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Relative Growth of Series in Systems of Functions and Laplace—Stieltjes-Type Integrals

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Abstract: For a regularly converging-in-$\mathbb{C}$ series $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$, where f is an entire transcendental function, the asymptotic behavior of the function $M_f^{-1}(M_A(r))$, where $M_f(r) = \max\{|f(z)| : |z| = r\}$, is investigated. It is proven that, under certain conditions on the functions f, α, and the coefficients a_n, the equality $\lim_{r\to+\infty} \frac{\alpha(M_f^{-1}(M_A(r)))}{\alpha(r)} = 1$ is correct. A similar result is obtained for the Laplace–Stiltjes-type integral $I(r) = \int_0^{\infty} a(x) f(rx) dF(x)$. Unresolved problems are formulated.

Keywords: relative growth; entire function; regularly converging series; Mittag–Leffler function

MSC: 30B50; 30D10; 30D20

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1. Introduction

Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \tag{1}$$

be an entire function, $M_f(r) = \max\{|f(z)| : |z| = r\}$, and $\Phi_f(r) = \ln M_f(r)$. For an entire function g with Taylor coefficients g_n, the study of growth of the function $\Phi_f^{-1}(\ln M_g(r))$ in terms of the exponential type was initiated in papers [1,2] and was continued in [3]. As a result, it is proven that, if $|f_{k-1}/f_k| \nearrow +\infty$ as $k \to \infty$, then

$$\varlimsup_{r\to+\infty} \frac{\Phi_f^{-1}(\ln M_g(r))}{r} = \varlimsup_{k\to\infty} \left(\frac{|g_n|}{|f_n|}\right)^{1/n}.$$

We remark that $\Phi_f^{-1}(x) = M_f^{-1}(e^x)$ and, thus, $\Phi_f^{-1}(\ln M_g(r)) = M_f^{-1}(M_g(r))$. The order $\rho[g]_g = \varlimsup_{r\to+\infty} \frac{\ln M_f^{-1}(M_g(r))}{\ln r}$ and the lower-order $\lambda[g]_f = \varliminf_{r\to+\infty} \frac{\ln M_f^{-1}(M_g(r))}{\ln r}$ of the function f with respect to the function g are used in Reference [4]. Research on the relative growth of entire functions was continued by many mathematicians (an incomplete bibliography is given in [5]).

Let (λ_n) be a sequence of positive numbers increasing to $+\infty$. Suppose that the series

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \tag{2}$$

in the system $f(\lambda_n z)$ is regularly convergent in $\mathbb{C}$, i.e., $\sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty$ for all $r \in [0, +\infty)$. Many authors have studied the representation of analytic functions by series in the system $f(\lambda_n z)$ and the growth of such functions. Here, we specify only the monographs of A.F. Leont'ev [6] and B.V. Vinnitskyi [3], which are references to other papers on this topic.

Since series (2) is regularly convergent in $\mathbb{C}$ and the function A is an entire function, a natural question arises about the asymptotic behavior of the function $M_f^{-1}(M_A(r))$.

We suppose that the function F is nonnegative, nondecreasing, unbounded, and continuous on the right on $[0,+\infty)$; that f is positive, increasing, and continuous on $[0,+\infty)$; and that a positive-on-$[0,+\infty)$ function a is such that the Laplace–Stietjes-type integral

$$I(r)=\int_0^{\infty} a(x)f(rx)dF(x) \tag{3}$$

exists for every $r\in[0,+\infty)$. The asymptotic behavior of such integrals in the case $f(x)=e^x$ is studied in the monograph [7]. A question arises again about the asymptotic behavior of the function $f^{-1}(I(r))$. Here, we present some results that indicate the possibility of solving these problems.

2. Relative Growth of Series in Systems of Functions

As in [8], by L, we denote a class of continuous nonnegative-on-$(-\infty,+\infty)$ functions α such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\leq x\to+\infty$. We say that $\alpha\in L^0$, if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_{si}$, if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for each $c\in(0,+\infty)$, i.e., α is a slowly increasing function. Clearly, $L_{si}\subset L^0$. We need the following lemma [9].

