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Multiple Optimal Solutions and the Best Lipschitz Constants Between an Aggregation Function and Associated Idempotized Aggregation Function

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Abstract: This paper presents and compares the optimal solutions and the theoretical and empirical best Lipschitz constants between an aggregation function and associated idempotized aggregation function. According to an exhaustive search we performed, the multiple optimal solutions and the empirical best Lipschitz constants are presented explicitly. The results indicate that differences of the multiple optimal solutions exist among the Minkowski norm, the number of steps, and the type of aggregation function. We demonstrate that these differences can affect the theoretical and empirical best Lipschitz constants of an aggregation function.

Keywords: aggregation; Lipschitz; computation; idempotent

MSC: 90C59; 65K05

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1. Introduction

Aggregation functions [1–8] find wide applications in almost all branches of engineering. Typical fields include mathematics and information sciences, especially in decision-making and the fusion of information problems. The aim of an aggregation function is to summarize an n-tuple of information by means of a single representative value. The fundamental axiom of an aggregation function is the non-decreasing monotonicity. Another axiomatic constraint of an aggregation function is that the aggregating of minimal or maximal inputs are, respectively, minimal and maximal outputs.

The prototypical example of aggregation functions is the arithmetic mean, which is the first modern definition of mean. The concept of arithmetic mean seems to have been proposed first by Cauchy in 1821 [9]. Since then, a large variety of aggregation functions have been proposed. The conjunctive, the disjunctive, the internal, and the mixed aggregation functions are four main classes of aggregation functions based on many-valued logics connectives [10]. The algebraic and analytical properties of an aggregation function are proposed and analyzed in the literature [1–8]. Associativity, symmetry, bisymmetry, idempotency, neutral element, and annihilator element are algebraic properties. Continuity, Lipschitzian, and additivity are analytical properties. A first review of papers reporting aggregation function results was undertaken by Xu and Da [8]. More details can be seen in the excellent reviews on the state of the art by Grabisch et al. [3–5], group decision making by Mohd and Abdullah [7] and Del Moral et al. [2], and construction methods by Khameneh and Kilicman [6]. Recently, many papers have been dedicated to an aggregation function in group decision making [11–13], multi-criteria decision making [14], two-side matching decision making [15], and others [16,17].

One characterization of the mean is the Chisini's equation [9], described as follows: a mean M, with respect to the function F, is that each input of F can be replaced with M without changing the overall aggregation. When F is considered as the sum and the product, the solution of Chisini′s equation is the arithmetic mean and the geometric mean, respectively. For the analytical properties, an aggregation function, which satisfies the Lipschitz condition is a continuous one. More details can be found in [1,3]. This paper analyzes and compares the Lipschitz behaviors between the sum and the arithmetic mean, and those of the product and the geometric mean. The Lipschitz constants for the sum, the arithmetic mean, the product, and the geometric mean can be obtained analytically by the triangular inequality and the Hölder inequality [1,3]. However, the best Lipschitz constants, which are the greatest lower bound of the Lipschitz constants, may or may not be attainable. The reason is that the feasible region of constraints for the mathematical programming model of the best Lipschitz constant is not compact. To the best knowledge of the authors, such a problem has not been considered in the literature. For the best Lipschitz constant, a mathematical model with non-Archimedean numbers is proposed. We also propose a discrete approximation of the mathematical model. We adopt an exhaustive analysis to empirically find and compare the optimal solutions and the empirical best Lipschitz constants for the sum and the arithmetic mean, and for the product and the geometric mean. The multiple optimal solutions and the empirical best Lipschitz constants are presented explicitly.

The organization of this paper is as follows. Section 2 briefly reviews an aggregation function. We analyze and compare the optimal solutions and the best Lipschitz constants between the sum function and the arithmetic mean in Section 3, and between the product function and the geometric mean in Section 4. Finally, some concluding remarks and future research are presented.

2. An Aggregation Function

We now recall the definition of an aggregation function [1–4,6,7]. Let $\mathbb{I} \subset \mathcal{R}$ be the closed unit interval [0,1], and $\mathbb{I}^n = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) | x_i \in \mathbb{I}, i = 1, 2, \ldots, n\}$. Furthermore, $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if $x_i \leq y_i, i = 1, 2, \ldots, n$.

Definition 1. *An n-ary aggregation function* $A^{(n)} : \mathbb{I}^n \to \mathbb{I}$ *satisfies:*

- $A^{(1)}(x) = x$, *for* $n = 1$ *and* $x \in \mathbb{I}$;
- *If* $\boldsymbol{x} \leq \boldsymbol{y}$, *then* $A^{(n)}(\boldsymbol{x}) \leq A^{(n)}(\boldsymbol{y})$ *for* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{I}^n$;
- $A^{(n)}(0, 0, \ldots, 0) = 0$ *and* $A^{(n)}(1, 1, \ldots, 1) = 1$.

The generalized inputs $\mathbb{I}$ of an aggregation function are a subdomain of the extended real line $[-\infty, \infty]$. They can be any type (open, closed, ...) of interval. For simplicity, we deal with the closed unit interval [0,1]. An extended aggregation function is a mapping $A : \cup_{n \in \mathcal{N}} \mathbb{I}^n \to \mathbb{I}$ whose restriction to $\mathbb{I}^n$ is the n-ary aggregation function $A^{(n)}$ for any $n \in \mathcal{N}$. When no confusion can arise, we use the convenient notation A to represent $A^{(n)}$.

For $x_i \in \mathbb{I}, i = 1, 2, \ldots, n$, some well-known examples of aggregation functions [1–4,6,7] are as follows:

1. Median Md defined by $\mathrm{Md}(x_1, x_2, \ldots, x_n) = x_{\left(\frac{n+1}{2}\right)}$ if n is odd and $\mathrm{Md}(x_1, x_2, \ldots, x_n) = \frac{1}{2}\left(x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}\right)$ if n is even where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$.
2. Arithmetic mean (AM) $\mathrm{AM}(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$.
3. Weighted arithmetic mean (WAM) $\mathrm{WAM}(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} w_i x_i$, where $w_i \in \mathbb{I}$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} w_i = 1$.
4. Geometric mean (GM) $\mathrm{GM}(x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$.
5. Harmonic mean (HM) $\mathrm{HM}(x_1, x_2, \ldots, x_n) = \frac{n}{\sum_{i=1}^{n} 1/x_i}$.
6. Minimum (min) $\min(x_1, x_2, \ldots, x_n) = \min_{i=1}^{n} x_i$ and maximum (max) $\max(x_1, x_2, \ldots, x_n) = \max_{i=1}^{n} x_i$.
7. Product function $\prod(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i$.
8. Projection function to the kth coordinate $P_k(x_1, x_2, \ldots, x_n) = x_k$.

