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Vector Fields and Differential Forms on the Orbit Space of a Proper Action

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Abstract: In this paper, we study differential forms and vector fields on the orbit space of a proper action of a Lie group on a smooth manifold, defining them as multilinear maps on the generators of infinitesimal diffeomorphisms, respectively. This yields an intrinsic view of vector fields and differential forms on the orbit space.

Keywords: proper action; orbit space; vector field on orbit space; differential forms on orbit space

MSC: 58A50



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1. Introduction

This paper is part of a series of papers devoted to the study of the geometry of singular spaces in terms of the theory of differential spaces, which were introduced by Sikorski [1], see also [2]. In this theory, geometric information about a space S is encoded in a ring $C^\infty(S)$ of real valued functions, which are deemed to be smooth. In particular, we are concerned with the class of subcartesian spaces introduced by Aronszajn [3]. A Hausdorff differential space S is *subcartesian* if every point x of S has a neighborhood U that is diffeomorphic to a subset V of a Euclidean (Cartesian) space \mathbb{R}^n . The restriction of $C^\infty(S)$ to U is isomorphic to the restriction of $C^\infty(\mathbb{R}^n)$ to V , see [4].

Palais [5] introduced the notion of a slice for an action of a not necessarily compact Lie group G on a manifold M . Since then, the structure of the space M/G of orbits of a proper action of G on M has been investigated by many mathematicians. In [6] Duistermaat showed that M/G is a subcartesian differential space with differential structure $C^\infty(M/G)$ consisting of push forwards of smooth G invariant functions on M by the G orbit mapping $\pi : M \rightarrow M/G$. On a smooth manifold M , there are two equivalent definitions of a vector field, namely as a derivation of $C^\infty(M)$, or as a generator of a local one parameter local group of diffeomorphisms of M . Choosing one, and proving the other is a matter of preference. On a subcartesian differential space S , which is not in general a manifold, these notions differ. We use the term *vector field* on S for a generator of a local one parameter group of local diffeomorphisms and denote the class of all vector fields on S by $\mathfrak{X}(S)$. A key reason for the choice made in this paper is the special case of the orbit space of a proper action. The class of derivations of $C^\infty(S)$ is, in general, larger than the class $\mathfrak{X}(S)$. For $S = M/G$ we show that a derivation Y of $C^\infty(S)$ is in $\mathfrak{X}(S)$, i.e., is a vector field on M/G , if and only if there exists a G invariant vector field X on M such that Y is π related to X , that is, $Y \circ \pi = T\pi \circ X$, where $\pi : M \rightarrow M/G$ is the G orbit map.

In the literature, there has been extensive discussion about the notion of a differential form on a singular space, see Smith [7], Marshall [8], and Sjamaar [9]. Here, in our search for an intrinsic notion of a differential form, we have been led to see them as multilinear maps on vector fields. In the case of a 1-form θ on M/G with θ a linear mapping

$$\theta : \mathfrak{X}(M/G) \rightarrow C^\infty(M/G) : Y \mapsto \langle \theta | Y \rangle$$

over the ring $C^\infty(M/G)$ of smooth functions on M/G , which is to say $\langle \theta | \bar{f} Y \rangle = \bar{f} \langle \theta | Y \rangle$ for every $\bar{f} \in C^\infty(M/G)$. With this definition we show that every differential 1-form on M/G pulls back under the orbit map π to a semi-basic G invariant 1-form on M . Furthermore, every G invariant semi-basic differential 1-form on M is the pull back by π of a differential 1-form on M/G .

We define a differential exterior algebra of differential forms on the orbit space, which satisfies a version of de Rham’s theorem. Our version is larger than Smith’s as it includes forms that are not Smith forms, see Section 6. It also handles singular orbit spaces of a proper action of a Lie group on a smooth manifold. The Lie group need not be compact and the orbit space need not be smooth, both of which Koszul hypothesized in [10].

We now give a section by section description of the contents of this paper.