Lemma 1. *If $\beta\in L$ and $B(\delta)=\overline{\lim}_{x\to+\infty}\frac{\beta((1+\delta)x)}{\beta(x)}$, $\delta>0$, then in order for $\beta\in L^0$, it is necessary and sufficient that $B(\delta)\to 1$ as $\delta\to+0$.*

We need also some well-known (see, for example, [10]) properties of the function $\ln M_f(r)$.

Lemma 2. *If a function f is transcendental, then the function $\ln M_f(r)$ is logarithmically convex and, thus,*

$$\Gamma_f(r):=\frac{d\ln M_f(r)}{d\ln r}\nearrow+\infty,\quad r\to+\infty,$$

(at the points where the derivative does not exist, where $\frac{d\ln M_f(r)}{d\ln r}$ means the right-hand derivative).

For $\alpha\in L$, $\beta\in L$, and entire functions f and g, we define the generalized (α,β)-order $\rho_{\alpha,\beta}[g]_f$ and the generalized lower (α,β)-order $\lambda_{\alpha,\beta}[g]_f$ of g with respect to f as follows:

$$\rho_{\alpha,\beta}[g]_f=\overline{\lim_{r\to+\infty}}\frac{\alpha(M_f^{-1}(M_g(r)))}{\beta(r)},\quad \lambda_{\alpha,\beta}[g]_f=\underline{\lim_{r\to+\infty}}\frac{\alpha(M_f^{-1}(M_g(r)))}{\beta(r)}.$$

Suppose that $a_n\geq 0$ for all $n\geq 1$. Since

$$A(z)=\sum_{n=1}^{\infty}a_n\sum_{k=0}^{\infty}f_k(z\lambda_n)^k=\sum_{k=0}^{\infty}f_k\left(\sum_{n=1}^{\infty}a_n\lambda_n^k\right)z^k,$$

in view of the Cauchy inequality, we have

$$M_A(r)\geq|f_k|\left(\sum_{n=1}^{\infty}a_n\lambda_n^k\right)r^k\geq a_n|f_k|(\lambda_n r)^k \tag{4}$$

for all $n\geq 1$, $k\geq 0$ and $r\in[0,+\infty)$. We also remark that, if $\mu_f(r)=\max\{|f_k|r^k:k\geq 0\}$ is the maximal term of series (1), then

$$M_f(r)\leq\sum_{k=0}^{\infty}|f_k|r^k=\sum_{k=0}^{\infty}|f_k|(2r)^k 2^{-k}\leq 2\mu_f(2r). \tag{5}$$

We choose $n_0 \geq 1$ such that $a_{n_0} > 0$ and $\lambda_{n_0} \geq 2$. Then, from (4) and (5), we get

$$M_A(r) \geq \max\{a_{n_0}|f_k|(\lambda_{n_0} r)^k : \ k \geq 0\} \geq a_{n_0}\mu_f(2r) \geq \frac{a_{n_0}}{2}M_f(r),$$

where $M_f^{-1}\left(\frac{2}{d_{n_0}}M_A(r)\right) \geq r$. By Lemma 2, $\frac{d\ln M_f^{-1}(x)}{d\ln x} \searrow 0$ as $x \to +\infty$ and, thus, for every $c > 1$

$$\ln M_f^{-1}(cx) - \ln M_f^{-1}(x) = \int_x^{cx} \frac{d\ln M_f^{-1}(t)}{d\ln t} d\ln t \leq \frac{d\ln M_f^{-1}(x)}{d\ln x} \to 0, \ x \to +\infty,$$

i.e., the function M_f^{-1} is slowly increasing. Therefore,

$$M_f^{-1}(M_A(r)) \geq (1+o(1))r, \ r \to +\infty. \tag{6}$$

On the other hand, since series (2) is regularly convergent in $\mathbb{C}$, for each $r \in [0, +\infty)$, there exists $\mu_A(r) = \max\{|an|M_f(r\lambda_n) : n \geq 1\}$ and, for every $r \in [0, +\infty)$ and $\tau > 0$, we have

$$M_A(r) \leq \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) \leq \mu_F((1+\tau)r) \sum_{n=1}^{\infty} \frac{M_f(r\lambda_n)}{M_f((1+\tau)r\lambda_n)}. \tag{7}$$