9. The weakest aggregation function $A_w(x_1, x_2, \ldots, x_n) = \begin{cases} 1, \text{if } (x_1,\ x_2, \ldots,\ x_n)\ =\ (1,\ 1, \ldots,\ 1) \\ 0, \text{else.} \end{cases}$
10. The strongest aggregation function $A_s(x_1, x_2, \ldots, x_n) = \begin{cases} 0,\ if\ (x_1,\ x_2, \ldots,\ x_n)\ =\ (0,\ 0, \ldots,\ 0) \\ 1,\ else. \end{cases}$
11. Operator $A_c(x_1, x_2, \ldots, x_n) = max(0,\ \min(1, c + \sum_{i=1}^{n}(x_i - c))$ for $c \in \mathbb{I}$.

For all $x_1, x_2, \ldots, x_n \in [0,1]$, the relationship between the arithmetic mean, the geometric mean, the harmonic mean, the minimum, the maximum, the product function, the weakest aggregation function, and the strongest aggregation function is

$A_w(x_1, x_2, \ldots, x_n) \leq \prod(x_1, x_2, \ldots, x_n) \leq\ \mathrm{Min}(x_1, x_2, \ldots, x_n) \leq\ \mathrm{HM}(x_1, x_2, \ldots, x_n) \leq \mathrm{GM}(x_1, x_2, \ldots, x_n) \leq\ \mathrm{AM}(x_1, x_2, \ldots, x_n) \leq\ \mathrm{Max}(x_1, x_2, \ldots, x_n) \leq A_s\ (x_1, x_2, \ldots, x_n)$.

The algebraic and analytical properties of aggregation functions [1–4,6,7] are described as follows:

Definition 2. *An aggregation function* $A : \mathbb{I}^{\,n} \to \mathbb{I}$ *is called*

- *having a neutral element* $e \in \mathbb{I}$, *if for* $i = 1, 2, \ldots, n$, *we have* $A(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_n) = A\ (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.
- *having an annihilator element* $a \in \mathbb{I}$, *if for* $i = 1, 2, \ldots, n$, *we have* $A\ (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) =\ a$.
- *additive, if for any* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y} \in \mathbb{I}^n$, *we have* $A(\boldsymbol{x} + \boldsymbol{y}) = A(\boldsymbol{x}) + A(\boldsymbol{y})$.
- *associative, if for all* $(x_1, x_2, x_3) \in \mathbb{I}^3$, *we have* $A\ (A\ (x_1, x_2), x_3) = A(x_1, A(x_2, x_3))$.
- *idempotent, if for all* $x \in \mathbb{I}$, *we have* $A(x, x, \ldots, x) = x$.
- *symmetric, if for all* $(x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ *and for any permutation* σ *of {1, 2, ... ,n}, we have* $A(x_1, x_2, \ldots, x_n) = A\Big(x_{\sigma\ (1)}, x_{\sigma\ (2)}, \ldots, x_{\sigma\ (n)}\Big)$.
- *bisymmetric, if for all* $x_{ij} \in \mathbb{I}, i, j \in \{1, 2, \ldots, n\}$, *we have* $A(A(x_{11}, x_{12}, \ldots, x_{1n}), \ldots, A(x_{n1}, x_{n2}, \ldots, x_{nn})) = A(A(x_{11}, x_{21}, \ldots, x_{n1}), \ldots, A(x_{1n}, x_{2n}, \ldots, x_{nn}))$.
- *continuous,* $\forall \varepsilon > 0, \exists \delta > 0$, *if* $|x_i - y_i| < \delta$ *for* $i \in \{1, 2, \ldots, n\}$, *then* $|A(x_1, x_2, \ldots, x_n) - A(y_1, y_2, \ldots, y_n)| < \varepsilon$.
- *c-Lipschitzian with respect to the norm* $\|.\|$, *if for some constant* $c \in (0, +\infty)$, *we have the Lipschitz condition* $|A(x_1, x_2, \ldots, x_n) - A(y_1, y_2, \ldots, y_n)| \leq c\|(x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)\|$ *for all* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{I}^n$ *and* $\|.\| : \mathcal{R}^{\,n} \to [0, +\infty)$.

The Minkowski norm of order $p \in [1, \infty)$, L_p-norm, defined by

$$\|x\|_p = \left(\sum_{i\,=\,1}^{n}|x_i|^{\,p}\right)^{1/p}$$

is a well-known norm. When $p = \infty$,

$$\|x\|_\infty =\ \max_{i=1}^{n}|x_i|$$

is called the Chebyshev norm. Since

$$\max_{i=1}^{n}|x_i - y_i| \leq \left(\sum_{i=1}^{n}|x_i\ -\ y_i|^{p+1}\right)^{\frac{1}{p+1}} \leq \left(\sum_{i\,=\,1}^{n}|x_i\ -\ y_i|^{p}\right)^{1/p}$$

for all $x_i, y_i \in \mathbb{I}$, $i \in \{1, 2, \ldots, n\}$ and $p \in [1, \infty)$, it follows that each d-Lipschitzian with respect to L_∞-norm implies d-Lipschitzian with respect to L_p-norm, $p \in [1, \infty)$. Additionally, each d-Lipschitzian with respect to L_{p+1}-norm implies d-Lipschitzian with respect to L_p-norm, $p \in [1, \infty)$.

The best Lipschitz constant is the greatest lower bound d of b such that $A\ (x_1, x_2, \ldots, x_n)$ is d-Lipschitzian but $A\ (x_1, x_2, \ldots, x_n)$ is not b–Lipschitzian for any $b \in (0, d)$. Two types of best Lipschitz constant are considered: theoretical best Lipschitz constant and empirical best Lipschitz constant. Theoretical best Lipschitz constant is obtained analytically by the triangular inequality and the Hölder inequality [1,3,4]. Empirical best Lipschitz constant is obtained by finding the maximum value of

$$\frac{|A\ (x_1, x_2, \ldots, x_n) - A\ (y_1, y_2, \ldots, y_n)|}{\|(x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)\|}$$

for $(x_1, x_2, \ldots, x_n) \neq (y_1, y_2, \ldots, y_n)$, $x_i, y_i \leq 1$ and $x_i, y_i \geq 0$, $i \in \{1, 2, \ldots, n\}$. For an aggregation function $A(x_1, x_2, \ldots, x_n)$, the mathematical programming model of the empirical best Lipschitz constant with respect to L_p-norm is

$$\begin{aligned} \text{Maximize } & \frac{|A(x_1,x_2,\ldots,x_n)-A(y_1,y_2,\ldots,y_n)|}{\left(\sum_{i=1}^{n}|x_i-y_i|^{\ p}\right)^{1/p}} \\ \text{subject to } & \left(\sum_{i=1}^{n}|x_i-y_i|^{\ p}\right)^{1/p} > 0 \\ & x_i, y_i \leq 1, i \in \{1, 2, \ldots, n\} \\ & x_i, y_i \geq 0, i \in \{1, 2, \ldots, n\} \end{aligned} \tag{1}$$