Section 2 deals with basic properties of a proper action of a Lie group G on a smooth manifold M and the differential structure of the orbit space M/G . We introduce the reader to the theory of subcartesian differential spaces in the context of the orbit space M/G . The differential geometry of M is described in terms of its smooth structure given by the ring $C^\infty(M)$ of smooth functions on M . The differential geometry of the orbit space M/G , which may have singularities, is similarly described in terms of the ring $C^\infty(M/G)$ of smooth functions on M/G , which is isomorphic to the ring $C^\infty(M)^G$ of smooth, G -invariant functions on M . Since a proper action has an invariant Riemannian metric, several results are proved using properties of the geodesics of the metric. Additionally, certain objects are shown to be smooth submanifolds.

In Section 3 we study vector fields on the orbit space M/G . In the case of the manifold M , derivations of the ring $C^\infty(M)$ are vector fields on M , and they generate a local one parameter group of local diffeomorphisms of M . In the case of the ring $C^\infty(M/G)$, not all derivations of $C^\infty(M/G)$ generate local one parameter groups of local diffeomorphisms of M/G . The derivations of $C^\infty(M/G)$, which generate local one parameter groups of local diffeomorphisms of M/G . This is the key idea of this paper. We establish that every vector field on the quotient M/G is covered by a G -invariant vector field on M . It is well known that the space M/G is stratified, see [6,11,12]. We show that every vector field on M/G defines a vector field tangent to each stratum of M/G .

In Section 4, we define differential 1-forms on the orbit space M/G as linear mappings on the space $\mathfrak{X}(M/G)$ of smooth vector fields on M/G . The most important consequence of this definition relates to pulling back 1-forms from M/G to M . In particular, our notion of a differential 1-form is intrinsic.

In Section 5 to prove a version of de Rham’s theorem we enlarge the algebra of differential 1-forms to k -forms with an exterior derivative operator. The key technical point is that everything is developed in terms of the Lie derivative of vector fields. Almost all of this section looks the same as that on manifolds.

In Section 6 we give all the details of the simplest nontrivial example. This example reveals that differential forms in our sense are not the same as those of Smith [7].

2. Basic Properties

This section gives some of the basic properties of smooth vector fields on the orbit space of a proper action of a Lie group on a smooth manifold.

Let M be a connected smooth manifold with a proper action

$$\Phi : G \times M \rightarrow M : (g, m) \mapsto \Phi_g(m) = g \cdot m \tag{1}$$

of a Lie group G on M , and let

$$\pi : M \rightarrow M/G : m \mapsto \bar{m} = G \cdot m = \{ \Phi_g(m) \in M \mid g \in G \} \tag{2}$$

be the orbit map of the G action Φ .

Let $C^\infty(M)^G$ be the algebra of smooth G invariant functions on M and let $C^\infty(M/G)$ be the algebra of functions \bar{f} on M/G such that $f = \pi^* \bar{f} = \bar{f} \circ \pi$ lies in $C^\infty(M)^G$. The map

$\pi^* : C^\infty(M/G) \rightarrow C^\infty(M)^G : \bar{f} \mapsto \bar{f} \circ \pi$ is a bijective algebra isomorphism, whose inverse is $\pi_* : C^\infty(M)^G \rightarrow C^\infty(M/G) : f \mapsto \bar{f}$.

Proposition 1. *The orbit space M/G with the differential structure $C^\infty(M/G)$ is a locally closed subcartesian differential space.*

Proof. See Corollary 4.11 of Duistermaat [6] and page 72 of [4]. \square

Let X be a smooth vector field on a manifold M . X gives rise to a map $L_X : C^\infty(M) \rightarrow C^\infty(M) : f \mapsto L_X f = X(f)$, called the *derivation* associated to X . If we want to emphasize this action of vector fields on M , we say that they form the space $\text{Der } C^\infty(M)$ of derivations of $C^\infty(M)$. If we want to emphasize that X generates a local one parameter group of local diffeomorphisms of M , we say that X is a *vector field* on M and write $\mathfrak{X}(M)$ for the set of vector fields on M . For each smooth manifold M we have $\mathfrak{X}(M) = \text{Der } C^\infty(M)$. However, these notions need not coincide for a subcartesian differential space.