Then, by Lemma 2, for $r \geq 1$, we have

$$\ln M_f((1+\tau)r\lambda_n) - \ln M_f(r\lambda_n) = \int_{r\lambda_n}^{(1+\tau)r\lambda_n} \frac{d\ln M_f(x)}{d\ln x} d\ln x = \int_{r\lambda_n}^{(1+\tau)r\lambda_n} \Gamma_f(x) d\ln x \geq$$
$$\geq \Gamma_f(r\lambda_n)\ln(1+\tau) \geq \Gamma_f(\lambda_n)\ln(1+\tau).$$

Therefore, if $\ln n \leq q\Gamma_f(\lambda_n)$ for all $n \geq n_0$ and $\ln(1+\tau) > q$, then

$$\sum_{n=n_0}^{\infty} \frac{M_f(r\lambda_n)}{M_f((1+\tau)r\lambda_n)} \leq \sum_{n=n_0}^{\infty} \exp\{-\Gamma_f(\lambda_n)\ln(1+\tau)\} \leq \sum_{n=n_0}^{\infty} \exp\left\{-\frac{\ln(1+\tau)}{q}\ln n\right\} < +\infty$$

and (7) implies, for $r \geq 1$,

$$M_A(r) \leq T\mu_A((1+\tau)r), \ T = \text{const} > 0. \tag{8}$$

Additionally, we have

$$\mu_A(r) \leq \max\left\{|a_n| \sum_{k=0}^{\infty} |f_k|(r\lambda_n)^k : \ n \geq 1\right\} \leq$$
$$\leq \sum_{k=0}^{\infty} \max\{|a_n|\lambda_n^k : \ n \geq 1\}|f_k|r^k = \sum_{k=0}^{\infty} \mu_D(k)|f_k|r^k, \tag{9}$$

where $\mu_D(\sigma) = \max\{|a_n|\exp\{\sigma\ln\lambda_n\} : \ n \geq 1\}$ is the maximal term of Dirichlet series

$$D(\sigma) = \sum_{n=1}^{\infty} |a_n|\exp\{\sigma\ln\lambda_n\}.$$

Using estimates (6), (8), and (9), we prove the following theorem.

Theorem 1. *Let f be an entire transcendental function, $a_n \geq 0$ for all $n \geq 1$, and series (2) be regularly convergent in $\mathbb{C}$. Suppose that $\ln n \leq q\Gamma_f(\lambda_n)$ for some $q > 0$ and all $n \geq n_0$ and that $\overline{\lim}_{\sigma\to+\infty} \frac{\ln\mu_D(\sigma)}{\sigma\ln M_f^{-1}(e^\sigma)} = \gamma$.*

If $\gamma < 1$, then $\lambda_{\alpha,\alpha}[F]_f = \rho_{\alpha,\alpha}[F]_f = 1$ for every function α such that $\alpha(e^x) \in L_{si}$. If $\gamma = 0$, then $\lambda_{\alpha,\alpha}[F]_f = \rho_{\alpha,\alpha}[F]_f = 1$ for every function α such that $\alpha(e^x) \in L^0$.

Proof. Since $\alpha \in L^0$, from (6), we get

$$\lambda_{\alpha,\alpha}[F]_f = \varliminf_{r\to+\infty} \frac{\alpha(M_f^{-1}(M_F(r)))}{\alpha(r)} \geq \varliminf_{r\to+\infty} \frac{\alpha((1+o(1))r)}{\alpha(r)} = 1.$$