The feasible region of constraints for the mathematical programming model (1) is not compact. The denominator of the objective function is required to be greater than a small positive number ε. Following the data envelopment analysis, this small number ε is called a non-Archimedean number [18]. The mathematical programming model (1) becomes

$$\begin{aligned} \text{Maximize } & \frac{|A(x_1,x_2,\ldots,x_n)-A(y_1,y_2,\ldots,y_n)|}{\left(\sum_{i=1}^{n}|x_i-y_i|^{\ p}\right)^{1/p}} \\ \text{subject to } & \left(\sum_{i=1}^{n}|x_i-y_i|^{\ p}\right)^{1/p} \geq \varepsilon \\ & x_i, y_i \leq 1, i \in \{1, 2, \ldots, n\} \\ & x_i, y_i \geq 0, i \in \{1, 2, \ldots, n\} \end{aligned} \tag{2}$$

It follows that the largest objective function of the mathematical programming model (2) is the empirical best Lipschitz constant. If $\left(\sum_{i\ =\ 1}^{n}|x_i-y_i|^{\ p}\right)^{1/p} \geq \varepsilon$ is a binding constraint, the value of objective function is dependent of the non-Archimedean number ε. Since the empirical best Lipschitz constants are the actual best Lipschitz constants, the analytical behaviors of the aggregation function can be analyzed by the behaviors of the empirical best Lipschitz constants.

The following definition establishes that a non-idempotent aggregation function can be transformed into an idempotent one [1,3,4].

Definition 3. *Let $A : \mathbb{I}^{\,n} \to \mathbb{I}$ be an aggregation function such that $\delta_A\ (x) = A\ (x,\ x, \ldots,\ x)$ is strictly increasing and*

$$\{\delta_A\ (x) | x \in \mathbb{I}\} = \{A(x_1, x_2, \ldots, x_n) | x_i \in \mathbb{I}, i = 1, \ldots, n\},$$

then the idempotent aggregation function is given by $AI\ (x_1, x_2, \ldots, x_n) = \delta_A^{-1}\ (A\ (x_1, x_2, \ldots, x_n))$, which is called idempotized A.

To characterize the mean $M : \mathbb{I}^{\,\mathrm{n}} \to \mathbb{I}$, the first one is Cauchy's internality property [9]. A mean M is an internal function, i.e., $\mathrm{Min}(x_1, x_2, \ldots, x_n) \leq M(x_1, x_2, \ldots, x_n) \leq \mathrm{Max}(x_1, x_2, \ldots, x_n)$. The second is the Chisini′s equation. A mean M with respect to the function $F : \mathbb{I}^{\,n} \to \mathbb{I}$ *is a number M such that*

$$F(M, M, \ldots, M) = F(x_1, x_2, \ldots, x_n).$$

The Chisini′s equation can be rewritten as

$$\delta_F(M) = F(x_1, x_2, \ldots, x_n).$$

Under some constraints, the mean that is obtained from the solution of Chisini's equation can fulfill Cauchy's internality property [1,3,4], described as follows:

Definition 4. *A function* $\mathrm{M} : \mathbb{I}^{\,n} \to \mathbb{I}$ *is an average associated with* F *in* $\mathbb{I}^{\,\mathrm{n}}$ *if there exists a nondecreasing and idempotizable function* $F : \mathbb{I}^{\,n} \to \Re$ *satisfying* $\delta_F(M) = F$.

From Definitions 3 and 4, it is implied that M is the idempotized F if and only if M is an average associated with F. When F is considered as the sum and the product, the idempotized F is the arithmetic mean and the geometric mean, respectively. The following sections will analyze and compare the optimal solutions and the best Lipschitz constants between an aggregation function and associated idempotized aggregation function.

3. The Best Lipschitz Constants of the Sum and Arithmetic Mean Functions

This section deals with the sum function $\sum(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i$ and the arithmetic mean $\mathrm{AM}(x_1, x_2, \ldots, x_n)$. The arithmetic mean is the idempotized sum function. Additionally, the arithmetic mean is an average associated with the sum function. The domain of the arithmetic mean is [0, 1], so the domain of the sum function is $[0, \infty)$. The arithmetic mean is an aggregation function with minimal Lipschitz constant with respect to L_1-norm, we will show related results for the other L_p-norms. It is evident that the sum function satisfies additive, associative, symmetric, bisymmetric, continuous, and Lipschitzian but non-idempotent. The sum function has neutral element $e = 0$ but no annihilator element [1,3,4].

A variant of the sum function is the bounded sum $\sum_L(x_1, x_2, \ldots, x_n) = \min\left(\sum_{i=1}^{n} x_i, 1\right)$. The bounded sum preserves some properties of the original sum function, such as the associativity, symmetry, bisymmetry, continuity, Lipschitzian, non-idempotency and neutral element $e = 0$. Two different properties exist between $\sum(x_1, x_2, \ldots, x_n)$ and $\sum_L(x_1, x_2, \ldots, x_n)$. The sum function possesses additivity and no annihilator element, while the bounded sum function dissatisfies additive and has annihilator element $a = 1$.

We now present the optimal solutions and the empirical best Lipschitz constant of an aggregation function empirically. This paper conducts some computational experiments to empirically study the influence of the number of variables, the Minkowski norm, the number of steps, and the type of aggregation function on the optimal inputs and the empirical best Lipschitz constant performance.