Let $(S, C^\infty(S))$ be a differential space with X a derivation of $C^\infty(S)$. Let $I_x \subseteq \mathbb{R} \rightarrow S : t \mapsto \varphi_t^X(x)$ be a maximal integral curve of X , which starts at x . Here I_x is an interval containing 0. If t, s , and $t + s$ lie in I_x , and if $s \in I_{\varphi_t^X(x)}$ and $t \in I_{\varphi_s^X(x)}$, then

$$\varphi_{t+s}^X(x) = \varphi_s^X(\varphi_t^X(x)) = \varphi_t^X(\varphi_s^X(x)).$$

The map φ_t^X may fail to be a local diffeomorphism of the differential space S , see example 3.2.7 in ([4], p. 37). A *vector field* on a subcartesian differential space S is a derivation X of $C^\infty(S)$ such that for every $x \in S$ there is an open neighborhood U of x and $\varepsilon > 0$ such that for every $t \in (-\varepsilon, \varepsilon)$ the map φ_t^X is defined on U and its restriction to U is a diffeomorphism from U onto an open subset of S . In other words, the derivation X is a vector field on S if $t \mapsto \varphi_t^X$ is a local one parameter group of local diffeomorphisms of S .

Example 1. Consider $\mathbb{Q} \subseteq \mathbb{R}$ with the structure of a differential subspace of \mathbb{R} . Let $\iota : \mathbb{Q} \rightarrow \mathbb{R}$ be the inclusion mapping. The differential structure $C^\infty(\mathbb{Q})$ of \mathbb{Q} consists of $\iota^* f$, which is the restriction of a smooth function f on \mathbb{R} to \mathbb{Q} . Let $X(x_1) = a_1(x_1) \frac{\partial}{\partial x_1}$ be a vector field on \mathbb{R} . Then for every $f \in C^\infty(\mathbb{R})$ and every $x_1 \in \mathbb{R}$ the function $x_1 \mapsto X(f)(x_1) = a_1(x_1) \frac{\partial f}{\partial x_1}(x_1)$ is smooth. Restricting to points x_1 in \mathbb{Q} we obtain $\iota^*(X(f)) = \iota^*(a_1) \iota^*(\frac{\partial f}{\partial x_1})$. We now show that we can obtain $\iota^*(\frac{\partial f}{\partial x_1})$ by operations on \mathbb{Q}^2 . Let $x_1^0 \in \mathbb{Q}$ and let $\{(x_1)_n\}$ be a sequence of points in \mathbb{Q} , which converges to x_1^0 . Then

$$\lim_{n \rightarrow \infty} \frac{\partial f}{\partial x_1}(x_1^0 + (x_1)_n) = \frac{\partial f}{\partial x_1}(x_1^0).$$

Thus, we show that $\iota^*(X(f)) = X|_{\mathbb{Q}}(\iota^* f)$ for every $f \in C^\infty(\mathbb{R})$. In other words, the restriction $X|_{\mathbb{Q}}$ of the vector field X to \mathbb{Q} is a derivation of $C^\infty(\mathbb{Q})$. Thus, $\text{Der } C^\infty(\mathbb{Q}) = \{X|_{\mathbb{Q}} \mid X \in \mathfrak{X}(\mathbb{R})\}$. However, no two distinct points of \mathbb{Q} can be joined by a smooth curve. Hence only the derivation of $C^\infty(\mathbb{Q})$ that is identically 0 on \mathbb{Q} admits integral curves, i.e., $\mathfrak{X}(\mathbb{Q}) = \{0\}$.

Let $\mathfrak{X}(M)^G$ be the set of smooth G invariant vector fields on M , that is,

$$\mathfrak{X}(M)^G = \{X \in \mathfrak{X}(M) \mid T_m \Phi_g X(m) = X(\Phi_g(m)) \text{ for every } (g, m) \in G \times M\}.$$

Since $\mathfrak{X}(M) = \text{Der } C^\infty(M)$, we have $\mathfrak{X}(M)^G = (\text{Der } C^\infty(M))^G$. Additionally, we may consider the space $\text{Der } C^\infty(M)^G$ of derivations of $C^\infty(M)^G$. Clearly, we have $(\text{Der } C^\infty(M))^G \subseteq \text{Der } C^\infty(M)^G$. For $X \in \mathfrak{X}(M)^G$ let

$$\varphi : D \subseteq \mathbb{R} \times M \rightarrow M : (t, m) \mapsto \varphi_t(m) \tag{3}$$