On the other hand, in view of the Cauchy inequality, we have $\ln|f_k| \leq \ln M_f(r) - k\ln r$ for all r and k. We choose $r = r_k = M_f^{-1}(e^k)$. Then, $\ln|f_k| \leq k - k\ln M_f^{-1}(e^k)$, i.e., $-\ln|f_k| \geq k(\ln M_f^{-1}(e^k) - 1)$. Therefore,

$$\varlimsup_{k\to\infty} \frac{\ln \mu_D(k)}{-\ln f_k} \leq \varlimsup_{k\to\infty} \frac{\ln \mu_D(k)}{k(\ln M_f^{-1}(e^k)-1)} \leq \varlimsup_{\sigma\to+\infty} \frac{\ln \mu_D(\sigma)}{\sigma \ln M_f^{-1}(e^\sigma)} = \sigma. \tag{10}$$

If $\gamma < 1$, then in view of (10), $\frac{\ln \mu_D(k)}{-\ln|f_k|} \leq p$ for each $p \in (\gamma, 1)$ and all $k \geq k_0$ and, thus, $\mu_D(k) \leq |f_k|^{-p}$ for all $k \geq k_0$. Therefore, in view of (9) and (5),

$$\begin{aligned}\mu_A(r) \leq \left(\sum_{k=0}^{k_0-1} + \sum_{k=k_0}^{\infty}\right)\mu_D(k)|f_k|r^k \leq O(r^{k_0-1}) + \sum_{k=k_0}^{\infty} |f_k|^{1-p}r^k \leq \\ \leq O(r^{k_0-1}) + 2\max\{f_k^{1-p}(2r)^k :\ k \geq 0\} = \\ = O(r^{k_0-1}) + 2\max\{(|f_k|(2r)^{k/(1-p)})^{1-p} :\ k \geq 0\} = \\ = O(r^{k_0-1}) + 2(\mu_f((2r)^{1/(1-p)}))^{1-p} \leq \mu_f((2r)^{1/(1-p)}),\ r \geq r_0,\end{aligned} \tag{11}$$

because $\ln r = o(\ln \mu_f(r))$ as $r \to +\infty$ for every entire transcendental function f and $1 - p < 1$. Therefore, from (8) and (11), we get

$$M_A(r) \leq T\mu_A((1+\tau)r) \leq T\mu_f((2(1+\tau)r)^{1/(1-p)}) \leq TM_f((2(1+\tau)r)^{1/(1-p)})$$

and, thus, $M_f^{-1}(M_A(r)) \leq (1+o(1))(2(1+\tau)r)^{1/(1-p)}$ as $r \to +\infty$. If $\alpha \in L_{si}$, then we obtain

$$\varlimsup_{r\to+\infty} \frac{\alpha(M_f^{-1}(M_A(r)))}{\alpha(r^{1/(1-p)})} \leq 1. \tag{12}$$

Suppose that $\alpha(e^x) \in L_{si}$. Then,

$$\alpha(r^{1/(1-p)}) = \alpha(\exp\left\{\frac{1}{1-p}\ln r\right\}) = (1+o(1))\alpha(\exp\{\ln r\}) = (1+o(1))\alpha(r)$$

as $r \to +\infty$. Therefore, (12) implies the inequality $\rho_{\alpha,\alpha}[A]_f \leq 1$, where in view of the inequality $\lambda_{\alpha,\alpha}[A]_f \geq 1$, we get $\lambda_{\alpha,\alpha}[A]_f = \rho_{\alpha,\alpha}[A]_f = 1$.

If $\gamma = 0$, then (12) holds for every $p \in (0,1)$ and all $r \geq r_0(p)$. If we put $\frac{1}{1-p} = 1+\delta$, then $\delta \to +0$ as $p \to +0$, and in view of the condition $\alpha(e^x) \in L^0$, by Lemma 1, we have

$$\varlimsup_{r\to+\infty} \frac{\alpha(r^{1/(1-p)})}{\alpha(r)} = \varlimsup_{r\to+\infty} \frac{\alpha(\exp\{(1+\delta)\ln r\})}{\alpha(\exp\{\ln r\})} = B(\delta) \to 1,\ \delta \to 1.$$