The first numerical experiment is conducted to find the forms of optimal solutions $\boldsymbol{x}$ and $\boldsymbol{y}$, and the empirical best Lipschitz constant for $\sum(x_1, x_2, \ldots, x_n)$. For L_p-norm, the mathematical programming model is

$$\begin{aligned} \text{Maximize } & \frac{|\sum(x_1,x_2,\ldots,x_n) - \sum(y_1,y_2,\ldots,y_n)|}{\left(\sum_{i=1}^{n}|x_i - y_i|^{\,p}\right)^{1/p}} \\ \text{subject to } & \left(\sum_{i=1}^{n}|x_i - y_i|^{\,p}\right)^{1/p} \geq \varepsilon \\ & x_i, y_i \leq 1, i \in \{1, 2, \ldots, n\} \\ & x_i, y_i \geq 0, i \in \{1, 2, \ldots, n\}. \end{aligned} \tag{3}$$

Since the sum function is a symmetric one, without loss of generality, let $\varepsilon = 1/m$, $m \in \{1000, 10{,}000, 100{,}000\}$, the mathematical programming model (3) becomes

$$\begin{aligned} \text{Maximize } & \frac{|\sum(x_1,x_2,\ldots,x_n) - \sum(y_1,y_2,\ldots,y_n)|}{\left(\sum_{i=1}^{n}|x_i - y_i|^{\,p}\right)^{1/p}} \\ \text{subject to } & y_k \geq x_k + \varepsilon, \text{for some } k \in \{1, 2, \ldots, n\} \\ & x_i, y_i \leq 1, i \in \{1, 2, \ldots, n\} \\ & x_i, y_i \geq 0, i \in \{1, 2, \ldots, n\} \end{aligned} \tag{4}$$

Let $m = 1/\varepsilon$ be the number of steps, the discrete approximation of the mathematical programming model (4) is

$$\begin{aligned} &\text{Maximize } \frac{|\sum(x_1,x_2,\ldots,x_n) - \sum(y_1,y_2,\ldots,y_n)|}{\left(\sum_{i=1}^{n} |x_i - y_i|^{\,p}\right)^{1/p}} \\ &\text{subject to } x_i, y_i \in \left\{0, \tfrac{1}{m}, \tfrac{2}{m}, \ldots, 1\right\}, i \in \{1,2,\ldots,n\} \\ &(x_1,x_2,\ldots,x_n) \neq (y_1,y_2,\ldots,y_n) \end{aligned} \tag{5}$$

For the number of variables $n \in \{2,3\}$, L_p-norm, $p \in \{1,2,3,\infty\}$ and the number of steps $m \in \{1000, 10{,}000, 100{,}000\}$, we perform an exhaustive search for all $x_i, y_i \in \left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\right\}$, $i \in \{1,2,\ldots,n\}$ and $(x_1,x_2,\ldots,x_n) \neq (y_1,y_2,\ldots,y_n)$ with the objective function

$$\text{Maximize } \frac{|\sum(x_1,x_2,\ldots,x_n) - \sum(y_1,y_2,\ldots,y_n)|}{\left(\sum_{i\,=\,1}^{n} |x_i \;-\; y_i|^{\,p}\right)^{1/p}}$$

For the two-variable $\sum(x_1,x_2)$ and L_1-norm, the optimal value for the objective function (5) is

$$\frac{|x_1 + x_2 - (x_1 \pm \alpha_1 + x_2 \pm \alpha_2)|}{\alpha_1 + \alpha_2} = 1$$

and is attained at the multiple solutions $\boldsymbol{x} = (x_1,x_2)$ and $\boldsymbol{y} = (x_1 \pm \alpha_1, x_2 \pm \alpha_2)$, $x_i, \alpha_i, x_i \pm \alpha_i \in [0,1]$, $i = 1,2$. For L_p-norm, $p \in \{2,3,\infty\}$, the multiple optimal solutions $\boldsymbol{x} = (x_1,x_2)$ and $\boldsymbol{y} = (x_1 \pm \alpha, x_2 \pm \alpha)$, $x_i, \alpha, x_i \pm \alpha \in [0,1]$, $i = 1,2$ yield the largest objective function

$$\frac{|x_1 + x_2 - (x_1 \pm \;\alpha\; + x_2 \pm \;\alpha)|}{(\alpha^p \;+\; \alpha^p)^{\,1/p}} = 2^{\,1-1/p}.$$

These optimal solutions are verified by applying the popular modelling language LINGO [19], which utilizes the power of linear and nonlinear optimization to solve mathematical problems (4). When the Chebyshev norm L_∞, the empirical best Lipschitz constant becomes 2. The empirical best Lipschitz constant $2^{\,1-1/p}$ will increase as the order p increases.

For the three-variable $\sum(x_1,x_2,x_3)$, the multiple optimal solutions are $\boldsymbol{x} = (x_1,x_2,x_3)$ and $\boldsymbol{y} = (x_1 \pm \alpha_1, x_2 \pm \alpha_2, x_3 \pm \alpha_3)$, $x_i, \alpha_i, x_i \pm \alpha_i \in [0,1]$, $i = 1,2,3$ and $\boldsymbol{x} = (x_1,x_2,x_3)$ and $\boldsymbol{y} = (x_1 \pm \alpha, x_2 \pm \alpha, x_3 \pm \alpha)$, $x_i, \alpha, x_i \pm \alpha \in [0,1]$, $i = 1,2,3$ with the associated empirical best Lipschitz constant 1 and $3^{1-1/p}$ for $p = 1$ and $p \in \{2,3,\infty\}$, respectively. These optimal solutions are verified by applying LINGO with $\varepsilon = 1/m$, $m \in \{1000, 10{,}000, 100{,}000\}$. If $p = \infty$, we find the empirical best Lipschitz constant 3.

Theoretically, applying the triangular inequality and the Hölder inequality, the result of a more general n-ary sum function $\sum(x_1,x_2,\ldots,x_n)$ is described as follows.

Theorem 1. *For the sum function $\sum(x_1,x_2,\ldots,x_n)$, the theoretical best Lipschitz constant is $n^{1-1/p}$ and n for $p \in [1,\infty)$ and $p = \infty$, respectively. The associated optimal solutions are $\boldsymbol{x} = (x_1,x_2,\ldots,x_n)$ and $\boldsymbol{y} = (x_1 \pm \alpha_1, x_2 \pm \alpha_2, \ldots, x_n \pm \alpha_n)$, $x_i, \alpha_i, x_i \pm \alpha_i \in [0,1]$, $i = 1,2,\ldots,n$ and $\boldsymbol{x} = (x_1,x_2,\ldots,x_n)$ and $\boldsymbol{y} = (x_1 \pm \alpha, x_2 \pm \alpha, \ldots, x_n \pm \alpha)$, $x_i, \alpha, x_i \pm \alpha \in [0,1]$, $i = 1,2,\ldots,n$ for $p = 1$ and $p \in [2,\infty]$,* respectively.