be the local flow of X , i.e., φ is a differentiable mapping such that

$$\frac{d\varphi_t}{dt}(m) = X(\varphi_t(m)), \text{ for all } (t, m) \in D.$$

Here D is a domain, i.e., D is the largest (in the sense of containment) open subset of $\mathbb{R} \times M$ such that for each $m \in M$ the set $\{t \in \mathbb{R} \mid (t, m) \in D\}$ is an open interval containing 0. Moreover, $\varphi(0, m) = m$ for every $m \in M$ and if $(t, m) \in D$, $(s, \varphi_t(m)) \in D$, and $(t + s, m) \in D$, then $\varphi_{s+t}(m) = \varphi_s(\varphi_t(m))$. Since $X \in \mathfrak{X}(M)^G$,

$$(\Phi_g \circ \varphi_t)(m) = \varphi_t(\Phi_g(m)), \text{ for all } (g, (t, m)) \in G \times D. \tag{4}$$

Thus, $(t, \Phi_g(m)) \in D$ for all $g \in G$, if $(t, m) \in D$.

Proposition 2. Let $X \in (\text{Der } C^\infty(M))^G$. Then X induces a derivation of $C^\infty(M/G)$ defined by

$$Y : C^\infty(M/G) \rightarrow C^\infty(M/G) : \bar{f} \mapsto \pi_*(X(\pi^*\bar{f})).$$

This leads to the module homomorphism

$$(\text{Der } C^\infty(M))^G \rightarrow \text{Der } C^\infty(M/G) : X \mapsto Y = \pi_* \circ X \circ \pi^*. \tag{5}$$

Proof. Let $Y = \pi_* \circ X \circ \pi^*$ and $\bar{f}, \bar{h} \in C^\infty(M/G)$. Then

$$\begin{aligned} Y(\bar{h}\bar{f}) &= \pi_*(X(\pi^*(\bar{h}\bar{f}))) = \pi_*((\pi^*\bar{h})X(\pi^*\bar{f}) + (\pi^*\bar{f})X(\pi^*\bar{h})) \\ &= ((\pi_* \circ \pi^*)\bar{h})\pi_*(X(\pi^*\bar{f})) + ((\pi_* \circ \pi^*)\bar{f})\pi_*(X(\pi^*\bar{h})) = \bar{h}Y(\bar{f}) + \bar{f}Y(\bar{h}). \end{aligned}$$

Since Y is a linear mapping of $C^\infty(M/G)$ into itself, it follows that it is a derivation of $C^\infty(M/G)$.

We now show that the map $X \rightarrow \pi_* \circ X \circ \pi^*$ is a module homomorphism. For $X, X' \in (\text{Der } C^\infty(M))^G$ and $\bar{f} \in C^\infty(M/G)$ we have

$$\begin{aligned} (\pi_* \circ (X + X') \circ \pi^*)\bar{f} &= \pi_*(X(\pi^*\bar{f}) + X'(\pi^*\bar{f})) \\ &= \pi_*(X(\pi^*\bar{f})) + \pi_*(X'(\pi^*\bar{f})) = (\pi_* \circ X \circ \pi^*)(\bar{f}) + (\pi_* \circ X' \circ \pi^*)(\bar{f}). \end{aligned}$$

Hence the map $X \rightarrow \pi_* \circ X \circ \pi^*$ is linear. For every $h \in C^\infty(M)^G$

$$(\pi_* \circ (hX) \circ \pi^*)\bar{f} = \pi_*(hX(\pi^*\bar{f})) = \pi_*(h)\pi_*(X(\pi^*\bar{f})) = \pi_*(h)(\pi_* \circ X \circ \pi^*)\bar{f}.$$

Therefore the map given by Equation (5) is a module homomorphism. \square

The importance of the module homomorphism (5) stems from the following result.

Proposition 3. Since M is a smooth manifold, $(\text{Der } C^\infty(M))^G = \mathfrak{X}(M)^G$. So $X \in \mathfrak{X}(M)^G$ implies that $Y = \pi_* \circ X \circ \pi^* \in \mathfrak{X}(M/G)$.

Proof. Because the orbit space M/G is locally closed and subcartesian, every maximal integral curve of X projects under the G orbit map to a maximal integral curve of Y . It follows that Y is a smooth vector field on M/G , see proposition 3.2.6 on page 34 of [4]. \square

The following example shows that not every derivation on $C^\infty(M/G)$ is a vector field on M/G .