Therefore,

$$\begin{aligned}1 \geq \varlimsup_{r\to+\infty} \frac{\alpha(M_f^{-1}(M_A(r)))}{\alpha(r^{1/(1-p)})} = \varlimsup_{r\to+\infty} \left(\frac{\alpha(M_f^{-1}(M_A(r)))}{\alpha(r)} \cdot \frac{\alpha(r)}{\alpha(r^{1+\delta})}\right) \geq \\ \geq \varlimsup_{r\to+\infty} \frac{\alpha(M_f^{-1}(M_A(r)))}{\alpha(r)} \varliminf_{r\to+\infty} \frac{\alpha(r)}{\alpha(r^{1+\delta})} = \frac{\rho_{\alpha,\alpha}[F]_f}{B(\delta)}.\end{aligned}$$

In view of the arbitrariness of δ, we get $\rho_{\alpha,\alpha}[A]_f \le 1$, and again, $\lambda_{\alpha,\alpha}[A]_f = \rho_{\alpha,\alpha}[A]_f = 1$. Theorem 1 is proven. $\square$

We remark that, if $f_k \ge 0$ for all $k \ge 0$, then $M_f(r) = f(r)$. Therefore, from Theorem 1, we obtain the following statement.

Corollary 1. *Let f be an entire transcendental function, $f_k \ge 0$ for all $k \ge 0$, $a_n \ge 0$ for all $n \ge 1$, and series (2) be regularly convergent in $\mathbb{C}$. Suppose that $f'(r)/f(r) \ge h > 0$ for all $r \ge r_0$, $\ln n = O(\lambda_n)$ as $n \to \infty$ and $\overline{\lim}_{\sigma\to+\infty} \frac{\ln \mu_D(\sigma)}{\sigma \ln f^{-1}(e^\sigma)} = \gamma$.*

If $\gamma < 1$, then $\lambda_{\alpha,\alpha}[A]_f = \rho_{\alpha,\alpha}[A]f = 1$ for every function α such that $\alpha(e^x) \in L_{si}$.

If $\gamma = 0$, then $\lambda_{\alpha,\alpha}[A]_f = \rho_{\alpha,\alpha}[A]_f = 1$ for every function α such that $\alpha(e^x) \in L^0$.

3. Relative Growth of Laplace–Stieltjes-Type Integrals

Suppose again that f is an entire transcendental functio,n $f_k \ge 0$ for all $k \ge 0$, and $x_0 > 1$ is such that $\int_1^{x_0} a(x)dF(x) \ge > 0$. Then,

$$I(r) \ge \int_1^{x_0} a(x)f(rx)dF(x) \ge f(r)c,$$

i.e., as above, $f^{-1}(I(r)) \ge (1+o(1))r$ as $r \to +\infty$, where for $\alpha \in L^0$,

$$\lambda_{\alpha,\alpha}[I]_f = \underline{\lim}_{r\to+\infty} \frac{\alpha(f^{-1}(I(r)))}{\alpha(r)} \ge 1.$$

On the other hand, if $\tau \ge e-1$, then as above, for $r \ge 1$, we have

$$\ln f((1+\tau)rx) - \ln f(rx) = \int_{rx}^{(1+\tau)rx} \frac{d\ln f(x)}{d\ln x} d\ln x = \int_{rx}^{(1+\tau)rx} \Gamma_f(x) d\ln x \ge \\ \ge \Gamma_f(x)\ln(1+x),$$

i.e., $\frac{f(rx)}{f((1+\tau)rx)} \le e^{-\Gamma_f(x)\ln(1+\tau)}$. Therefore, if $\mu_I(r) = \max\{a(x)f(rx):\ x \ge 0\}$ is the maximum of the integrand and $\ln F(x) \le q\Gamma_f(x)$ for some $q > 0$ and all $x \ge x_0$, then for $\ln(1+\tau) > q$ (for simplicity assuming $x_0 = 0$), we get

