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Banach Contraction Principle and Meir–Keeler Type of Fixed Point Theorems for Pre-Metric Spaces

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Abstract: The fixed point theorems in so-called pre-metric spaces is investigated in this paper. The main issue in the pre-metric space is that the symmetric condition is not assumed to be satisfied, which can result in four different forms of triangle inequalities. In this case, the fixed point theorems in pre-metric space will have many different styles based on the different forms of triangle inequalities.

Keywords: Cauchy sequence; fixed point; pre-metric space; triangle inequality; weakly uniformly strict contraction

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1. Introduction

In this paper, we consider a so-called pre-metric space that does not assume the symmetric condition. We first recall the basic concept of (conventional) metric space as follows.

Given a nonempty universal set X, let $d : X \times X \to \mathbb{R}_+$ be a nonnegative real-valued function defined on the product set $X \times X$. Recall that (X, d) is a metric space when the following conditions are satisfied:

- $d(x, y) = 0$ implies $x = y$ for any $x, y \in X$;
- $d(x, x) = 0$ for any $x \in X$;
- $d(x, y) = d(y, x)$ for any $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

Different kinds of spaces that weaken the above conditions have been proposed. Wilson [1] says that (X, d) is a quasi-metric space when the symmetric condition is not satisfied. More precisely, (X, d) is a quasi-metric space when the following conditions are satisfied:

- $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

Many authors (by referring to [2–16] and the references therein) also defined the different type of quasi-metric space as follows:

- $d(x, y) = 0 = d(y, x)$ if and only if $x = y$ for any $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

In the Wilson's sense, it is obvious that we also have $d(y, x) = 0$ if and only if $y = x$. Wilson [17] also says that (X, d) is a semi-metric space when the triangle inequality is not satisfied. More precisely, the following conditions are satisfied:

- $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$;
- $d(x, y) = d(y, x)$ for any $x, y \in X$.

Matthews [11] proposed the concept of partial metric space that satisfies the following conditions:

- $x = y$ if and only if $d(x, x) = d(x, y) = d(y, y)$ for any $x, y \in X$;

- $d(x,x) \leq d(x,y)$ for any $x,y \in X$;
- $d(x,y) = d(y,x)$ for any $x,y \in X$.
- $d(x,z) \leq d(x,y) + d(y,z) - d(y,y)$ for any $x,y,z \in X$.

The partial metric space does not assume the self-distance condition $d(x,x) = 0$. In this paper, we consider a so-called pre-metric space by assuming that

$$d(x,y) = 0 \text{ implies } x = y \text{ for any } x,y \in X.$$

The triangle inequality always plays a very important role in the study of metric space. Without considering the symmetric condition, the triangle inequalities can be considered in four different forms by referring to Wu [18]. The purpose of this paper is to establish the fixed point theorems in pre-metric space based on the different forms of triangle inequalities. We separately study the Banach contraction principle and Meir–Keeler type of fixed point theorems for pre-metric spaces. On the other hand, three types of contraction functions are considered in this paper. We also mention that the Meir–Keeler type of fixed point theorems in the context of b-metric spaces have been studied by Pavlović and Radenović [19].

This paper is organized as follows. In Section 2, four different forms of triangle inequalities in pre-metric space are presented. Many basic properties are also provided for further study. In Section 3, based on the different forms of triangle inequalities, many concepts of Cauchy sequences in pre-metric space are proposed in order to establish the fixed point theorems in pre-metric space. In Section 4, three different types of contraction functions are considered to establish the fixed point theorems using the different forms of triangle inequalities.

2. Pre-Metric Spaces

We formally introduce the basic concept of pre-metric space by considering four different forms of triangle inequalities as follows.

Definition 1. *Given a nonempty universal set X, let d be a mapping from $X \times X$ into $\mathbb{R}_+$.*

- *The metric d is said to satisfy the* $\bowtie$-triangle inequality *when the following inequality is satisfied:*
$$d(x,y) + d(y,z) \geq d(x,z) \text{ for all } x,y,z \in X.$$
- *The metric d is said to satisfy the* $\triangleright$-triangle inequality *when the following inequality is satisfied:*
$$d(x,y) + d(z,y) \geq d(x,z) \text{ for all } x,y,z \in X.$$
- *The metric d is said to satisfy the* $\triangleleft$-triangle inequality *when the following inequality is satisfied:*
$$d(y,x) + d(y,z) \geq d(x,z) \text{ for all } x,y,z \in X.$$
- *The metric d is said to satisfy the* $\diamond$-triangle inequality *when the following inequality is satisfied:*
$$d(y,x) + d(z,y) \geq d(x,z) \text{ for all } x,y,z \in X.$$

Suppose that d satisfies the symmetric condition. It is clear to see that all the concepts of $\bowtie$-triangle inequality, $\triangleright$-triangle inequality, $\triangleleft$-triangle inequality and $\diamond$-triangle inequality described above are all equivalent. This means that the pre-metric space extends the concept of (conventional) metric space.

Remark 1. *Now, we represent some interesting observations that are used in the study.*

- *Suppose that the metric d satisfies the $\bowtie$-triangle inequality. Then, we have*
$$d(a,b) + d(b,c) + d(c,d) \geq d(a,c) + d(c,d) \geq d(a,d).$$

We also see that

$$d(b,a) + d(c,b) = d(c,b) + d(b,a) \geq d(c,a),$$

which implies

$$d(b,a) + d(c,b) + d(d,c) \geq d(d,a).$$

In general, we can have the following inequalities

$$d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_p,x_{p+1}) \geq d(x_1,x_{p+1})$$

and

$$d(x_2,x_1) + d(x_3,x_2) + \cdots + d(x_{p+1},x_p) \geq d(x_{p+1},x_1).$$

- *Suppose that the metric d satisfies the $\triangleright$-triangle inequality. Since*

$$d(a,b) + d(c,b) \geq d(a,c) \text{ and } d(c,b) + d(a,b) \geq d(c,a),$$

we see that

$$d(a,b) + d(c,b) = d(c,b) + d(a,b) \geq \max\{d(a,c), d(c,a)\}.$$

Therefore, we obtain

$$d(a,b) + d(c,b) + d(d,c) \geq \max\{d(a,d), d(d,a)\}. \tag{1}$$

In general, we can have the following inequalities

$$d(x_1,x_2) + d(x_3,x_2) + d(x_4,x_3) + \cdots + d(x_{p+1},x_p) \geq \max\{d(x_1,x_{p+1}), d(x_{p+1},x_1)\}.$$

- *Suppose that the metric d satisfies the $\triangleleft$-triangle inequality. Since*

$$d(b,a) + d(b,c) = d(b,c) + d(b,a) \geq \max\{d(a,c), d(c,a)\},$$

we see that

$$d(b,a) + d(b,c) + d(c,d) \geq \max\{d(a,d), d(d,a)\}. \tag{2}$$

In general, we can have the following inequalities

$$d(x_2,x_1) + d(x_2,x_3) + d(x_3,x_4) + \cdots + d(x_p,x_{p+1}) \geq \max\{d(x_1,x_{p+1}), d(x_{p+1},x_1)\}.$$

- *Suppose that the metric d satisfies the $\diamond$-triangle inequality. Then, we have*

$$d(a,b) + d(b,c) + d(d,c) = d(b,c) + d(a,b) + d(d,c) \geq d(c,a) + d(d,c) \geq d(a,d) \tag{3}$$

and

$$d(b,a) + d(c,b) + d(c,d) \geq d(a,c) + d(c,d) = d(c,d) + d(a,c) \geq d(d,a). \tag{4}$$

In general, we consider the following cases.

(a) Suppose that p is an even number. Then, we have the following inequalities

$$\begin{aligned} &d(x_1,x_2) + d(x_2,x_3) + d(x_4,x_3) + d(x_4,x_5) + d(x_6,x_5) \\ &\quad + d(x_6,x_7) + \cdots + d(x_p,x_{p+1}) \geq d(x_{p+1},x_1) \end{aligned}$$

and

$$\begin{aligned} &d(x_2,x_1) + d(x_3,x_2) + d(x_3,x_4) + d(x_5,x_4) + d(x_5,x_6) \\ &\quad + d(x_7,x_6) + \cdots + d(x_{p+1},x_p) \geq d(x_1,x_{p+1}). \end{aligned}$$

(b) *Suppose that p is an odd number. Then, we have the following inequalities*

$$d(x_1,x_2)+d(x_2,x_3)+d(x_4,x_3)+d(x_4,x_5)+d(x_6,x_5)+d(x_6,x_7)+\cdots+d(x_p,x_{p+1})\geq d(x_1,x_{p+1})$$

and

$$d(x_2,x_1)+d(x_3,x_2)+d(x_3,x_4)+d(x_5,x_4)+d(x_5,x_6)+d(x_7,x_6)+\cdots+d(x_{p+1},x_p)\geq d(x_{p+1},x_1).$$

Definition 2 (Wu [18]). *Given a nonempty universal set X, let d be a mapping from $X\times X$ into $\mathbb{R}_+$. We say that (X,d) is a* pre-metric space *when $d(x,y)=0$ implies $x=y$ for any $x,y\in X$.*

Proposition 1 (Wu [18]). *Given a nonempty universal set X, let d be a mapping from $X\times X$ into $\mathbb{R}_+$. Suppose that the following conditions are satisfied:*

- *$d(x,x)=0$ for all $x\in X$;*
- *d satisfies the $\triangleright$-triangle inequality or the $\triangleleft$-triangle inequality or the $\diamond$-triangle inequality.*

Then d satisfies the symmetric condition.

