u_i \in \mathcal{C}_i, \ \forall i = 1, 2, \cdots, m,$$

wile $u^*[u_i]$ is obtain from u^* by letting ζ_i^* with u_i. Furthermore, we consider $f(u, y) := \Delta(u, y) - \Delta(u, u)$ while $\Delta(u, y) := -\sum_{i=1}^{m} F_i(u[y_i])$. An equilibrium level of the model is defined by

$$\text{Find } u^* \in \mathcal{C} : f(u^*, z) \geq 0, \ \forall\, z \in \mathcal{C}.$$

Bifunction f converts into the following form (see [23]):

$$f(u, y) = \langle Pu + Qy + c, y - u\rangle$$

where $c \in \mathcal{R}^m$ and P, Q matrices of order m. Matrix P is positive semidefinite, and matrix $Q - P$ is negative semidefinite with Lipschitz-type constants $c_1 = c_2 = \frac{1}{2}\|P - Q\|$ (see [23]) for details. P, Q are taken randomly. (Two diagonal matrices randomly A_1 and A_2 take elements from $[0, 2]$ and $[-2, 0]$ respectively. Randomly $O_1 = RandOrthMat(m)$ and $O_2 = RandOrthMat(m)$ orthogonal matrices are generated. Then, a positive semidefinite matrix $B_1 = O_1 A_1 O_1^T$ and a negative semidefinite matrix $B_2 = O_2 A_2 O_2^T$ are achieved. Lastly, set $Q = B_1 + B_1^T$, $S = B_2 + B_2^T$ and $P = Q - S$.). The constraint set $C \subset \mathcal{R}^m$ be defined by

$$\mathcal{C} := \{u \in \mathcal{R}^m : -10 \leq u_i \leq 10\}.$$

*Numerical explanations for the first 200 iterations of three methods are considered in Figures 1–6 and Table 1 by letting initial points $u_0 = u_{-1} = (1, 1, \cdots, 1, 1)^T$. For Algorithm 3.2 (**mAlg2**) in [34]: $\chi = \frac{1}{4c_1}$ and $\rho_n = \frac{1}{100(n+2)}$; For Algorithm (**mAlg3**) in (60): $\chi_0 = 0.20$, $\mu = 0.70$, $\varrho_n = \frac{1}{100(n+2)}$, $\rho_n = 0.5(1 - \varrho_n)$; For Algorithm 1 (**mAlg1**): $\chi_0 = 0.20$, $\mu = 0.70$, $\phi = 0.60$, $\varsigma_n = \frac{1}{(n+1)^2}$, $\varrho_n = \frac{1}{100(n+2)}$ and $\rho_n = 0.5(1 - \varrho_n)$.*

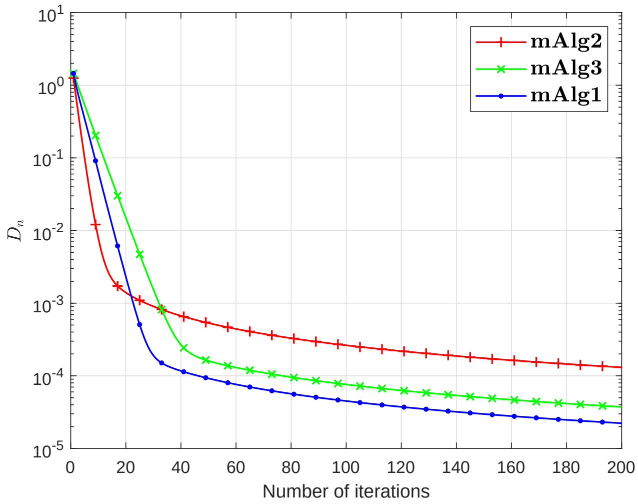

Figure 1. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $\mathcal{R}^5$.