Proof of Theorem 1. From the triangular inequality, we have

$$\left|\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i\right| = \left|\sum_{i=1}^{n} (x_i - y_i)\right| \leq \|\boldsymbol{x} - \boldsymbol{y}\|_1$$

for $x_i, y_i \in [0,1]$, $i = 1,2,\ldots,n$ [2,3,5]. From the Hölder inequality, for $\frac{1}{p} + \frac{1}{q} = 1$, $p,q \in [1,\infty)$, we obtain

$$\left|\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \boldsymbol{y}_i\right| \leq \|\boldsymbol{x} - y\|_1 \leq (1,1,\ldots,1)_q \times x - y_1 = n^{\,1-1/p}\|\boldsymbol{x} - \boldsymbol{y}\|_p$$

It follows that the theoretical best Lipschitz constant is $n^{\,1-1/p}$. The theoretical best Lipschitz constants of the solutions $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (x_1 \pm \alpha_1, x_2 \pm \alpha_2, \ldots, x_n \pm \alpha_n)$, $x_i, \alpha_i, x_i \pm \alpha_i \in [0,1]$, $i = 1, 2, \ldots, n$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (x_1 \pm \alpha, x_2 \pm \alpha, \ldots, x_n \pm \alpha)$, $x_i, \alpha, x_i \pm \alpha \in [0,1]$, $i = 1, 2, \ldots, n$, are 1 and $n^{1-1/p}$ for $p = 1$ and $p \in [2, \infty]$, respectively. Therefore, these solutions are the optimal ones. □

From Theorem 1, it is implied that the theoretical best Lipschitz constants are the same as those of the empirical best Lipschitz constants. Therefore, the theoretical and empirical best Lipschitz constant of the sum function is $n^{\,1-1/p}$. The best Lipschitz constant $n^{1-1/p}$ increases with increases in either the order p, or the number of variables n. Moreover, our numerical experiment indicates that the optimal solutions are multiple and the theoretical best Lipschitz constants are attainable.

According to the experiment we perform on a bounded sum function, the empirical best Lipschitz constants and associated optimal solutions $\boldsymbol{x}$ and $\boldsymbol{y}$ of the sum function and those of the bounded sum function coincide.

For the arithmetic mean $\mathrm{AM}(x_1, x_2, \ldots, x_n)$, it is evident that AM fulfills additive, idempotent, symmetric, bisymmetric, continuous, and Lipschitzian, but non-associative and has no neutral element and no annihilator element.

We now present the optimal values of $\boldsymbol{x}$ and $\boldsymbol{y}$ and the empirical best Lipschitz constant of $\mathrm{AM}(x_1, x_2, \ldots, x_n)$. Since

$$\text{Maximize } \frac{|\mathrm{AM}\,(x_1, x_2, \ldots, x_n) - \mathrm{AM}\,(y_1, y_2, \ldots, y_n)|}{\left(\sum_{i=1}^{n} |x_i - y_i|^{\,p}\right)^{1/p}} = \text{Maximize } \frac{1}{n} \frac{|\sum(x_1, x_2, \ldots, x_n) - \sum(y_1, y_2, \ldots, y_n)|}{\left(\sum_{i=1}^{n} |x_i - y_i|^{\,p}\right)^{1/p}} \tag{6}$$

for $(x_1, x_2, \ldots, x_n) \neq (y_1, y_2, \ldots, y_n)$, $x_i, y_i \leq 1$ and $x_i, y_i \geq 0$, $i \in \{1, 2, \ldots, n\}$. The result of AM $(x_1, x_2, \ldots, x_n)$ are directly linked to related results of the sum function described as follows.

For the $\mathrm{AM}(x_1, x_2, \ldots, x_n)$, the theoretical best Lipschitz constant is $n^{-1/p}$ and 1 for $p \in [1, \infty)$ and $p = \infty$, respectively. The associated multiple optimal solutions are $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (x_1 \pm \alpha_1, x_2 \pm \alpha_2, \ldots, x_n \pm \alpha_n)$, $x_i, \alpha_i, x_i \pm \alpha_i \in [0,1]$, $i \in \{1, 2, \ldots, n\}$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (x_1 \pm \alpha, x_2 \pm \alpha, \ldots, x_n \pm \alpha)$, $x_i, \alpha, x_i \pm \alpha \in [0,1]$, $i \in \{1, 2, \ldots, n\}$ for $p = 1$ and $p \in [2, \infty]$, respectively.

It implies that the theoretical best Lipschitz constants, which are the same as those of the empirical best Lipschitz constants. Therefore, the theoretical and empirical best Lipschitz constant is $n^{-1/p}$. The best Lipschitz constant $n^{-1/p}$ increases for either the number of variables n increasing or the order p increasing. Moreover, the optimal solutions are multiple and the theoretical best Lipschitz constants are attainable.

We compare the algebraic and analytical properties of $\sum(x_1, x_2, \ldots, x_n)$ and $\mathrm{AM}(x_1, x_2, \ldots, x_n)$ head to head. The differences of both kinds of aggregation functions exist among the idempotency, associativity, and neutral element. The sum function satisfies associative and non-idempotent, and has neutral element $e = 0$. While the arithmetic mean satisfies non-associative and idempotent and has no neutral element. For the sum and arithmetic mean functions, the associated multiple optimal solutions of the empirical best Lipschitz constants are identical and are $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (x_1 \pm \alpha_1, x_2 \pm \alpha_2, \ldots, x_n \pm \alpha_n)$, $x_i, \alpha_i, x_i \pm \alpha_i \in [0,1]$, $i \in \{1, 2, \ldots, n\}$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (x_1 \pm \alpha, x_2 \pm \alpha, \ldots, x_n \pm \alpha)$, $x_i, \alpha, x_i \pm \alpha \in [0,1]$, $i \in \{1, 2, \ldots, n\}$ for $p = 1$ and $p \in [2, \infty]$, respectively. For L_p-norm, $p \in [1, \infty]$, the empirical best Lipschitz constant is $n^{1-1/p}$ for the sum function and $n^{\,-1/p}$ for the arithmetic mean, which are the same as those of analytical method. The ratio of the best Lipschitz constant of the sum to that of the arithmetic mean is n, which is independent of p. Moreover, our numerical experiments indicate that the optimal solutions are multiple, and the theoretical best Lipschitz constants are attainable. The multiple optimal solutions can be expected, since $\sum(x_1, x_2, \ldots, x_n)$ and AM $(x_1, x_2, \ldots, x_n)$ satisfy symmetry. More precisely, if $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ is an optimal solution, then $\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right)$ and $\left(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}\right)$ is also an optimal solution for any permutation σ of {1, 2, … ,n}. The $\mathrm{AM}(x_1, x_2, \ldots, x_n)$ is a kernel aggregation function, which is a maximally stable aggregation function with respect to possible input errors [20]. The theoretical best Lipschitz constants of $\mathrm{AM}(x_1, x_2, \ldots, x_n)$ and associated $\sum(x_1, x_2, \ldots, x_n)$ are attainable.