We also remark that Proposition 4.4 in Wu [18] is redundant and it can be omitted.

3. Cauchy Sequences in Pre-Metric Space

Let (X,d) be a pre-metric space. Many different concepts of limit are proposed below because of lacking the symmetric condition.

Definition 3. *Let (X,d) be a pre-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.*

- *We write $x_n \xrightarrow{d^{\triangleright}} x$ as $n\to\infty$ when $d(x_n,x)\to 0$ as $n\to\infty$.*
- *We write $x_n \xrightarrow{d^{\triangleleft}} x$ as $n\to\infty$ when $d(x,x_n)\to 0$ as $n\to\infty$.*
- *We write $x_n \xrightarrow{d} x$ as $n\to\infty$ when*

$$\lim_{n\to\infty} d(x_n,x)=\lim_{n\to\infty} d(x,x_n)=0.$$

The uniqueness of limits are given below.

Proposition 2 (Wu [18]). *Let (X,d) be a pre-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.*

(i) *Suppose that the metric d satisfies the $\bowtie$-triangle inequality or $\diamond$-triangle inequality. If $x_n \xrightarrow{d^{\triangleleft}} x$ and $x_n \xrightarrow{d^{\triangleright}} y$, then $x=y$.*

(ii) *Suppose that the metric d satisfies the $\triangleleft$-triangle inequality. If $x_n \xrightarrow{d^{\triangleright}} x$ and $x_n \xrightarrow{d^{\triangleright}} y$, then $x=y$. In other words, the $d^{\triangleright}$-limit is unique.*

(iii) *Suppose that the metric d satisfies the $\triangleright$-triangle inequality. If $x_n \xrightarrow{d^{\triangleleft}} x$ and $x_n \xrightarrow{d^{\triangleleft}} y$, then $x=y$. In other words, the $d^{\triangleleft}$-limit is unique.*

Without the symmetric condition, the different concepts of Cauchy sequences are also presented below.

Definition 4. *Let (X,d) be a pre-metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X.*

- *We say that $\{x_n\}_{n=1}^{\infty}$ is a $>$-*Cauchy sequence *when, given any $\epsilon>0$, there exists an integer N such that $d(x_m,x_n)<\epsilon$ for all pairs (m,n) of integers m and n with $m>n\geq N$.*
- *We say that $\{x_n\}_{n=1}^{\infty}$ is a $<$-*Cauchy sequence *when, given any $\epsilon>0$, there exists an integer N such that $d(x_n,x_m)<\epsilon$ for all pairs (m,n) of integers m and n with $m>n\geq N$.*

- *We say that* $\{x_n\}_{n=1}^{\infty}$ *is a* Cauchy sequence *when, given any* $\epsilon > 0$*, there exists an integer* N *such that* $d(x_m, x_n) < \epsilon$ *and* $d(x_n, x_m) < \epsilon$ *for all pairs* (m, n) *of integers* m *and* n *with* $m, n \geq N$ *and* $m \neq n$*.*

We can also consider the different concepts of completeness for pre-metric space.

Definition 5. *Let* (X, d) *be a pre-metric space.*

- *We say that* (X, d) *is* $(>, \triangleright)$-complete *when each* $>$*-Cauchy sequence is convergent in the sense of* $x_n \xrightarrow{d^{\triangleright}} x$*.*
- *We say that* (X, d) *is* $(>, \triangleleft)$-complete *when each* $>$*-Cauchy sequence is convergent in the sense of* $x_n \xrightarrow{d^{\triangleleft}} x$*.*
- *We say that* (X, d) *is* $(<, \triangleright)$-complete *when each* $<$*-Cauchy sequence is convergent in the sense of* $x_n \xrightarrow{d^{\triangleright}} x$*.*
- *We say that* (X, d) *is* $(<, \triangleleft)$-complete *when each* $<$*-Cauchy sequence is convergent in the sense of* $x_n \xrightarrow{d^{\triangleleft}} x$*.*
- *We say that* (X, d) *is* $\triangleleft$-complete *when each Cauchy sequence is convergent in the sense of* $x_n \xrightarrow{d^{\triangleleft}} x$*.*
- *We say that* (X, d) *is* $\triangleright$-complete *when each Cauchy sequence is convergent in the sense of* $x_n \xrightarrow{d^{\triangleright}} x$*.*

Based on the above different concepts of completeness, we establish many fixed point theorems in pre-metric space by using the different types of triangle inequalities. Next, we present some examples to demonstrate the completeness.

Let S be a bounded subset S of $\mathbb{R}^k$ containing infinitely many points. The Bolzano–Weierstrass theorem says that there exists at least one accumulation point of S, where the concept of accumulation point is based on the usual topology induced by the conventional metric. When the metric does not satisfy the symmetric condition, Wu [18] has proposed two different concepts of open balls given by

$$B^{\triangleleft}(x; r) = \{y \in X : d(x, y) < r\}$$

and

$$B^{\triangleright}(x; r) = \{y \in X : d(y, x) < r\},$$

which can induces two respective topologies as follows

$$\tau^{\triangleleft} = \{O^{\triangleleft} \subseteq X : x \in O^{\triangleleft} \text{ if and only if there exist } r > 0 \text{ such that } x \in B^{\triangleleft}(x; r) \subseteq O^{\triangleleft}\}.$$

and

$$\tau^{\triangleright} = \{O^{\triangleright} \subseteq X : x \in O^{\triangleright} \text{ if and only if there exist } r > 0 \text{ such that } x \in B^{\triangleright}(x; r) \subseteq O^{\triangleright}\}.$$

In this case, we can similarly define the concepts of $\triangleleft$-accumulation point and $\triangleright$-accumulation point based on the open balls $B^{\triangleleft}(x; r)$ and $B^{\triangleright}(x; r)$, respectively. Therefore, we can similarly obtain the $\triangleleft$-type of Bolzano–Weierstrass theorem and $\triangleright$-type of Bolzano–Weierstrass theorem by considering the $\triangleleft$-accumulation point and $\triangleright$-accumulation point, respectively, which is used to present the completeness in $\mathbb{R}$.

Example 1. *We are going to claim that every* $>$*-Cauchy sequence in* $\mathbb{R}_+$ *is convergent in the sense of* $x_n \xrightarrow{d^{\triangleright}} x$ *with respect to a pre-metric* $d : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *defined by*

$$d(x,y) = \begin{cases} x & \text{if } x > y \\ 0 & \text{if } x = y \\ 2y - x & \text{if } x < y, \end{cases}$$

where the symmetric condition is not satisfied and d satisfies the $\bowtie$*-triangle inequality.*

Let $T = \{x_1, x_2, \cdots, x_n, \cdots\}$ *be a* $>$*-Cauchy sequence in* $\mathbb{R}_+$*. We are going to show that T is* $\triangleright$*-bounded. Given* $\epsilon = 1$*, there is an integer N such that* $d(x_n, x_N) < 1$ *for each* $n > N$*. This means that* $x_n \in B^{\triangleright}(x_N; 1)$ *for each* $n \geq N$*. We define*

$$r = 1 + \max\{d(x_1, 0), \cdots, d(x_N, 0)\}.$$

- *For* $n \leq N$*, we have*

$$d(x_n, 0) \leq \max\{d(x_1, 0), \cdots, d(x_N, 0)\} < r.$$

- *For* $n > N$*, using the* $\bowtie$*-triangle inequality, we have*

$$d(x_n, 0) \leq d(x_n, x_N) + d(x_N, 0) < 1 + d(x_N, 0) \leq r.$$

Then, we see that $T \subseteq B^{\triangleright}(0; r)$*, which says that T is* $\triangleright$*-bounded. Using the Bolzano–Weierstrass theorem, the sequence T has a* $\triangleright$*-accumulation point* $x^* \in \mathbb{R}_+$*. Next we are going to show that* $x_n \xrightarrow{d^{\triangleright}} x^*$*. Given any* $\epsilon > 0$*, there exists an integer N such that* $n > m \geq N$ *implies* $d(x_n, x_m) < \epsilon/2$*. Since* x^* *is a* $\triangleright$*-accumulation point of the sequence T, it follows that the open ball* $B^{\triangleright}(x^*; \epsilon/2)$ *contains a point* x_m *for* $m \geq N$*, i.e.,* $d(x_m, x^*) < \epsilon/2$*. Therefore, for* $n > N$*, using the* $\bowtie$*-triangle inequality, we have*

$$d(x_n, x^*) \leq d(x_n, x_m) + d(x_m, x^*) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which shows that $x_n \xrightarrow{d^{\triangleright}} x^*$*. In other words, the pre-metric space* $(\mathbb{R}, d)$ *is* $(>, \triangleright)$*-complete*