$$\begin{aligned} I(r) = \int_0^\infty a(x) f((1+\tau)rx) \frac{f(rx)}{f((1+\tau)rx)} dF(x) \le \mu_I((1+\tau)r) \int_0^\infty \frac{f(rx)}{f((1+\tau)rx)} dF(x) \le \\ \le \mu_I((1+\tau)r) \int_0^\infty e^{-\Gamma_f(x)\ln(1+\tau)} dF(x \le \\ \le \mu_I((1+\tau)r)\ln(1+\tau) \int_0^\infty e^{-\Gamma_f(x)\ln(1+\tau)+\ln F(x)} d\Gamma_f(x) \le \\ \le \mu_I((1+\tau)r)\ln(1+\tau) \int_0^\infty e^{-\Gamma_f(x)(\ln(1+\tau)-q)} d\Gamma_f(x) = \mu_I((1+\tau)r)\frac{\ln(1+\tau)}{\ln(1+\tau)-q} = \\ = T\mu_I((1+\tau)r). \end{aligned} \tag{13}$$

Additionally, as above, we have

$$\begin{aligned} \mu_I(r) = \max\left\{ a(x) \sum_{k=0}^\infty f_k (xr)^k :\ x \ge 0 \right\} \le \\ \le \sum_{k=0}^\infty \max\{a(x)x^k :\ x \ge 0\} f_k r^k = \sum_{k=0}^\infty \mu_J(k) f_k r^k, \end{aligned} \tag{14}$$

where $\mu_J(\sigma) = \max\{a(x)e^{\sigma \ln x} : x \geq 0\} = \max\{a(x)x^{\ln x} : x \geq 0\}$ is the maximum of the integrand for the Laplace integral

$$J(\sigma) = \int_0^{\infty} a(x)e^{\sigma \ln x} dF(x).$$

Using estimates (13) and (14), and $\lambda_{\alpha,\alpha}[I]_f \geq 1$, we prove the following analog of Theorem 1.

Theorem 2. *Let $\ln F(x) \leq q\Gamma_f(x)$ for some $q > 0$ and all $x \geq x_0$, and $\overline{\lim}_{\sigma\to+\infty} \frac{\ln \mu_J(\sigma)}{\gamma \ln f^{-1}(e^{\sigma})} = \gamma$.*

If $\gamma < 1$, then $\lambda_{\alpha,\alpha}[I]_f = \rho_{\alpha,\alpha}[I]_f = 1$ for every function α such that $\alpha(e^x) \in L_{si}$.

If $\gamma = 0$, then $\lambda_{\alpha,\alpha}[I]_f = \rho_{\alpha,\alpha}[I]_f = 1$ for every function α such that $\alpha(e^x) \in L^0$.

Proof. As in the proof of Theorem 1, we obtain $-\ln|f_k| \geq k(\ln f^{-1}(e^k) - 1)$ and $\overline{\lim}_{k\to\infty} \frac{\ln \mu_J(k)}{-\ln f_k} \leq \gamma$. Therefore, if $\gamma < 1$, then $\mu_D(k) \leq |f_k|^{-p}$ for each $p \in (\gamma, 1)$ and all $k \geq k_0$, and in view of (14) and (5), as in the proof of Theorem 1, we get $\mu_I(r) \leq \mu_f((2r)^{1/(1-p)})$ for $r \geq r_0$. Therefore, in view of (13), we get

$$I(r) \leq T\mu_I((1+\tau)r) \leq Tf((2(1+\tau)r)^{1/(1-p)}),$$

where $f^{-1}(I(r)) \leq (1+o(1))(2(1+\tau)r)^{1/(1-p)}$ as $r \to +\infty$. If $\alpha \in L_{si}$, then we obtain

$$\overline{\lim_{r\to+\infty}} \frac{\alpha(f^{-1}(I(r)))}{\alpha(r^{1/(1-p)})} \leq 1.$$

Further proof of Theorem 2 is the same as that of Theorem 1. □

Theorem 2 implies the following statement.