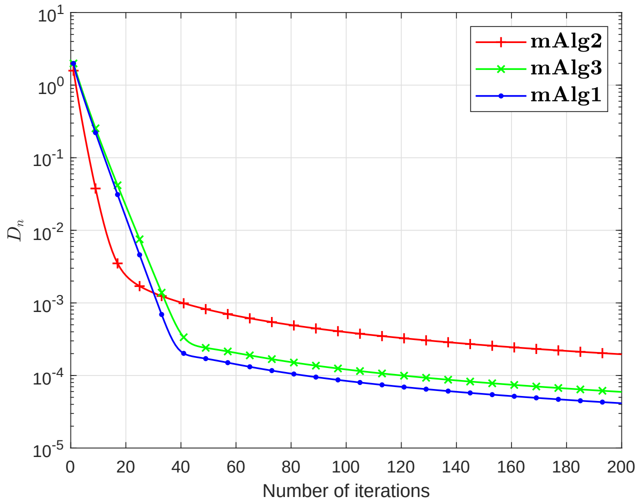

Figure 2. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $\mathcal{R}^{10}$.

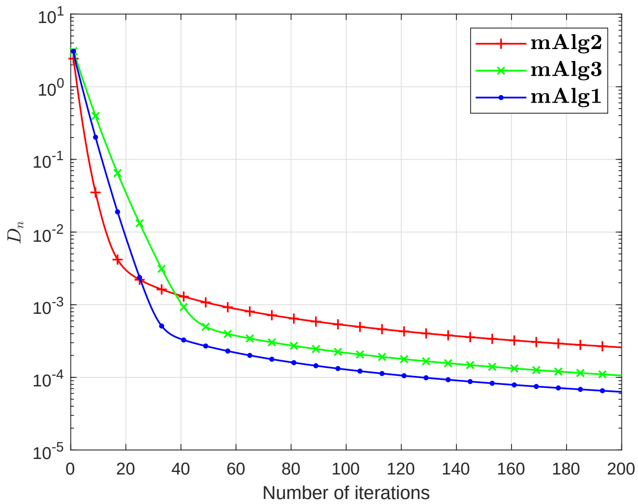

Figure 3. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $\mathcal{R}^{20}$.

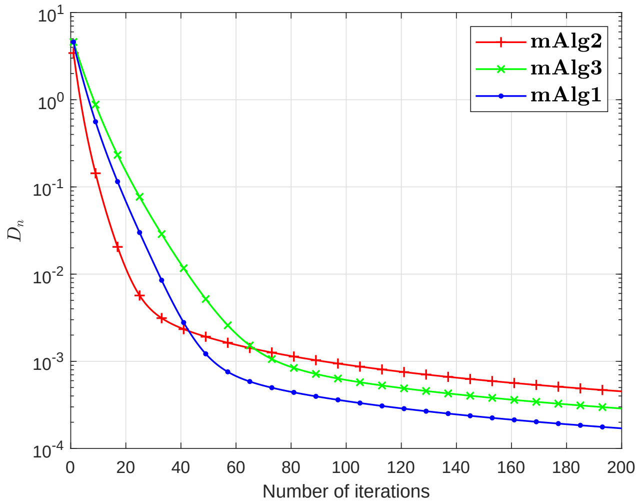

Figure 4. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $\mathcal{R}^{50}$.

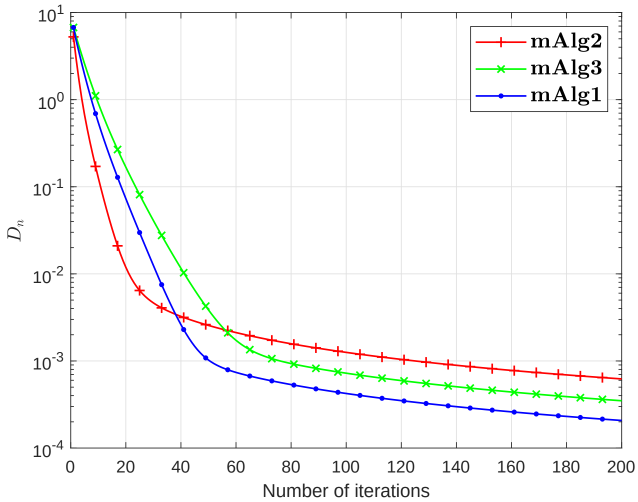

Figure 5. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $\mathcal{R}^{100}$.