Example 2. *Continued from Example 1, we are going to claim that every* $<$*-Cauchy sequence in* $\mathbb{R}_+$ *is convergent in the sense of* $x_n \xrightarrow{d^{\triangleleft}} x$*. Let* $T = \{x_1, x_2, \cdots, x_n, \cdots\}$ *be a* $<$*-Cauchy sequence in* $\mathbb{R}_+$*. We are going to show that T is* $\triangleleft$*-bounded. Given* $\epsilon = 1$*, there is an integer N such that* $d(x_N, x_n) < 1$ *for each* $n > N$*. This means that* $x_n \in B^{\triangleleft}(x_N; 1)$ *for each* $n \geq N$*. We define*

$$r = 1 + \max\{d(0, x_1), \cdots, d(0, x_N)\}.$$

- *For* $n \leq N$*, we have*

$$d(0, x_n) \leq \max\{d(0, x_1), \cdots, d(0, x_N)\} < r.$$

- *For* $n > N$*, using the* $\bowtie$*-triangle inequality, we have*

$$d(0, x_n) \leq d(0, x_N) + d(x_N, x_n) < 1 + d(0, x_N) \leq r.$$

Then, we see that $T \subseteq B^{\triangleleft}(0; r)$*, which says that T is* $\triangleleft$*-bounded. Using the Bolzano–Weierstrass theorem, the sequence T has a* $\triangleleft$*-accumulation point* $x^\circ \in \mathbb{R}_+$*. Next we are going to show that* $x_n \xrightarrow{d^{\triangleleft}} x^\circ$*. Given any* $\epsilon > 0$*, there exists an integer N such that* $n > m \geq N$ *implies* $d(x_m, x_n) < \epsilon/2$*. Since* x° *is a* $\triangleleft$*-accumulation point of the sequence T, it follows that the open ball* $B^{\triangleleft}(x^\circ; \epsilon/2)$ *contains a point* x_m *for* $m \geq N$*, i.e.,* $d(x^\circ, x_m) < \epsilon/2$*. Therefore, for* $n > N$*, using the* $\bowtie$*-triangle inequality, we have*

$$d(x^\circ, x_n) \leq d(x^\circ, x_m) + d(x_m, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which shows that $x_n \xrightarrow{d^\triangleleft} x^\circ$. *In other words, the pre-metric space* $(\mathbb{R}, d)$ *is* $(<, \triangleleft)$*-complete*

Example 3. *Continued from Examples 1 and 2, we are going to claim that the pre-metric space* $(\mathbb{R}, d)$ *is simultaneously* $\triangleright$*-complete and* $\triangleleft$*-complete. Let* $T = \{x_1, x_2, \cdots, x_n, \cdots\}$ *be a Cauchy sequence in* $\mathbb{R}_+$. *It means that* T *is both a* $>$*-Cauchy sequence and* $<$*-Cauchy sequence in* $\mathbb{R}_+$. *Examples 1 and 2 say that there exist* x^* *and* x° *satisfying* $x_n \xrightarrow{d^\triangleright} x^*$ *and* $x_n \xrightarrow{d^\triangleleft} x^\circ$. *In other words, the pre-metric space* $(\mathbb{R}, d)$ *is simultaneously* $\triangleright$*-complete and* $\triangleleft$*-complete. We also remark that* $x^* \neq x^\circ$ *in general.*

4. Banach Contraction Principle for Pre-Metric Spaces

Let $T : X \to X$ be a function from a nonempty set X into itself. If $T(x) = x$, we say that $x \in X$ is a fixed point of T. The well-known Banach contraction principle says that any functions that are a contraction on X has a fixed point when X is taken to be a complete metric space. In this paper, we study the Banach contraction principle when X is taken to be a complete pre-metric space.

Definition 6. *Let* (X, d) *be a pre-metric space. A function* $T : (X, d) \to (X, d)$ *is called a* contraction on X *when there is a real number* $0 < \alpha < 1$ *satisfying*

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

for any $x, y \in X$.

Given any initial element $x_0 \in X$, using the function T, we consider the iterative sequence $\{x_n\}_{n=1}^\infty$ as follows:

$$x_1 = T(x_0), x_2 = T(x_1) = T^2(x_0), \cdots, x_n = T^n(x_0), \cdots. \tag{5}$$

We are going to show that the sequence $\{x_n\}_{n=1}^\infty$ can converge to a fixed point of T under some suitable conditions.

Theorem 1 (Banach Contraction Principle Using the $\triangleleft$-Triangle Inequality)**.** *Let* (X, d) *be a* $(>, \triangleright)$*-complete pre-metric space or* $(<, \triangleright)$*-complete pre-metric space such that the* $\triangleleft$*-triangle inequality is satisfied. Suppose that the function* $T : (X, d) \to (X, d)$ *is a contraction on* X. *Then* T *has a unique fixed point* $x \in X$. *Moreover, the fixed point* x *is obtained by the following limit*

$$d(x_n, x) \to 0 \text{ as } n \to \infty,$$

where the sequence $\{x_n\}_{n=1}^\infty$ *is generated according to* (5).

Proof. Given any initial element $x_0 \in X$, according to (5), we can generate the iterative sequence $\{x_n\}_{n=1}^\infty$. The purpose is to show that $\{x_n\}_{n=1}^\infty$ is both a $>$-Cauchy sequence and $<$-Cauchy sequence. Since T is a contraction on X, without having the symmetric condition, we have the two cases as follows:

$$\begin{aligned} d(x_{m+1}, x_m) &= d(T(x_m), T(x_{m-1})) \leq \alpha d(x_m, x_{m-1}) \\ &= \alpha d(T(x_{m-1}), T(x_{m-2})) \leq \alpha^2 d(x_{m-1}, x_{m-2}) \\ &\leq \cdots \leq \alpha^m d(x_1, x_0) \end{aligned}$$

and

$$d(x_m, x_{m+1}) \leq \alpha^m d(x_0, x_1).$$

For $m > n$, since the $\triangleleft$-triangle inequality is assumed to be satisfied, according to the third observation in Remark 1, we obtain

$$\begin{aligned} d(x_m, x_n) &\leq d(x_{m-1}, x_m) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \alpha^{m-1} \cdot d(x_0, x_1) + \left(\alpha^{m-2} + \cdots + \alpha^n\right) \cdot d(x_1, x_0) \\ &\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right) \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \\ &= \alpha^n \cdot \frac{1-\alpha^{m-n}}{1-\alpha} \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \end{aligned}$$

and

$$d(x_n, x_m) \leq \alpha^n \cdot \frac{1-\alpha^{m-n}}{1-\alpha} \cdot \max\{d(x_0, x_1), d(x_1, x_0)\}.$$

Since $0 < \alpha < 1$, we have $1 - \alpha^{m-n} < 1$ in the numerator. Therefore, we obtain

$$d(x_m, x_n) \leq \frac{\alpha^n}{1-\alpha} \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \to 0 \text{ as } n \to \infty$$

and

$$d(x_n, x_m) \to 0 \text{ as } n \to \infty,$$

which shows that $\{x_n\}_{n=1}^{\infty}$ is both a $>$-Cauchy sequence and $<$-Cauchy sequence. Since X is $(>, \triangleright)$-complete or $(<, \triangleright)$-complete, there exists $x \in X$ satisfying $d(x_n, x) \to 0$ as $n \to \infty$.