Corollary 2. *Let $f'(x)/f(x) \geq h$, $h > 0$, $\ln F(x) \leq qx$ for some $q > 0$ and all $x \geq 0$, and $\overline{\lim}_{r\to+\infty} \frac{\ln \mu_J(\sigma)}{\sigma f^{-1}(e^{\sigma})} = \gamma$.*

If $\gamma < 1$, then $\lambda_{\alpha,\alpha}[I]_f = \rho_{\alpha,\alpha}[I]_f = 1$ for every function α such that $\alpha(e^x) \in L_{si}$.

If $\gamma = 0$, then $\lambda_{\alpha,\alpha}[I]_f = \rho_{\alpha,\alpha}[I]_f = 1$ for every function α such that $\alpha(e^x) \in L^0$.

4. Examples

Here, we consider the case when $f(z) = E_\rho(z)$, where

$$E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\frac{k}{\rho})},\ 0 < \rho < +\infty,$$

is the Mittag–Leffler function. The properties of this function have been used in many problems in the theory of entire functions. We only need the following property of the Mittag–Leffler function: if $0 < \rho < +\infty$, then ([11] p. 85)

$$M_{E_\rho}(r) = E_\rho(r) = (1+o(1))\rho e^{r^\rho},\ r \to +\infty \tag{15}$$

and, if $1/2 < \rho < +\infty$, then [12]

$$E'_\rho(r)/E_\rho(r) = \rho r^{\rho-1} + O(r^{\rho-2}e^{-r^\rho}),\ \ r \to +\infty. \tag{16}$$

From (15), it follows that $E_\rho^{-1}(x) = (1+o(1))\ln^{1/\rho} x$ as $x \to +\infty$. Therefore, for $f(x) = E_\rho(x)$, we have $\sigma \ln f^{-1}(e^\sigma) = \frac{1+o(1)}{\rho}\sigma\ln\sigma$ as $\sigma \to +\infty$. Since in (16), $\Gamma_{E_\rho}(r) = \rho r^\rho + o(1)$ as $r \to +\infty$, then if $\ln F(x) \le q\rho x^\rho$ for some $q > 0$ and all $x \ge x_0$, and

$$\overline{\lim_{\sigma\to+\infty}} \frac{\ln \mu_J(\sigma)}{\sigma \ln \sigma} = 0, \tag{17}$$

then for $\alpha(x) = \ln x$ $(x \ge e)$, by Theorem 2, we get

$$\lim_{r\to+\infty} \frac{\ln E_\rho^{-1}(I_\rho(r))}{\ln r} = 1, \quad I_\rho(r) = \int_0^\infty a(x)E_\rho(rx)dF(x). \tag{18}$$

Let us now find out under what conditions (17) holds on $a(x)$. For this, as in ([7] p. 29), by Ω, we denote a class of positive unbounded functions Φ on $(-\infty, +\infty)$ such that the derivative Φ_0 is positive, continuously differentiable, and increasing to $+\infty$ on $(-\infty, +\infty)$. For $\Phi \in \Omega$, let φ be the inverse function to Φ' and $\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}$ be the function associated with Φ in the sense of Newton.

By Theorem 2.2.1 from ([7] p. 30), $\ln\max\{a(x)e^{\sigma x} : x \ge 0\} \le \Phi(\sigma) \in \Omega$ for all $\sigma \ge \sigma_0$ if and only if $\ln a(x) \le -x\Psi(\varphi(x))$ for all $x \ge x_0$. Choosing $\Phi(\sigma) = \epsilon\sigma\ln\sigma$ for $\sigma \ge \sigma_0$, we obtain $\Phi'(\sigma) = \epsilon(\ln\sigma + 1)$, $\varphi(x) = \exp\{x/\epsilon - 1\}$ and $x\Psi(\varphi(x)) = x\varphi(x) - \Phi(\varphi(x)) = \epsilon\exp\{x/\epsilon - 1\}$ for $x \ge x_0$. Therefore, $\ln\mu_J(\sigma) \le \epsilon\sigma\ln\sigma$ for all $\sigma \ge \sigma_0$ if and only if $\ln a(x) \le -\varepsilon\exp\{\ln x/\varepsilon - 1\}$ for $x \ge x_0$. Hence, it follows that, if $\ln x = o(\ln\ln(1/a(x)))$ as $x \to +\infty$, then (17) holds. Thus, the following statement is true.