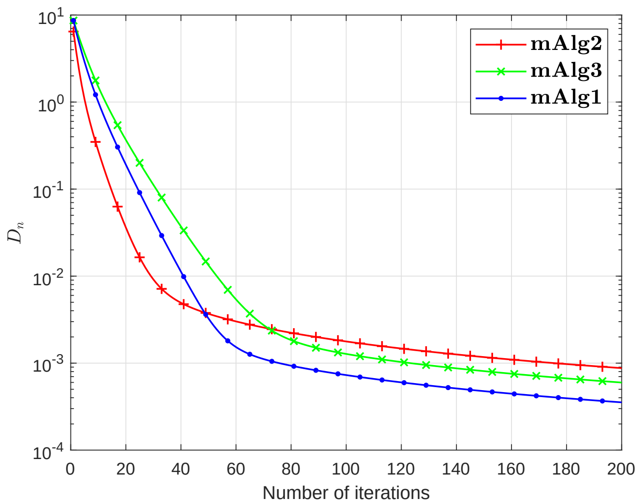

Figure 6. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $\mathcal{R}^{200}$.

Table 1. Figures 1–6 execution time required for first 200 iterations.

	Execution Time in Seconds		
m	**mAlg2**	**mAlg3**	**mAlg1**
5	2.55846812	2.73622248	2.923849848
10	2.89823133	2.99853685	3.341848537
20	3.23847254	3.51835212	3.332562246
50	3.93645046	4.05462157	4.084188882
100	4.57837436	5.32873548	5.723835682
200	5.86241836	6.28194713	6.825465869

Example 2. *Assume that set $\mathcal{C} \subset L^2([0,1])$ is defined by*

$$\mathcal{C} := \{u \in L^2([0,1]) : \|u\| \leq 1\}.$$

Let us define an operator $\mathcal{L} : \mathcal{C} \to \mathcal{H}$, such that

$$\mathcal{L}(u)(t) = \int_0^1 [u(t) - H(t,s)f(u(s))]ds + g(t),$$

*where $H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}$, $f(u) = \cos(u)$ and $g(t) = \frac{2te^t}{e\sqrt{e^2-1}}$. In the above $\mathcal{H} = L^2([0,1])$ is a Hilbert space with inner product $\langle u, y\rangle = \int_0^1 u(t)y(t)dt$, $\forall u, y \in \mathcal{H}$ and induced norm is $\|u\| = \sqrt{\int_0^1 |u(t)|^2 dt}$. Numerical explanations for the first 200 iterations of three methods are considered in Figures 7–10 by letting initial points $u_0 = u_{-1} = (1, 1, \cdots, 1, 1)^T$. For Algorithm 3.2 (**mAlg2**) in [34]: $\chi = \frac{1}{3c_1}$ and $\rho_n = \frac{1}{100(n+2)}$; For Algorithm (**mAlg3**) in (60): $\chi_0 = 0.50$, $\mu = 0.50$, $\varrho_n = \frac{1}{100(n+2)}$, $\rho_n = 0.7(1-\varrho_n)$; For Algorithm 1 (**mAlg1**): $\chi_0 = 0.50$, $\mu = 0.50$, $\phi = 0.70$, $\varsigma_n = \frac{1}{(n+1)^2}$, $\varrho_n = \frac{1}{100(n+2)}$ and $\rho_n = 0.7(1-\varrho_n)$.*

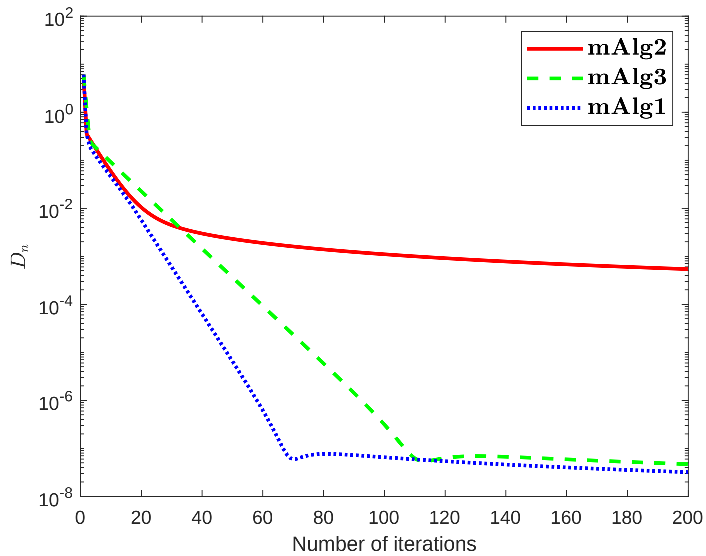

Figure 7. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $u_0 = 1 + t + 2t^2$.