Now, we are going to claim that x is indeed a fixed point. We have

$$\begin{aligned} d(x, T(x)) &\leq d(x_m, x) + d(x_m, T(x)) \text{ (using the } \triangleleft\text{-triangle inequality)} \\ &= d(x_m, x) + d(T(x_{m-1}), T(x)) \\ &\leq d(x_m, x) + \alpha d(x_{m-1}, x) \text{ (using the contraction)} \end{aligned}$$

which implies $d(x, T(x)) = 0$ as $m \to \infty$. We conclude that $T(x) = x$ by the condition of pre-metric space. The uniqueness will also be obtained. Assume that there is another fixed point $\bar{x}$ of T, i.e., $\bar{x} = T(\bar{x})$. The contraction of function T says that

$$d(\bar{x}, x) = d(T(\bar{x}), T(x)) \leq \alpha d(\bar{x}, x).$$

Since $0 < \alpha < 1$, we conclude that $d(\bar{x}, x) = 0$, i.e., $\bar{x} = x$. This completes the proof. □

Theorem 2 (Banach Contraction Principle Using the $\triangleright$-Triangle Inequality)**.** *Let (X, d) be a $(>, \triangleleft)$-complete pre-metric space or $(<, \triangleleft)$-complete pre-metric space such that the $\triangleright$-triangle inequality is satisfied. Suppose that the function $T : (X, d) \to (X, d)$ is a contraction on X. Then T has a unique fixed point $x \in X$. Moreover, the fixed point x is obtained by the following limit*

$$d(x, x_n) \to 0 \text{ as } n \to \infty,$$

where the sequence $\{x_n\}_{n=1}^{\infty}$ is generated according to (5).

Proof. Given any initial element $x_0 \in X$, according to (5), we can generate the iterative sequence $\{x_n\}_{n=1}^{\infty}$. The purpose is to show that $\{x_n\}_{n=1}^{\infty}$ is both a $>$-Cauchy sequence and $<$-Cauchy sequence. From the proof of Theorem 1, the contraction of function T says that

$$d(x_{m+1}, x_m) \leq \alpha^m d(x_1, x_0) \text{ and } d(x_m, x_{m+1}) \leq \alpha^m d(x_0, x_1).$$

For $m > n$, since the $\triangleright$-triangle inequality is satisfied, according to the second observation in Remark 1, we obtain

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-2}, x_{m-1}) + \cdots + d(x_n, x_{n+1}) \\ &\leq \alpha^{m-1} \cdot d(x_1, x_0) + \left(\alpha^{m-2} + \cdots + \alpha^n\right) \cdot d(x_0, x_1) \\ &\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right) \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \\ &= \alpha^n \cdot \frac{1-\alpha^{m-n}}{1-\alpha} \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \end{aligned}$$

and

$$d(x_n, x_m) \leq \alpha^n \cdot \frac{1-\alpha^{m-n}}{1-\alpha} \cdot \max\{d(x_0, x_1), d(x_1, x_0)\},$$

which also imply

$$d(x_m, x_n) \to 0 \text{ and } d(x_n, x_m) \to 0 \text{ as } n \to \infty.$$

Therefore $\{x_n\}_{n=1}^{\infty}$ is both a $>$-Cauchy sequence and $<$-Cauchy sequence. Since X is $(>, \triangleleft)$-complete or $(<, \triangleleft)$-complete, there exists $x \in X$ satisfying $d(x, x_n) \to 0$ as $n \to \infty$.

Regarding the uniqueness, we have

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_m) + d(T(x), x_m) \text{ (using the } \triangleright\text{-triangle inequality)} \\ &= d(x, x_m) + d(T(x), T(x_{m-1})) \\ &\leq d(x, x_m) + \alpha d(x, x_{m-1}) \text{ (using the contraction)} \end{aligned}$$

which implies $d(x, T(x)) = 0$ as $m \to \infty$. We conclude that $T(x) = x$. The uniqueness can also be obtained from the argument in the proof of Theorem 1. This completes the proof. □

Theorem 3 (Banach Contraction Principle Using the $\bowtie$-Triangle Inequality)**.** *Let (X, d) be a pre-metric space such that the $\bowtie$-triangle inequality is satisfied. We also assume that any one of the following conditions is satisfied:*

- *(X, d) is simultaneously $(>, \triangleright)$-complete and $(>, \triangleleft)$-complete;*
- *(X, d) is simultaneously $(>, \triangleright)$-complete and $(<, \triangleleft)$-complete;*
- *(X, d) is simultaneously $(<, \triangleright)$-complete and $(>, \triangleleft)$-complete;*
- *(X, d) is simultaneously $(<, \triangleright)$-complete and $(<, \triangleleft)$-complete;*
- *(X, d) is simultaneously $\triangleright$-complete and $\triangleleft$-complete.*

Suppose that the function $T : (X, d) \to (X, d)$ is a contraction on X. Then T has a unique fixed point $x \in X$. Moreover, the fixed point x is obtained by the following limits

$$d(x_n, x) \to 0 \text{ or } d(x, x_n) \to 0 \text{ as } n \to \infty,$$

where the sequence $\{x_n\}_{n=1}^{\infty}$ is generated according to (5).

Proof. Given any initial element $x_0 \in X$, according to (5), we can generate the iterative sequence $\{x_n\}_{n=1}^{\infty}$. The purpose is to show that $\{x_n\}_{n=1}^{\infty}$ is both a $>$-Cauchy sequence and $<$-Cauchy sequence. From the proof of Theorem 1, the contraction of function T says that

$$d(x_{m+1}, x_m) \leq \alpha^m d(x_1, x_0) \text{ and } d(x_m, x_{m+1}) \leq \alpha^m d(x_0, x_1).$$

For $m > n$, since the $\bowtie$-triangle inequality is satisfied, according to the first observation in Remark 1, we obtain

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right) \cdot d(x_1, x_0) \\ &= \alpha^n \cdot \frac{1-\alpha^{m-n}}{1-\alpha} \cdot d(x_1, x_0) \end{aligned}$$

and

$$\begin{aligned} d(x_n, x_m) &\leq d(x_{m-1}, x_m) + d(x_{m-2}, x_{m-1}) + \cdots + d(x_n, x_{n+1}) \\ &\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right) \cdot d(x_0, x_1) \\ &= \alpha^n \cdot \frac{1-\alpha^{m-n}}{1-\alpha} \cdot d(x_0, x_1), \end{aligned}$$

which imply

$$d(x_m, x_n) \to 0 \text{ and } d(x_n, x_m) \to 0 \text{ as } n \to \infty.$$

This proves that $\{x_n\}_{n=1}^{\infty}$ is both a $>$-Cauchy sequence and $<$-Cauchy sequence. It follows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Assume that X is simultaneously $(>, \triangleright)$-complete and $(<, \triangleleft)$-complete. Then there exists $x^*, x^\circ \in X$ satisfying $d(x_n, x^*) \to 0$ and $d(x^\circ, x_n) \to 0$ as $n \to \infty$. Now, we have

$$\begin{aligned} d(x^\circ, T(x^*)) &\leq d(x^\circ, x_m) + d(x_m, T(x^*)) \text{ (using the } \bowtie\text{-triangle inequality)} \\ &= d(x^\circ, x_m) + d(T(x_{m-1}), T(x^*)) \\ &\leq d(x^\circ, x_m) + \alpha d(x_{m-1}, x^*) \text{ (using the contraction)}, \end{aligned}$$

which implies $d(x^\circ, T(x^*)) = 0$ as $m \to \infty$. Therefore, we obtain $T(x^*) = x^\circ$. Now, we have

$$\begin{aligned} d(T(x^\circ), x^*) &\leq d(T(x^\circ), x_m) + d(x_m, x^*) \text{ (using the } \bowtie\text{-triangle inequality)} \\ &= d(T(x^\circ), T(x_{m-1})) + d(x_m, x^*) \\ &\leq \alpha d(x^\circ, x_{m-1}) + d(x_m, x^*) \text{ (using the contraction)}, \end{aligned}$$

which implies $d(T(x^\circ), x^*) = 0$ as $m \to \infty$. We also obtain $T(x^\circ) = x^*$. Now, we have

$$T^2(x^*) = T(T(x^*)) = T(x^\circ) = x^* \text{ and } T^2(x^\circ) = T(T(x^\circ)) = T(x^*) = x^\circ. \tag{6}$$

This shows that x^* and x° are fixed points of the composition mapping $T \circ T \equiv T^2$. The contraction of function T says that

$$d(x^*, x^\circ) = d(T^2(x^*), T^2(x^\circ)) \leq \alpha d(T(x^*), T(x^\circ)) \leq \alpha^2 d(x^*, x^\circ).$$

Since $0 < \alpha < 1$, i.e., $0 < \alpha^2 < 1$, we conclude that $d(x^*, x^\circ) = 0$, i.e., $x^* = x^\circ$. This also says that $x^* = x^\circ$ is a fixed point of T.