4. The Best Lipschitz Constants of the Product and Geometric Mean Functions

This section is devoted to the product function $\prod(x_1, x_2, \ldots, x_n)$ and the geometric mean GM $(x_1, x_2, \ldots, x_n)$. The geometric mean is the idempotized product function. Additionally, the geometric mean is an average associated with the product function. The domains of the product function $\prod(x_1, x_2, \ldots, x_n)$ and the geometric mean GM $(x_1, x_2, \ldots, x_n)$ are $[0,1]^n$. Evidently, the product function satisfies associative, symmetric, bisymmetric, continuous and Lipschitzian, but non-additive and non-idempotent. The product function has neutral element $e = 1$ and annihilator element $a = 0$.

The second experiment is concerned with an exhaustive search for a product function $\prod(x_1, x_2, \ldots, x_n)$, with the objective of maximizing the empirical Lipschitz constant performance. For the number of variables $n \in \{2, 3\}$, L_p-norm, $p \in \{1, 2, 3, \infty\}$ and the number of steps $m \in \{1000, 10{,}000, 100{,}000\}$, we perform an exhaustive search for all $x_i, y_i \in \left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\right\}$, $i \in \{1, 2, \ldots, n\}$ and $(x_1, x_2, \ldots, x_n) \neq (y_1, y_2, \ldots, y_n)$ to find the optimal value of the objective function

$$\text{Maximize } \frac{|\prod(x_1, x_2, \ldots, x_n) - \prod(y_1, y_2, \ldots, y_n)|}{\left(\sum_{i=1}^{n} |x_i - y_i|^{p}\right)^{1/p}} \tag{7}$$

Consider the two-variable programming problem. For L_1-norm, the optimal value of the objective function (7) is

$$\frac{|x_1 - y_1|}{|x_1 - y_1|} = 1$$

and the associated multiple optimal solutions are $\boldsymbol{x} = (x_1, 1)$ and $\boldsymbol{y} = (y_1, 1)$, $x_1, y_1 \in [0, 1]$. For L_p-norm, $p \in \{2, 3, \infty\}$, the unique optimal solution $\boldsymbol{x} = \left(1 - \frac{1}{m}, 1 - \frac{1}{m}\right)$ and $\boldsymbol{y} = (1, 1)$, $m \in \{1000, 10{,}000, 100{,}000\}$, yields the largest objective function

$$\frac{\left|\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m}\right) - 1\right|}{\left(\left(\frac{1}{m}\right)^{p} + \left(\frac{1}{m}\right)^{p}\right)^{1/p}} = 2^{\,1-1/p} - \frac{2^{-1/p}}{m}.$$

These optimal solutions are verified by adopting LINGO with $\varepsilon = 1/m$, $m \in \{1000, 10{,}000, 100{,}000\}$. The limit of the largest objective function is equal to $2^{\,1-1/p}$ as the number of steps m approaches ∞. Since $2^{\,1-1/p} - \frac{2^{-\frac{1}{p}}}{m} < 2^{\,1-1/p}$, $m \in \mathcal{N}$, the limit of the empirical best Lipschitz constant $2^{\,1-1/p}$ is unattainable. The value of $2^{\,1-1/p}$ grows with increases in p. When $p = 1$, the limit value 1 is the same as that of L_1-norm. Furthermore, if $p = \infty$, the limit of the empirical best Lipschitz constant becomes 2.

For the three-variable product function $\prod(x_1, x_2, x_3)$, we get the multiple optimal solutions $\boldsymbol{x} = (x_1, 1, 1)$, $\boldsymbol{y} = (y_1, 1, 1)$, $x_1, y_1 \in [0, 1]$ and the unique optimal solution $\boldsymbol{x} = \left(1 - \frac{1}{m}, 1 - \frac{1}{m}, 1 - \frac{1}{m}\right)$, $\boldsymbol{y} = (1, 1, 1)$, $m \in \{1000, 10{,}000, 100{,}000\}$ with the associated empirical best Lipschitz constant 1 and $3^{\,1-1/p} - \frac{3^{\,1-\frac{1}{p}}}{m} + \frac{3^{-1/p}}{m^{\,2}}$ for $p = 1$ and $p \in \{2, 3, \infty\}$, respectively. These optimal solutions are verified by adopting LINGO with $\varepsilon = 1/m$, $m \in \{1000, 10{,}000, 100{,}000\}$. For $p \in \{2, 3, \infty\}$, we can make the empirical best Lipschitz constant as close to $3^{\,1-1/p}$ as we please, provided we choose m sufficiently close to ∞. When $p = 1$, the limit of the empirical best Lipschitz constant becomes 1, which is the same as that of L_1-norm. Furthermore, if $p = \infty$, the limit of the empirical best Lipschitz constant is 3.

By induction on n, the empirical best Lipschitz constant is 1 and $n^{\,1-1/p} + n^{\,-1/p} \sum_{i=2}^{n} C_i^n \left(-\frac{1}{m}\right)^{i-1}$ for $p = 1$ and $p \in (1, \infty]$, respectively. The associated optimal solutions are

$\boldsymbol{x} = (x_1, 1, \ldots, 1)$ and $\boldsymbol{y} = (y_1, 1, \ldots, 1)$, $x_1, y_1 \in [0,1]$ and $\boldsymbol{x} = \left(1 - \frac{1}{m}, 1 - \frac{1}{m}, \ldots, 1 - \frac{1}{m}\right)$ and $\boldsymbol{y} = (1, 1, \ldots, 1)$, $m \in \mathcal{N}$ for $p = 1$ and $p \in (1, \infty]$, respectively. The empirical best Lipschitz constant of $p \in (1, \infty]$ can be made close to $n^{1-1/p}$ by taking m sufficiently close to ∞.

Theoretically, the result of a more general n-ary product function $\prod(x_1, x_2, \ldots, x_n)$ is presented as follows.

Theorem 2. *For an n-ary product function* $\prod(x_1, x_2, \ldots, x_n)$*, the theoretical best Lipschitz constant is* $n^{\,1-1/p}$ *for* $p \in [1, \infty]$.