Example 2. Consider the \mathbb{Z}_2 action on \mathbb{R} generated by $\zeta : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto -x$. The algebra $C^\infty(\mathbb{R})^{\mathbb{Z}_2}$ of smooth \mathbb{Z}_2 invariant functions is generated by the polynomial $\sigma(x) = x^2$. The orbit map of the \mathbb{Z}_2 action is $\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}_2 \subseteq \mathbb{R} : x \mapsto \sigma(x) = x^2$. The derivation $\frac{\partial}{\partial \sigma}$ of $C^\infty(\mathbb{R})^{\mathbb{Z}_2}$ is

not a smooth vector field on \mathbb{R}/\mathbb{Z}_2 , because its maximal integral curve γ_{σ_0} starting at $\sigma_0 \in \mathbb{R}/\mathbb{Z}_2$, given by $\gamma_{\sigma_0}(t) = t + \sigma_0$, is defined on $[-\sigma_0, \infty)$, which is not an open interval that contains 0.

Fix $m \in M$. Then $G_m = \{g \in G \mid \Phi_g(m) = m\}$ is the isotropy group of the action Φ at m . It is a compact subgroup of G , see Duistermaat and Kolk [13]. Let H be a compact subgroup of G . The set

$$M_H = \{m \in M \mid G_m = H\} \tag{6}$$

is a submanifold of M , which is not necessarily connected. Hence its connected component are submanifolds. Connected components of M_H are H invariant submanifolds of M , see Duistermaat and Kolk [13]. The conjugacy class in G of a closed subgroup H is denoted by $(H) = \{gHg^{-1} \in G \mid g \in G\}$ and is called a *type*. The set

$$M_{(H)} = \{m \in M \mid G_m \in (H)\} = G \cdot M_H \tag{7}$$

is called an *orbit type* (H) . Moreover, the G invariance of (H) implies that each connected component of $M_{(H)}$ is G invariant. The orbit type $M_{(H)}$ is associated to the type (H) . Let (H_1) and (H_2) be two types. Define the partial order \leq by the condition: $(H_1) \leq (H_2)$ if some G conjugate of H_2 is a subgroup of H_1 . Since the orbit space \bar{M} is connected, there is a unique maximal orbit type $M_{(K)}$.

Proposition 4. *The maximal orbit type $M_{(K)}$ is open and dense in M .*

Proof. See page 118 of Duistermaat and Kolk [13]. \square

Proposition 5. *The orbit type $M_{(H)}$ is a smooth invariant submanifold of every vector field $X \in \mathfrak{X}(M)^G$.*

Proof. Let $m \in M_{(H)}$. Then $G_m = gHg^{-1}$ for some $g \in G$. Let φ_t be the local 1 parameter group of local diffeomorphisms of M generated by $X \in \mathfrak{X}(M)^G$. Then

$$\Phi_{gHg^{-1}}(\varphi_t(m)) = \varphi_t(\Phi_{gHg^{-1}}(m)) = \varphi_t(m),$$

since $gHg^{-1} = G_m$. Thus, $gHg^{-1} \subseteq G_{\varphi_t(m)}$. Conversely, suppose that $k \in G_{\varphi_t(m)}$. Then $\varphi_t(m) = \Phi_k(\varphi_t(m)) = \varphi_t(\Phi_k(m))$. So

$$m = \varphi_{-t}(\varphi_t(m)) = \varphi_{-t}(\varphi_t(\Phi_k(m))) = \Phi_k(m),$$

that is, $k \in G_m = gHg^{-1}$. Thus, $G_{\varphi_t(m)} \subseteq gHg^{-1}$. Consequently, $G_{\varphi_t(m)} = gHg^{-1}$, i.e., $\varphi_t(m) \in M_{(H)}$. \square

Let Y be a smooth vector field on \bar{M} , which is π related to the smooth G invariant vector field X on M , i.e., $T_m\pi X(m) = Y(\pi(m))$ for $m \in M$. Then $\bar{\varphi}_t(\bar{m}) = \pi(\varphi_t(m))$ for $(t, \bar{m}) \in \mathcal{D} = \{(t, \bar{m}) \in \mathbb{R} \times \bar{M} \mid (t, m) \in D \ \& \ \pi(m) = \bar{m}\}$. The set \mathcal{D} is well defined, because $(t, g \cdot m) \in D$ for every $g \in G$. So \mathcal{D} is the domain of a local generator of Y , i.e., $\bar{\varphi} : \mathcal{D} \subseteq \mathbb{R} \times \bar{M} \mapsto \bar{M}$ is a differentiable mapping such that $\frac{d\bar{\varphi}_t}{dt}(\bar{m}) = Y(\bar{\varphi}_t(\bar{m}))$ for $(t, \bar{m}) \in \mathcal{D}$.