The uniqueness can be obtained using the argument in the proof of Theorem 1. For the other three conditions, we can similarly obtain the desired results. This completes the proof. □

Example 4. *Continued from Example 3, since the pre-metric space $(\mathbb{R}, d)$ is simultaneously $\triangleright$-complete and $\triangleleft$-complete, any function $T : (\mathbb{R}, d) \to (\mathbb{R}, d)$ that is a contraction on $\mathbb{R}$ has a unique fixed point. The concrete examples regarding functions that are contraction on $\mathbb{R}$ can be obtained from the literature.*

Theorem 4 (Banach Contraction Principle Using the $\diamond$-Triangle Inequality)**.** *Let (X, d) be a pre-metric space such that the $\diamond$-triangle inequality is satisfied. We also assume that any one of the following conditions is satisfied:*

- *(X, d) is simultaneously $(>, \triangleright)$-complete and $(>, \triangleleft)$-complete;*
- *(X, d) is simultaneously $(>, \triangleright)$-complete and $(<, \triangleleft)$-complete;*
- *(X, d) is simultaneously $(<, \triangleright)$-complete and $(>, \triangleleft)$-complete;*
- *(X, d) is simultaneously $(<, \triangleright)$-complete and $(<, \triangleleft)$-complete.*

Suppose that the function $T : (X, d) \to (X, d)$ is a contraction on X. Then T has a unique fixed point $x \in X$. Moreover, the fixed point x is obtained by the following limits

$$d(x_n, x) \to 0 \text{ or } d(x, x_n) \to 0 \text{ as } n \to \infty,$$

where the sequence $\{x_n\}_{n=1}^{\infty}$ is generated according to (5).

Proof. Given any initial element $x_0 \in X$, according to (5), we can generate the iterative sequence $\{x_n\}_{n=1}^{\infty}$. The purpose is to show that $\{x_n\}_{n=1}^{\infty}$ is both a $>$-Cauchy sequence and $<$-Cauchy sequence. From the proof of Theorem 1, the contraction of function T says that

$$d(x_{m+1}, x_m) \leq \alpha^m d(x_1, x_0) \text{ and } d(x_m, x_{m+1}) \leq \alpha^m d(x_0, x_1).$$

For $m > n$, since the $\diamond$-triangle inequality is satisfied, according to the fourth observation in Remark 1 by assuming $m - n \equiv p$ is an even number, we obtain

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+3}, x_{n+2}) + d(x_{n+3}, x_{n+4}) + d(x_{n+5}, x_{n+4}) \\ &\quad + d(x_{n+5}, x_{n+6}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1) + \alpha^{n+2} d(x_1, x_0) + \alpha^{n+3} d(x_0, x_1) + \alpha^{n+4} d(x_1, x_0) \\ &\quad + \alpha^{n+5} d(x_0, x_1) + \cdots + \alpha^{m-1} d(x_0, x_1) \\ &\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right) \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \\ &= \alpha^n \cdot \frac{1 - \alpha^{m-n}}{1 - \alpha} \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \end{aligned}$$

and

$$\begin{aligned} d(x_n, x_m) &\leq d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1}) + d(x_{n+2}, x_{n+3}) + d(x_{n+4}, x_{n+3}) + d(x_{n+4}, x_{n+5}) \\ &\quad + d(x_{n+6}, x_{n+5}) + \cdots + d(x_m, x_{m-1}) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n+1} d(x_1, x_0) + \alpha^{n+2} d(x_0, x_1) + \alpha^{n+3} d(x_1, x_0) + \alpha^{n+4} d(x_0, x_1) \\ &\quad + \alpha^{n+5} d(x_1, x_0) + \cdots + \alpha^{m-1} d(x_1, x_0) \\ &\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right) \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \\ &= \alpha^n \cdot \frac{1 - \alpha^{m-n}}{1 - \alpha} \cdot \max\{d(x_0, x_1), d(x_1, x_0)\} \end{aligned}$$

which imply

$$d(x_m, x_n) \to 0 \text{ and } d(x_n, x_m) \to 0 \text{ as } n \to \infty.$$

This proves that $\{x_n\}_{n=1}^{\infty}$ is both a $>$-Cauchy sequence and $<$-Cauchy sequence.

Assume that X is simultaneously $(>, \triangleright)$-complete and $(<, \triangleleft)$-complete. Then there exists $x^*, x^\circ \in X$ satisfying $d(x_n, x^*) \to 0$ and $d(x^\circ, x_n) \to 0$ as $n \to \infty$. Now, we have

$$\begin{aligned} d(x^*, T(x^\circ)) &\leq d(x_m, x^*) + d(T(x^\circ), x_m) \text{ (using the } \diamond\text{-triangle inequality)} \\ &= d(x_m, x^*) + d(T(x^\circ), T(x_{m-1})) \\ &\leq d(x_m, x^*) + \alpha d(x^\circ, x_{m-1}) \text{ (using the contraction),} \end{aligned}$$

which implies $d(x^*, T(x^\circ)) = 0$ as $m \to \infty$. We conclude that $T(x^\circ) = x^*$. We also have

$$\begin{aligned} d(T(x^*), x^\circ) &\leq d(x_m, T(x^*)) + d(x^\circ, x_m) \text{ (using the } \diamond\text{-triangle inequality)} \\ &= d(T(x_{m-1}), T(x^*)) + d(x^\circ, x_m) \\ &\leq \alpha d(x_{m-1}, x^*) + d(x^\circ, x_m) \text{ (using the contraction),} \end{aligned}$$

which implies $d(T(x^*), x^\circ) = 0$ as $m \to \infty$. We conclude that $T(x^*) = x^\circ$. The remaining proof follows from the same argument in the proof of Theorem 3. This completes the proof. □

5. Meir–Keeler Type of Fixed Point Theorems for Pre-Metric Spaces

In the sequel, we are going to establish the Meir–Keeler type of fixed point theorems for pre-metric spaces. First of all, we consider the different contraction.

Definition 7. *Let (X, d) be a pre-metric space. A function $T : (X, d) \to (X, d)$ is called a weakly strict contraction on X when the following conditions are satisfied:*

- *$d(x, y) = 0$ implies $d(T(x), T(y)) = 0$;*
- *$d(x, y) \neq 0$ implies $d(T(x), T(y)) < d(x, y)$.*

It is clear to see that if T is a contraction on X, then it is also a weakly strict contraction on X.

Theorem 5 (Fixed Points Using the $\triangleleft$-Triangle Inequality)**.** *Let (X, d) be a $(>, \triangleright)$-complete (resp. $(<, \triangleright)$-complete) pre-metric space such that the $\triangleleft$-triangle inequality is satisfied. Suppose that the function $T : (X, d) \to (X, d)$ is a weakly strict contraction on X, and that $\{T^n(x_0)\}_{n=1}^{\infty}$ forms a $>$-Cauchy sequence (resp. $<$-Cauchy sequence) for some $x_0 \in X$. Then, the function T has a unique fixed point $x \in X$. Moreover, the fixed point x is obtained by the following limit*

$$d(T^n(x_0), x) \to 0 \text{ as } n \to \infty.$$

Proof. Since $\{T^n(x_0)\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence, the $(>, \triangleright)$-completeness says that there exists $x \in X$ satisfying $d(T^n(x_0), x) \to 0$ as $n \to \infty$. In other words, given any $\epsilon > 0$, there exists an integer N satisfying $d(T^n(x_0), x) < \epsilon$ for $n \geq N$. Regarding $d(T^n(x_0), x)$, we consider two different cases as follows.

- Suppose that $d(T^n(x_0), x) = 0$. Then, the weakly strict contraction of T says that

$$d(T^{n+1}(x_0), T(x)) = 0 < \epsilon.$$

- Suppose that $d(T^n(x_0), x) \neq 0$. Then, the weakly strict contraction of T says that

$$d(T^{n+1}(x_0), T(x)) < d(T^n(x_0), x) < \epsilon \text{ for } n \geq N.$$

The above two cases conclude that $d(T^{n+1}(x_0), T(x)) \to 0$ as $n \to \infty$. Using the $\triangleleft$-triangle inequality, we obtain

$$d(T(x), x) \leq d\Big(T^{n+1}(x_0), T(x)\Big) + d\Big(T^{n+1}(x_0), x\Big) \to 0 \text{ as } n \to \infty,$$

which shows that $d(T(x), x) = 0$, i.e., $T(x) = x$. In other words, x is a fixed point.

Regarding the uniqueness, suppose that $\bar{x}$ is another fixed point of T. i.e., $T(\bar{x}) = \bar{x}$. Since $x \neq \bar{x}$, the weakly strict contraction of T says that

$$d(x, \bar{x}) = d(T(x), T(\bar{x})) < d(x, \bar{x}).$$

This contradiction shows that $\bar{x}$ cannot be a fixed point of T.