Proof of Theorem 2. From the triangular inequality, the Hölder inequality and $x_i, y_i \in [0,1]$, $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
|x_1x_2\ldots x_n - y_1y_2\ldots y_n| &= |x_n(x_1x_2\ldots x_{n-1} - y_1y_2\ldots y_{n-1}) + y_1y_2\ldots y_{n-1}(x_n - y_n)| \\
&\le |x_1x_2\ldots x_{n-1} - y_1y_2\ldots y_{n-1}| + |x_n - y_n| \\
&= |x_{n-1}(x_1x_2\ldots x_{n-2} - y_1y_2\ldots y_{n-2}) + y_1y_2\ldots y_{n-2}(x_{n-1} - y_{n-1})| + |x_n - y_n| \\
&\le |x_1x_2\ldots x_{n-2} - y_1y_2\ldots y_{n-2}| + |x_{n-1} - y_{n-1}| + |x_n - y_n| \\
&\le \|\boldsymbol{x} - \boldsymbol{y}\|_1 \le n^{\,1-1/p}\|\boldsymbol{x} - \boldsymbol{y}\|_p.
\end{aligned}$$

It follows that the theoretical best Lipschitz constant is $n^{\,1-1/p}$. □

From Theorem 2, the theoretical best Lipschitz constant $n^{\,1-1/p}$ increases for either the number of variables n increasing or the order p increasing. The theoretical best Lipschitz constant coincides with the limit of the empirical best Lipschitz constant. However, our numerical experiment indicates that the theoretical best Lipschitz constant $n^{\,1-1/p}$, $p \in (1, \infty]$, is unattainable because $n^{\,1-1/p} + n^{-1/p}\sum_{i=2}^{n} C_i^n\left(-\frac{1}{m}\right)^{i-1} < n^{\,1-1/p}$ for all $m \in \mathcal{N}$. Therefore, the actual best Lipschitz constant of the product function is 1 and $n^{\,1-1/p} + n^{-1/p}\sum_{i=2}^{n} C_i^n\left(-\frac{1}{m}\right)^{i-1}$ for $p = 1$ and $p \in (1, \infty]$, respectively.

The geometric mean GM $(x_1, x_2, \ldots, x_n)$ satisfies idempotent, symmetric, bisymmetric, and continuous, but non-associative, non-additive, and non-Lipschitzian. The geometric function has annihilator element $a = 0$ but no neutral element.

The third computational experiment is empirically studying the empirical best Lipschitz constant of GM $(x_1, x_2, \ldots, x_n)$. For the number of variables $n \in \{2, 3\}$, L_p-norm, $p \in \{1, 2, 3, \infty\}$, and the number of steps $m \in \{1000, 10{,}000, 100{,}000\}$, we perform an exhaustive search for all $x_i, y_i \in \left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\right\}$, $i \in \{1, 2, \ldots, n\}$, and $(x_1, x_2, \ldots, x_n) \neq (y_1, y_2, \ldots, y_n)$ with the objective of maximizing the Lipschitz constant

$$\text{Maximize } \frac{|\mathrm{GM}(x_1, x_2, \ldots, x_n) - \mathrm{GM}(y_1, y_2, \ldots, y_n)|}{\left(\sum_{i=1}^{n}|x_i - y_i|^p\right)^{1/p}} \tag{8}$$

For the two-variable GM (x_1, x_2), the optimal value for the objective function (8) is $\sqrt{m}$. For L_p-norm, $p \in \{1, 2, 3\}$, the associated unique optimal solution is $\boldsymbol{x} = (0, 1)$ and $\boldsymbol{y} = \left(\frac{1}{m}, 1\right)$, $m \in \{1000, 10{,}000, 100{,}000\}$. For L_∞-norm, the associated multiple optimal solutions are $\boldsymbol{x} = (0, 1)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1\right)$ and $\boldsymbol{x} = \left(0, 1 - \frac{1}{m}\right)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1\right)$, $m \in \{1000, 10{,}000, 100{,}000\}$. These optimal solutions are verified by applying LINGO with $\varepsilon = 1/m$, $m \in \{1000, 10{,}000, 100{,}000\}$. The empirical best Lipschitz constant $\sqrt{m}$ tends to infinity as m takes on arbitrarily large positive value. Therefore, the two-variable $\mathrm{GM}(x_1, x_2)$ does not satisfy the Lipschitz condition.

For the three-variable $\mathrm{GM}(x_1, x_2, x_3)$, the unique optimal solution $\boldsymbol{x} = (0, 1, 1)$ and $\boldsymbol{y} = \left(\frac{1}{m}, 1, 1\right)$, $m \in \{1000, 10{,}000, 100{,}000\}$, has the largest Lipschitz constant $m^{\,2/3}$ for

L_p-norm, $p \in \{1,2,3\}$. For L_∞-norm, the multiple optimal solutions are $\boldsymbol{x} = (0,1,\ 1)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1,\ 1\right)$, and $\boldsymbol{x} = \left(0, 1-\frac{1}{m}, 1-\frac{1}{m}\right)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1,\ 1\right)$, $m \in \{1000, 10{,}000, 100{,}000\}$ with associated empirical best Lipschitz constant $m^{\ 2/3}$. These optimal solutions are verified by applying LINGO with $\varepsilon = 1/m$, $m \in \{1000, 10{,}000, 100{,}000\}$. For $p \in \{1,2,3,\infty\}$, we can make the empirical best Lipschitz constant $m^{\ 2/3}$ as close to ∞ as we please, provided we choose m sufficiently close to ∞. It implies that the three-variable GM (x_1, x_2, x_3) does not fulfill the Lipschitz condition.

By induction on n, the empirical best Lipschitz constant of an n-ary geometric mean function GM $(x_1, x_2, \ldots, x_n)$ is $m^{\ 1-1/n}$ for all $p \in [1,\infty]$. The associated optimal solutions are $\boldsymbol{x} = (0,1,\ldots,\ 1)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1, \ldots,\ 1\right)$, $m \in \mathcal{N}$ for $p \in [1,\infty]$ and $\boldsymbol{x} = (0,1,\ldots,\ 1)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1, \ldots, 1\right)$ and $\boldsymbol{x} = \left(0, 1-\frac{1}{m}, \ldots, 1-\frac{1}{m}\right)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1, \ldots,\ 1\right)$, $m \in \mathcal{N}$ for $p = \infty$. For $n \geq 2$ and L_p-norm $p \in [1,\infty]$, the best Lipschitz constant $m^{\ 1-1/n}$ approaches plus infinity as m approaches plus infinity. This can be expected since GM $(x_1, x_2, \ldots, x_n)$, $n \geq 2$, is not differentiable at $\boldsymbol{x} = (0,0,\ldots,\ 0)$. Additionally, GM $(x_1, x_2, \ldots, x_n)$, $n \geq 2$, is not uniformly continuous on $[0,1]^{\ n}$. Note that the best Lipschitz constant, $m^{\ 1-1/n}$ is independent of order p.