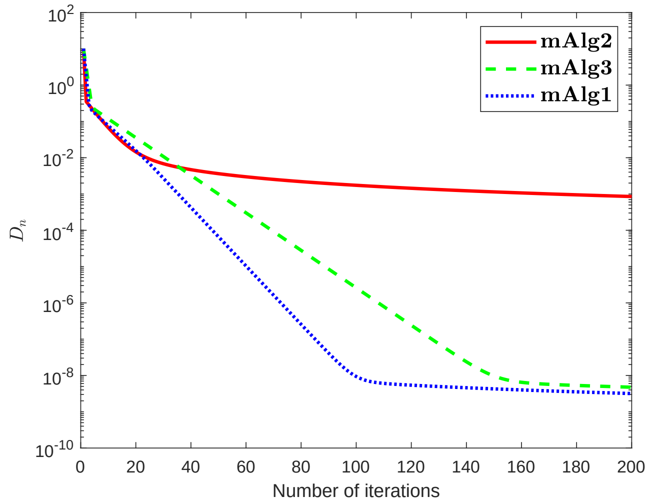

Figure 8. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $u_0 = 1 + 2t + 3e^t$.

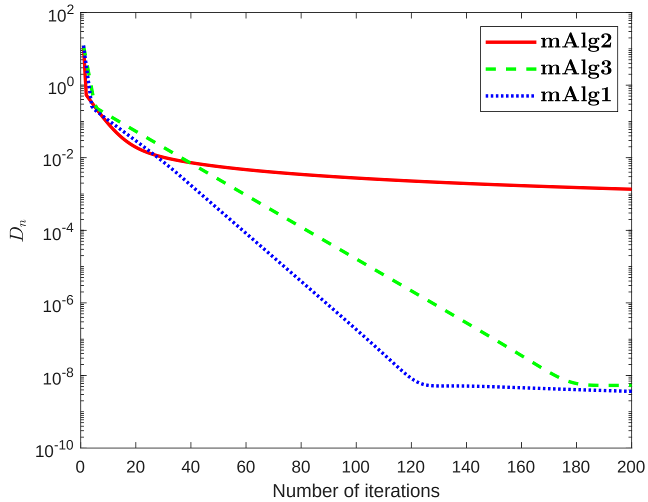

Figure 9. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $u_0 = 1 + 2t + \sin(t)$.

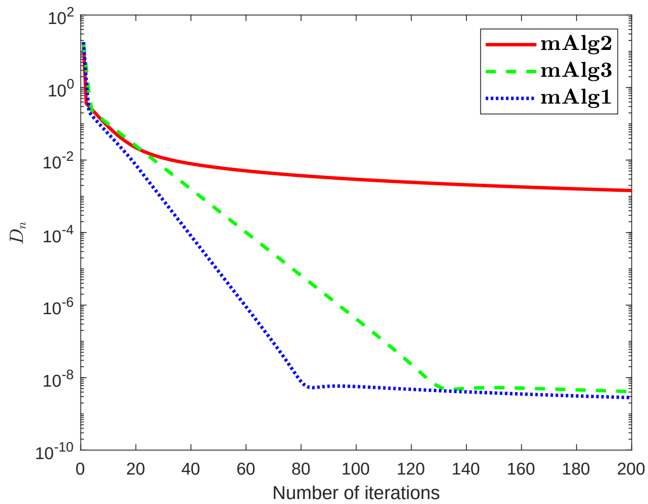

Figure 10. Algorithm 1 compared to Algorithm (60) and Algorithm 3.2 in [34] for $u_0 = 1 + 3t^2 + \cos(t)$.

7. Conclusions

We studied a Mann-type extragradient-like scheme for determining the numerical solution of equilibrium problem involving pseudomonotone function and also prove a strong convergent theorem. Computational conclusions were established to illustrate the computational performance of our algorithms relative to other approaches. Such computational experiments showed that the inertial effect increases the efficacy of the iterative method in this sense.

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