When $\{T^n(x_0)\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence, using the $(<, \triangleright)$-completeness, we can similarly obtain the desired results. This completes the proof. □

Theorem 6 (Fixed Points Using the $\triangleright$-Triangle Inequality)**.** *Let (X, d) be a $(>, \triangleleft)$-complete (resp. $(<, \triangleleft)$-complete) pre-metric space such that the $\triangleright$-triangle inequality is satisfied. Suppose that the function $T : (X, d) \to (X, d)$ is a weakly strict contraction on X, and that $\{T^n(x_0)\}_{n=1}^{\infty}$*

forms a $>$-Cauchy sequence (resp. $<$-Cauchy sequence) for some $x_0 \in X$. Then, the function T has a unique fixed point $x \in X$. Moreover, the fixed point x is obtained by the following limit

$$d(x, T^n(x_0)) \to 0 \text{ as } n \to \infty.$$

Proof. Since $\{T^n(x_0)\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence, the $(>, \triangleleft)$-completeness says that there exists $x \in X$ satisfying $d(x, T^n(x_0)) \to 0$ as $n \to \infty$. From the proof of Theorem 5, the weakly strict contraction of T can similarly show that $d(T(x), T^{n+1}(x_0)) \to 0$ as $n \to \infty$. Using the $\triangleright$-triangle inequality, we obtain

$$d(T(x), x) \leq d\Big(T(x), T^{n+1}(x_0)\Big) + d\Big(x, T^{n+1}(x_0)\Big) \to 0 \text{ as } n \to \infty,$$

which says that $d(T(x), x) = 0$, i.e., $T(x) = x$. This shows that x is a fixed point. The remaining proof follows from the proof of Theorem 5. This completes the proof. □

Theorem 7 (Fixed Points Using the $\bowtie$-Triangle Inequality)**.** *Let (X, d) be a simultaneously $(>, \triangleright)$-complete and $(>, \triangleleft)$-complete (resp. $(<, \triangleright)$-complete and $(<, \triangleleft)$-complete) pre-metric space such that the $\bowtie$-triangle inequality is satisfied. Suppose that the function $T : (X, d) \to (X, d)$ is a weakly strict contraction on X, and that $\{T^n(x_0)\}_{n=1}^{\infty}$ forms a $>$-Cauchy sequence (resp. $<$-Cauchy sequence) for some $x_0 \in X$, then T has a unique fixed point $x \in X$. Moreover, the fixed point x is obtained by the following limits*

$$d(T^n(x_0), x) \to 0 \text{ or } d(x, T^n(x_0)) \to 0 \text{ as } n \to \infty.$$

Proof. Since $\{T^n(x_0)\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence, the $(>, \triangleright)$-completeness says that there exists $x^* \in X$ satisfying $d(T^n(x_0), x^*) \to 0$ as $n \to \infty$. The $(>, \triangleleft)$-completeness also says that there exists another $x^\circ \in X$ satisfying $d(x^\circ, T^n(x_0)) \to 0$ as $n \to \infty$. From the proof of Theorem 5, the weakly strict contraction of T can similarly show that $d(T^{n+1}(x_0), T(x^*)) \to 0$ and $d(T(x^\circ), T^{n+1}(x_0)) \to 0$ as $n \to \infty$. Using the $\bowtie$-triangle inequality, we obtain

$$d(T(x^\circ), x^*) \leq d\Big(T(x^\circ), T^{n+1}(x_0)\Big) + d\Big(T^{n+1}(x_0), x^*\Big) \to 0 \text{ as } n \to \infty,$$

which says that $d(T(x^\circ), x^*) = 0$, i.e., $T(x^\circ) = x^*$. On the other hand, we also have

$$d(x^\circ, T(x^*)) \leq d\Big(x^\circ, T^{n+1}(x_0)\Big) + d\Big(T^{n+1}(x_0), T(x^*)\Big) \to 0 \text{ as } n \to \infty,$$

which says that $T(x^*) = x^\circ$. By referring to (6), it follows that x^* and x° are fixed points of the composition mapping $T \circ T \equiv T^2$. Suppose that $x^* \neq x^\circ$. We want to claim $T(x^*) \neq T(x^\circ)$. Assume that it is not true, i.e., $T(x^*) = T(x^\circ)$. Then, we shall have

$$x^\circ = T(x^*) = T(x^\circ) = x^*,$$

which contradicts $x^* \neq x^\circ$. The weakly strict contraction of T also says that

$$d(x^*, x^\circ) = d(T^2(x^*), T^2(x^\circ)) < d(T(x^*), T(x^\circ)) < d(x^*, x^\circ).$$

This contradiction shows that $x^* = x^\circ$, and says that $x^* = x^\circ$ is a fixed point of T. The uniqueness can be obtained from the proof of Theorem 5

When $\{T^n(x_0)\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence, using the $(<, \triangleright)$-completeness and $(<, \triangleleft)$-completeness, we can similarly obtain the desired results. This completes the proof. □

Theorem 8 (Fixed Points Using the $\diamond$-Triangle Inequality)**.** *Let (X, d) be a simultaneously $(>, \triangleright)$-complete and $(>, \triangleleft)$-complete (resp. $(<, \triangleright)$-complete and $(<, \triangleleft)$-complete) pre-metric space such that the $\diamond$-triangle inequality is satisfied. Suppose that the function $T : (X, d) \to$*

(X,d) is a weakly strict contraction on X, and that $\{T^n(x_0)\}_{n=1}^{\infty}$ forms a $>$-Cauchy sequence (resp. $<$-Cauchy sequence) for some $x_0 \in X$. Then T has a unique fixed point $x \in X$. Moreover, the fixed point x is obtained by the following limits

$$d(T^n(x_0), x) \to 0 \text{ or } d(x, T^n(x_0)) \to 0 \text{ as } n \to \infty.$$

Proof. From the proof of Theorem 7, there exist $x^*, x^\circ \in X$ satisfying $d(T^n(x_0), x^*) \to 0$, $d(x^\circ, T^n(x_0)) \to 0$, $d(T^{n+1}(x_0), T(x^*)) \to 0$ and $d(T(x^\circ), T^{n+1}(x_0)) \to 0$ as $n \to \infty$. Using the $\diamond$-triangle inequality, we can obtain

$$d(x^*, T(x^\circ)) \leq d\Big(T^{n+1}(x_0), x^*\Big) + d\Big(T(x^\circ), T^{n+1}(x_0)\Big) \to 0 \text{ as } n \to \infty,$$

which says that $d(x^*, T(x^\circ)) = 0$, i.e., $T(x^\circ) = x^*$. We also have

$$d(T(x^*), x^\circ) \leq d\Big(T^{n+1}(x_0), T(x^*)\Big) + d\Big(x^\circ, T^{n+1}(x_0)\Big) \to 0 \text{ as } n \to \infty,$$

which says that $T(x^*) = x^\circ$. The remaining proof follows from the similar argument in the proof of Theorem 7. This completes the proof. □

Next, we consider the different fixed point theorems based on the weakly uniformly strict contraction that was proposed by Meir and Keeler [12].

Definition 8. *Let (X,d) be a pre-metric space. A function $T : (X,d) \to (X,d)$ is called a weakly uniformly strict contraction on X when the following conditions are satisfied:*

- *$d(x,y) = 0$ implies $d(T(x), T(y)) = 0$;*
- *given any $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon \leq d(x,y) < \epsilon + \delta$ implies $d(T(x), T(y)) < \epsilon$ for any $x, y \in X$ with $d(x,y) \neq 0$.*

Remark 2. *We observe that if T is a weakly uniformly strict contraction on X, then T is also a weakly strict contraction on X.*

Lemma 1. *Let (X,d) be a pre-metric space, and let $T : (X,d) \to (X,d)$ be a weakly uniformly strict contraction on X. Then, the sequences $\{d(T^n(x), T^{n+1}(x))\}_{n=1}^{\infty}$ and $\{d(T^{n+1}(x), T^n(x))\}_{n=1}^{\infty}$ are decreasing to zero for any $x \in X$.*

Proof. For convenience, we simply write $T^n(x) = x_n$ for all n. Let $c_n = d(x_n, x_{n+1})$. Regarding $d(x_{n-1}, x_n) \geq 0$, we consider two different cases as follows.

- Suppose that $d(x_{n-1}, x_n) \neq 0$. By Remark 2, we have

$$c_n = d(x_n, x_{n+1}) = d(T^n(x), T^{n+1}(x)) < d(T^{n-1}(x), T^n(x)) = d(x_{n-1}, x_n) = c_{n-1}.$$

- Suppose that $d(x_{n-1}, x_n) = 0$. Then, by the first condition of Definition 8, we have

$$c_n = d(T^n(x), T^{n+1}(x)) = d(T(x_{n-1}), T(x_n)) = 0 \leq c_{n-1}.$$

The above two cases conclude that the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing. We also consider the following two cases.