Proposition 1. *If $\rho > 1/2$, $\ln F(x) = O(x^\rho)$ and $\ln x = o(\ln\ln(1/a(x)))$ as $x \to +\infty$, then (18) holds.*

Remark 1. *If $\rho = 1$, then $E_\rho(r) = E_1(r) = e^r$, and we have a usual Laplace–Stieltjes integral $I_1(r) = \int_0^\infty a(x)e^{rx}dF(x)$. Therefore, if $\ln F(x) = O(x)$ and $\ln x = o(\ln\ln(1/a(x)))$ as $x \to +\infty$, then $p_R[I_1] := \lim_{r\to+\infty} \frac{\ln\ln I_1(r)}{\ln r} = 1$. On the other hand, the quantity $p_R[I_1]$ is called the logarithmic R-order of I_1, and in ([7] p. 83), it is proven that, if $\ln F(x) = O(x)$ as $x \to +\infty$, then $p_R[I_1] = \overline{\lim}_{x\to+\infty} \frac{\ln x}{\ln(\frac{1}{x}\ln\frac{1}{a(x)})} = 1$, i.e., if $\ln F(x) = O(x)$ and $\ln x = o(\ln\ln(1/a(x)))$ as $x \to +\infty$, then $p_R[I_1] = 1$.*

Similarly, we can prove the following statement.

Proposition 2. *Let $\rho \ge 1/2$, $\ln n = O(\lambda_n^\rho)$ as $n \to \infty$, $a_n \ge 0$ for all $n \ge 1$ and series $A_\rho(z) = \sum_{n=1}^\infty a_n E_\rho(\lambda_n z)$be regularly convergent in $\mathbb{C}$. If $\ln n = o(\ln\ln(1/a_n))$ as $n \to \infty$, then $\lim_{r\to+\infty} \frac{\ln E_\rho^{-1}(M_{A_\rho}(r))}{\ln r} = 1$.*

Remark 2. *If $\rho = 1$, then we have a Dirichlet series $A_1(z) = \sum_{n=1}^\infty a_n e^{\lambda_n z}$. Therefore, if this Dirichlet series is absolutely convergent in $\mathbb{C}$, $a_n \ge 0$ for all $n \ge 1$, $\ln n = O(\lambda_n)$, and $\ln n = o(\ln\ln(1/a_n))$ as $n \to \infty$, then $p_R[A_1] := \lim_{r\to+\infty} \frac{\ln\ln M_{A_1}(r)}{\ln r} = 1$. On the other hand, the quantity $p_R[A_1]$ is called the logarithmic R-order of A_1 and $p_R[A_1] = \overline{\lim}_{n\to+\infty} \frac{\ln\lambda_n}{\ln(\frac{1}{\lambda_n}\ln\frac{1}{a_n})} = 1$ provided $\ln n = O(\lambda_n)$ as $n \to \infty$ [13], i.e., if $\ln n = O(\lambda_n)$ and $\ln\lambda_n = o(\ln\ln(1/a_n))$ as $n \to \infty$, then $p_R[A_1] = 1$.*

5. Discussion Open Problems

1. The natural problem studied was the relative growth when the domain of regular convergence of series (2) is the disk $D_R = \{z : |z| < R < +\infty\}$ and the function f is either entire or analytic in D_R.

2. It is well known that the study of the growth of entire functions of many complex variables involves many options. The following problem is the simplest.

Let f be an entire function and the series $A(z,w) = \sum_{m=1,n=1}^{\infty} a_{m,n} f(\lambda_m z + \mu_n w)$ be regularly convergent in $\mathbb{C}^2$. A question arises about the asymptotic behavior of the function $M_f^{-1}(M_A(r,\rho))$, where $M_A(r,\rho) = \max\{|A(z,w)| : |z| \le r, |w| \le \rho\}$.

3. The condition $\rho \ge 1/2$ in Propositions 1 and 2 arose in connection to the application of Equation (16). Probably, it is superfluous in the above statements.

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