Comparing the algebraic and analytical properties of $\prod(x_1, x_2, \ldots, x_n)$ and GM $(x_1, x_2, \ldots, x_n)$, the differences of the adopted properties exist among the associativity, idempotency, Lipschitzian, and neutral element. The product function satisfies associative, non-idempotent, Lipschitzian, and has neutral element $e = 1$. While the geometric mean satisfies non-associative, idempotent, non-Lipschitzian, and has no neutral element. The associated optimal solutions of the product function and those of the geometric mean function are different. The associated optimal solutions of the product function are $\boldsymbol{x} = (x_1, 1, \ldots, 1)$, $\boldsymbol{y} = (y_1, 1, \ldots, 1)$, $x_1, y_1 \in [0,1]$ and $\boldsymbol{x} = \left(1-\frac{1}{m}, 1-\frac{1}{m}, \ldots, 1-\frac{1}{m}\right)$, $\boldsymbol{y} = (1,1,\ldots,1)$, $m \in \mathcal{N}$ for $p = 1$ and $p \in (1,\infty]$, respectively. The associated optimal solutions of the geometric mean function are $\boldsymbol{x} = (0,1,\ldots,\ 1)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1, \ldots,\ 1\right)$, $m \in \mathcal{N}$ for $p \in [1,\infty)$ and $\boldsymbol{x} = (0,1,\ldots,\ 1)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1, \ldots,\ 1\right)$, and $\boldsymbol{x} = \left(0, 1-\frac{1}{m}, \ldots, 1-\frac{1}{m}\right)$, $\boldsymbol{y} = \left(\frac{1}{m}, 1, \ldots,\ 1\right)$, $m \in \mathcal{N}$ for $p = \infty$. The empirical best Lipschitz constant of the product function is 1 and $n^{\ 1-1/p} + n^{-1/p} \sum_{i=2}^{n} C_i^n \left(-\frac{1}{m}\right)^{\ i-1}$ for $p = 1$ and $p \in (1,\infty]$, respectively. The empirical best Lipschitz constant of the geometric mean function is $m^{\ 1-1/n}$ for all $p \in [1,\infty]$. As m approaches infinity, the empirical best Lipschitz constant approaches $n^{\ 1-1/p}$ and infinity for the product function and the geometric mean function, respectively. Moreover, our numerical experiments indicate that the limits of the empirical best Lipschitz constants of the product and geometric mean functions are unattainable as m approaches infinity. The reason is because the non-kernel aggregation functions of the product and geometric mean functions. Moreover, the product function do not fulfill the Lipschitz condition.

5. Conclusions and Future Research

This paper analyzes and compares the optimal solutions and the theoretical and empirical best Lipschitz constants between an aggregation function and associated idempotized aggregation function. We conduct some computational experiments to empirically study the influence of the number of variables, the Minkowski norm, the number of steps and the type of aggregation function on the optimal solutions and the theoretical and empirical best Lipschitz constant performance.

For the sum function and the arithmetic mean, the differences of the adopted algebraic and analytical properties exist among the idempotency, associativity, and neutral element. Our numerical experiments indicate that for both sum and arithmetic mean functions, the associated optimal solutions are multiple and identical. For L_p-norm, $p \in [1,\infty]$, the theoretical and empirical best Lipschitz constant is $n^{\ 1-1/p}$ for the sum function and $n^{\ -1/p}$ for the arithmetic mean. These theoretical best Lipschitz constants are attainable. The ratio of the best Lipschitz constant of the sum to that of the arithmetic mean is n, which is independent of p.

For the product function and geometric mean, the differences exist among the associativity, idempotency, Lipschitzian, and neutral element. Our numerical experiments indicate that the associated optimal solutions of the product function and those of the geometric mean are different and dependent of m. The empirical best Lipschitz constant of the product function is 1 and $n^{1-1/p} + n^{-1/p} \sum_{i=2}^{n} C_i^n \left(-\frac{1}{m}\right)^{i-1}$ for $p = 1$ and $p \in (1, \infty]$, respectively. The empirical best Lipschitz constant of the geometric mean function is $m^{1-1/n}$ for all $p \in [1, \infty]$. As the number of steps m approaches ∞, the empirical best Lipschitz constant approaches $n^{1-1/p}$ and ∞ for the product function and the geometric mean function, respectively. These limits of the empirical best Lipschitz constants are unattainable.

For an aggregation function and associated idempotized aggregation function, the differences of the adopted algebraic and analytical properties always exist among the associativity, idempotency, and neutral element. However, the associated optimal solutions of the best Lipschitz constant for the product function and those of the geometric mean are different. It follows that the product function satisfies Lipschitzian, but the geometric mean do not. While the optimal solutions of the sum and arithmetic mean functions are multiple and identical. Both sum and arithmetic mean functions satisfy Lipschitzian and attain the theoretical best Lipschitz constant.

The results of this paper can be considered to apply in group decision making or two-sided decision making matching problems. For a group decision making problem, a group of experts are usually required to express preference information over a set of alternatives according to their knowledge and experience. By adopting the results of this paper, we aggregate the individual preference information to obtain collective preference information. Then, a solution is obtained. By considering the leadership and bounded confidence levels of experts, Zhang et al. [13] proposed a new consensus reaching algorithm for social network group decision making problems with interval fuzzy preference relations. For a two-sided matching decision making (TSMDM) problem, people aim to find an appropriate matching between two sets of objects, such as marriage matching, colleges admissions, person–job matching, and knowledge service matching. Due to the imprecise knowledge of matching objects and the different culture of decision makers, TSMDM problems with different preference structures are proposed. It is natural that matching objects will provide linguistic assessments. Aggregation of the linguistic assessments is the main process, especially for the multi-criteria TSMDM problems with multi-granular hesitant fuzzy linguistic term sets and incomplete criteria weight information [15].

In the future, we will analyze the best Lipschitz constants for all aggregation functions theoretically and empirically. In particular, the theoretical and empirical analysis can be extended to the conjunctive, the disjunctive, and the mixed aggregation functions. Thus, the Lipschitz analysis for the conjunctive, the disjunctive, and the mixed aggregation functions is a subject of considerable ongoing research.

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