- Let m be the first index in the sequence $\{x_n\}_{n=1}^{\infty}$ satisfying $d(x_{m-1}, x_m) = 0$. Then, we want to claim

$$c_{m-1} = c_m = c_{m+1} = \cdots = 0.$$

Using the first condition of Definition 8, we have

$$0 = d(T(x_{m-1}), T(x_m)) = d(T^m(x), T^{m+1}(x)) = d(x_m, x_{m+1}) = c_m.$$

Since $d(x_m, x_{m+1}) = 0$, using the similar argument, we can also obtain $c_{m+1} = 0$ and $d(x_{m+1}, x_{m+2}) = 0$. This shows that the sequence $\{c_n\}_{n=1}^{\infty}$ is indeed decreasing to zero.

- Suppose that $d(x_m, x_{m+1}) \neq 0$ for all $m \geq 1$. Since the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing, we assume that $c_n \downarrow \epsilon > 0$, i.e., $c_n \geq \epsilon > 0$ for all n. Therefore, there exists $\delta > 0$ satisfying $\epsilon \leq c_m < \epsilon + \delta$ for some m, i.e., $\epsilon \leq d(x_m, x_{m+1}) < \epsilon + \delta$. Using the second condition of Definition 8, it follows that

$$c_{m+1} = d(x_{m+1}, x_{m+2}) = d(T^{m+1}(x), T^{m+2}(x)) = d(T(x_m), T(x_{m+1})) < \epsilon,$$

which contradicts $c_{m+1} \geq \epsilon$.

Therefore, we conclude that the sequence $\{d(T^n(x), T^{n+1}(x))\}_{n=1}^{\infty}$ is indeed decreasing to zero for any $x \in X$. We can similarly show that the sequence $\{d(T^{n+1}(x), T^n(x))\}_{n=1}^{\infty}$ is decreasing to zero for any $x \in X$. This completes the proof. □

Theorem 9 (Meir–Keeler Type of Fixed Points Using the $\triangleleft$-Triangle Inequality)**.** *Let (X, d) be a $(>, \triangleright)$-complete pre-metric space such that the $\triangleleft$-triangle inequality is satisfied, and let $T : (X, d) \to (X, d)$ be a weakly uniformly strict contraction on X. Then T has a unique fixed point. Moreover, the fixed point x is obtained by the following limit*

$$d(T^n(x_0), x) \to 0 \text{ as } n \to \infty \text{ for some } x_0.$$

Proof. According to Theorem 5 and Remark 2, we just need to claim that if T is a weakly uniformly strict contraction, then $\{T^n(x_0)\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence for $x_0 \in X$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is not a $>$-Cauchy sequence. Then, there exists $2\epsilon > 0$ such that, given any integer N, there exist $n > m \geq N$ satisfying $d(x_n, x_m) > 2\epsilon$. We are going to lead to a contradiction. The weakly uniformly strict contraction of T says that there exists $\delta > 0$ satisfying

$$\epsilon \leq d(x, y) < \epsilon + \delta \text{ implies } d(T(x), T(y)) < \epsilon \text{ for any } x, y \in X \text{ with } d(x, y) \neq 0.$$

Let $\delta' = \min\{\delta, \epsilon\}$. We want to show

$$\epsilon \leq d(x, y) < \epsilon + \delta' \text{ implies } d(T(x), T(y)) < \epsilon \text{ for any } x, y \in X \text{ with } d(x, y) \neq 0. \tag{7}$$

Indeed, when $\delta' = \epsilon$, i.e., $\epsilon < \delta$, we have $\epsilon + \delta' = \epsilon + \epsilon < \epsilon + \delta$.

Let $c_n = d(x_n, x_{n+1})$ and $\bar{c}_n = d(x_{n+1}, x_n)$. Since the sequences $\{c_n\}_{n=1}^{\infty}$ and $\{\bar{c}_n\}_{n=1}^{\infty}$ are decreasing to zero by Lemma 1, we can find a common integer N satisfying

$$c_N < \delta'/3 \text{ and } \bar{c}_N < \delta'/3. \tag{8}$$

For $n > m \geq N$, we have

$$d(x_n, x_m) > 2\epsilon \geq \epsilon + \delta', \tag{9}$$

which implicitly says that $d(x_n, x_m) \neq 0$. Since the sequence $\{\bar{c}_n\}_{n=1}^{\infty}$ is decreasing by Lemma 1 again, we can obtain

$$d(x_{m+1}, x_m) = \bar{c}_m \leq \bar{c}_N < \frac{\delta'}{3} \leq \frac{\epsilon}{3} < \epsilon. \tag{10}$$

For j with $m < j \leq n$, using the $\triangleleft$-triangle inequality, it follows that

$$d(x_{j+1}, x_m) \leq d(x_j, x_{j+1}) + d(x_j, x_m). \tag{11}$$

We want to cliam that there exists an integer j with $m < j \leq n$ satisfying $d(x_j, x_m) \neq 0$ and

$$\epsilon + \frac{2\delta'}{3} < d(x_j, x_m) < \epsilon + \delta'. \tag{12}$$

Let $\gamma_j = d(x_j, x_m)$ for $j = m+1, \cdots, n$. Using (9) and (10), we have

$$\gamma_{m+1} < \epsilon \text{ and } \gamma_n > \epsilon + \delta'. \tag{13}$$

Let j_0 be an index satisfying

$$j_0 = \max\left\{ j \in [m+1, n] : \gamma_j \leq \epsilon + \frac{2\delta'}{3} \right\}. \tag{14}$$

Then, from (13), we see that $m+1 \leq j_0 < n$, which also says that j_0 is well-defined. By the definition of j_0, it follows that $j_0 + 1 \leq n$ and $\gamma_{j_0+1} > \epsilon + \frac{2\delta'}{3}$, which also says that $d(x_{j_0+1}, x_m) \neq 0$. Therefore, the expression (12) will be sound if we can show

$$\epsilon + \frac{2\delta'}{3} < \gamma_{j_0+1} < \epsilon + \delta'.$$

Suppose that this is not true, i.e., $\gamma_{j_0+1} \geq \epsilon + \delta'$. From (11), we have

$$\frac{\delta'}{3} > c_N \geq c_{j_0} = d(x_{j_0}, x_{j_0+1}) \geq \gamma_{j_0+1} - \gamma_{j_0} \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \frac{\delta'}{3}.$$

This contradiction says that the expression (12) is sound. Since $d(x_j, x_m) \neq 0$, using (7), it follows that (12) implies

$$d(x_{j+1}, x_{m+1}) = d(T(x_j), T(x_m)) < \epsilon. \tag{15}$$

Using the $\triangleleft$-triangle inequality and referring to (2), we can obtain

$$\begin{aligned} d(x_j, x_m) &\leq d(x_{j+1}, x_j) + d(x_{j+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &= \bar{c}_j + d(x_{j+1}, x_{m+1}) + \bar{c}_m < \bar{c}_j + \epsilon + \bar{c}_m \text{ (by (15))} \\ &\leq \bar{c}_N + \epsilon + \bar{c}_N \text{ (since } \{\bar{c}_n\}_{n=1}^{\infty} \text{ is decreasing)} \\ &< \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} \text{ (by (8))} \\ &= \epsilon + \frac{2\delta'}{3}, \end{aligned}$$

which contradicts (12). This contradiction shows that every sequence $\{T^n(x)\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence. Using Theorem 5, the proof is complete. □

Theorem 10 (Meir–Keeler Type of Fixed Points Using the $\triangleright$-Triangle Inequality)**.** *Let (X, d) be a $(>, \triangleleft)$-complete pre-metric space such that the $\triangleright$-triangle inequality is satisfied, and let $T : (X, d) \to (X, d)$ be a weakly uniformly strict contraction on X. Then T has a unique fixed point. Moreover, the fixed point x is obtained by the following limit*

$$d(x, T^n(x_0)) \to 0 \text{ as } n \to \infty \text{ for some } x_0.$$

Proof. According to Theorem 6 and Remark 2, we just need to claim that if T is a weakly uniformly strict contraction, then $\{T^n(x_0)\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence for $x_0 \in X$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is not a $<$-Cauchy sequence. Then, there exists $2\epsilon > 0$

such that, given any integer N, there exist $n > m \geq N$ satisfying $d(x_m, x_n) > 2\epsilon$. Let $\delta' = \min\{\delta, \epsilon\}$. For $n > m \geq N$, we have

$$d(x_m, x_n) > 2\epsilon \geq \epsilon + \delta', \tag{16}$$

which implicitly says that $d(x_m, x_n) \neq 0$. Let $c_n = d(x_n, x_{n+1})$ and $\bar{c}_n = d(x_{n+1}, x_n)$. Since the sequence $\{c_n\}_{n=1}^{\infty}$ is decreasing by Lemma 1, we obtain

$$d(x_m, x_{m+1}) = c_m \leq c_N < \frac{\delta'}{3} \leq \frac{\epsilon}{3} < \epsilon. \tag{17}$$

For j with $m < j \leq n$, using the $\triangleright$-triangle inequality, we also have

$$d(x_m, x_{j+1}) \leq d(x_m, x_j) + d(x_{j+1}, x_j). \tag{18}$$

We want to cliam that there exists an integer j with $m < j \leq n$ satisfying $d(x_m, x_j) \neq 0$ and

$$\epsilon + \frac{2\delta'}{3} < d(x_m, x_j) < \epsilon + \delta'. \tag{19}$$

Let $\gamma_j = d(x_m, x_j)$ for $j = m+1, \cdots, n$. From (16) and (17), we can also obtain (13). Let j_0 be defined in (14). Then, the expression (19) will be sound if we can show that

$$\epsilon + \frac{2\delta'}{3} < \gamma_{j_0+1} < \epsilon + \delta'.$$

Suppose that this is not true, i.e., $\gamma_{j_0+1} \geq \epsilon + \delta'$. From (18) and (8), it follows that

$$\frac{\delta'}{3} > \bar{c}_N \geq \bar{c}_{j_0} = d(x_{j_0+1}, x_{j_0}) \geq \gamma_{j_0+1} - \gamma_{j_0} \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \frac{\delta'}{3}.$$

This contradiction says that (19) is sound. Since $d(x_m, x_j) \neq 0$, using (7), it follows that (19) implies

$$d(x_{m+1}, x_{j+1}) = d(T(x_m), T(x_j)) < \epsilon. \tag{20}$$

Using the $\triangleright$-triangle inequality and referring to (1), we can obtain

$$\begin{aligned} d(x_m, x_j) &\leq d(x_j, x_{j+1}) + d(x_{m+1}, x_{j+1}) + d(x_m, x_{m+1}) \\ &= c_j + d(x_{m+1}, x_{j+1}) + c_m < c_j + \epsilon + c_m \text{ (using Equation (20))} \\ &\leq c_N + \epsilon + c_N \text{ (since } \{c_n\}_{n=1}^{\infty} \text{ is decreasing)} \\ &< \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} \text{ (using Equation (8))} \\ &= \epsilon + \frac{2\delta'}{3}, \end{aligned}$$

which contradicts (19). This contradiction shows that every sequence $\{T^n(x)\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence. Using Theorem 6, the proof is complete. □

Theorem 11 (Meir–Keeler Type of Fixed Points Using the $\bowtie$-Triangle Inequality). *Let (X, d) be a pre-metric space such that the $\bowtie$-triangle inequality is satisfied. We also assume that any one of the following conditions is satisfied:*

- *(X, d) is simultaneously $(>, \triangleright)$-complete and $(>, \triangleleft)$-complete;*
- *(X, d) is simultaneously $(<, \triangleright)$-complete and $(<, \triangleleft)$-complete;*
- *(X, d) is simultaneously $\triangleright$-complete and $\triangleleft$-complete.*

Suppose that $T : (X, d) \to (X, d)$ is a weakly uniformly strict contraction on X. Then T has a unique fixed point. Moreover, the fixed point x is obtained by the following limits

$$d(T^n(x_0), x) \to 0 \text{ or } d(x, T^n(x_0)) \to 0 \text{ as } n \to \infty.$$

Proof. According to Theorem 7 and Remark 2, we just need to claim that if T is a weakly uniformly strict contraction, then $\{T^n(x_0)\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is both a $<$-Cauchy sequence and $>$-Cauchy sequence for $x_0 \in X$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is not a $<$-Cauchy sequence. Then, there exists $2\epsilon > 0$ such that, given any integer N, there exist $n > m \geq N$ satisfying $d(x_m, x_n) > 2\epsilon$. We are going to follow the similar proof of Theorem 10.

Let $\delta' = \min\{\delta, \epsilon\}$, and let $\gamma_j = d(x_m, x_j)$ for $j = m+1, \cdots, n$. For j with $m < j \leq n$, using the $\bowtie$-triangle inequality, we have

$$d(x_m, x_{j+1}) \leq d(x_m, x_j) + d(x_j, x_{j+1}),$$

which implies

$$\frac{\delta'}{3} > c_N \geq c_{j_0} = d(x_{j_0}, x_{j_0+1}) \geq \gamma_{j_0+1} - \gamma_{j_0} \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \frac{\delta'}{3}.$$

This contradiction shows that there exists an integer j with $m < j \leq n$ satisfying $d(x_m, x_j) \neq 0$ and

$$\epsilon + \frac{2\delta'}{3} < d(x_m, x_j) < \epsilon + \delta', \tag{21}$$

which implies

$$d(x_{m+1}, x_{j+1}) = d(T(x_m), T(x_j)) < \epsilon. \tag{22}$$

Using the $\bowtie$-triangle inequality, we can obtain

$$\begin{aligned} d(x_m, x_j) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{j+1}) + d(x_{j+1}, x_j) \\ &= c_m + d(x_{m+1}, x_{j+1}) + \bar{c}_j < c_m + \epsilon + \bar{c}_j \text{ (by (22))} \\ &\leq c_N + \epsilon + \bar{c}_N \text{ (since } \{c_n\}_{n=1}^{\infty} \text{ and } \{\bar{c}_n\}_{n=1}^{\infty} \text{ are decreasing)} \\ &< \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} \text{ (by (8))} \\ &= \epsilon + \frac{2\delta'}{3}, \end{aligned}$$

which contradicts (21). This contradiction shows that every sequence $\{T^n(x)\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is a $<$-Cauchy sequence.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is not a $>$-Cauchy sequence. Then, there exists $2\epsilon > 0$ such that, given any integer N, there exist $n > m \geq N$ satisfying $d(x_n, x_m) > 2\epsilon$. Let $\delta' = \min\{\delta, \epsilon\}$, and let $\gamma_j = d(x_j, x_m)$ for $j = m+1, \cdots, n$. For j with $m < j \leq n$, using the $\bowtie$-triangle inequality, we have

$$d(x_{j+1}, x_m) \leq d(x_{j+1}, x_j) + d(x_j, x_m),$$

which implies

$$\frac{\delta'}{3} > \bar{c}_N \geq \bar{c}_{j_0} = d(x_{j_0+1}, x_{j_0}) \geq \gamma_{j_0+1} - \gamma_{j_0} \geq \epsilon + \delta' - \epsilon - \frac{2\delta'}{3} = \frac{\delta'}{3}.$$

This contradiction shows that there exists an integer j with $m < j \leq n$ satisfying $d(x_j, x_m) \neq 0$ and

$$\epsilon + \frac{2\delta'}{3} < d(x_j, x_m) < \epsilon + \delta', \tag{23}$$

which implies

$$d(x_{j+1}, x_{m+1}) = d(T(x_j), T(x_m)) < \epsilon. \tag{24}$$

Using the $\bowtie$-triangle inequality, we can obtain

$$\begin{aligned} d(x_j, x_m) &\leq d(x_j, x_{j+1}) + d(x_{j+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &= c_j + d(x_{j+1}, x_{m+1}) + \bar{c}_m < c_j + \epsilon + \bar{c}_m \text{ (by (24))} \\ &\leq c_N + \epsilon + \bar{c}_N \text{ (since } \{c_n\}_{n=1}^{\infty} \text{ and } \{\bar{c}_n\}_{n=1}^{\infty} \text{ are decreasing)} \\ &< \frac{\delta'}{3} + \epsilon + \frac{\delta'}{3} \text{ (by (8))} \\ &= \epsilon + \frac{2\delta'}{3}, \end{aligned}$$

which contradicts (23). This contradiction shows that every sequence $\{T^n(x)\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is a $>$-Cauchy sequence. Using Theorem 7, the proof is complete. □

Example 5. *Continued from Example 3, since the pre-metric space $(\mathbb{R}, d)$ is simultaneously $\triangleright$-complete and $\triangleleft$-complete, any function $T : (\mathbb{R}, d) \to (\mathbb{R}, d)$ that is a weakly uniformly strict contraction on $\mathbb{R}$ has a unique fixed point. The concrete examples regarding functions that are weakly uniformly strict contraction on $\mathbb{R}$ can be obtained from the literature.*

We finally remark that the Meir–Keeler type of fixed point theorem based on the $\diamond$-triangle inequality cannot be obtained by using an argument similar to Theorem 11. In other words, we need to design a different argument to obtain the Meir–Keeler type of fixed point theorem based on the $\diamond$-triangle inequality. It is also possible that we cannot establish the Meir–Keeler type of fixed point theorem based on the $\diamond$-triangle inequality. Therefore, this problem remains open and could be the subject future research.

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