Please note that the orbit type $M_{(H)}$ need not be connected and its connected components may be of different dimensions. In the following we concentrate our attention on the properties of the connected components of $M_{(H)}$, which we denote by $M_{(H)}$.

Proposition 6. *For every compact subgroup H of G the image of each connected component $M_{(H)}$ of the orbit type $M_{(H)}$ under the orbit mapping $\pi : M \rightarrow M/G$ is a smooth submanifold of the differential space $(M/G, C^\infty(M/G))$.*

Proof. See page 74 of [4]. \square

Proposition 7. *The connected component $\overline{M}_{(H)} = \pi(M_{(H)})$ of the orbit type $\overline{M}_{(H)}$ of M/G is an invariant manifold of every smooth vector field Y on M/G .*

Proof. Let $\overline{\varphi} : \overline{U} \subseteq \mathbb{R} \times M/G \rightarrow M/G$ be the local flow of the vector field Y . For each point $y \in \overline{M}_{(H)}$ there is an open neighborhood \overline{V} of y in M/G such that for every $t \in [0, \varepsilon)$ the map $\overline{\varphi}_t$ is a local diffeomorphism onto its image. Hence $\overline{\varphi}_t(y) \in \overline{M}_{(H)}$ for every $t \in [0, \varepsilon)$. So $\overline{M}_{(H)}$ is an invariant manifold of the vector field Y . \square

Theorem 1. *Let H be a compact subgroup of the Lie group G . Let Y be a smooth vector field on M/G . Then on every connected component $M_{(H)}$ of the orbit type $M_{(H)}$ there is a smooth G invariant vector field X , which is $\pi|_{M_{(H)}}$ related to $Y|_{\overline{M}_{(H)}}$.*

We need the next few results to prove this theorem, which is the main result of this section.

Lemma 1. *The G orbit $G \cdot m = \mathcal{O}$ through $m \in M$ is a submanifold of M .*

Proof. Let S_m be a slice to the G action Φ at m . By Bochner’s lemma, see ([14], p. 306), there is a local diffeomorphism $\psi : T_m M \rightarrow M$, which sends $0_m \in T_m M$ to $m \in M$ and intertwines the $H = G_m$ action

$$\Psi_m : H \times T_m M \rightarrow T_m M : (h, v_m) \mapsto h \star v_m = T_m \Phi_h v_m$$

on $T_m M$ with the H action $\Phi_H : H \times M \rightarrow M : (h, m) \mapsto \Phi_h(m)$. Since $T_m S_m$ is H invariant, it follows that $\psi : T_m S_m \rightarrow S_m$ is a local diffeomorphism which sends 0_m to m . Let L be a complement of \mathfrak{h} in \mathfrak{g} , where \mathfrak{h} is the Lie algebra of H . The map

$$\phi : L \times T_m S_m \rightarrow M : (\xi, v_m) \mapsto \Phi_{\exp \xi}(\psi(v_m)),$$

which sends $(0_L, 0_m)$ to m is a local diffeomorphism that sends an open neighborhood of $(0_L, 0_m)$ in $L \times \{0_m\}$ onto an open neighborhood of m in \mathcal{O} . Thus, \mathcal{O} is a smooth submanifold of M near m . For every $g \in G$, since Φ_g is a diffeomorphism of M , the map $\Phi_g \circ \phi$ is a local diffeomorphism of $(0_L, 0_m)$ in $L \times \{0_m\}$ onto an open neighborhood of $g \cdot m$ in \mathcal{O} . Thus, \mathcal{O} is a submanifold of M . \square

Lemma 2. *For each connected component $M_{(H)}$ of the orbit type $M_{(H)}$ the map*

$$\pi|_{M_{(H)}} : M_{(H)} \rightarrow \overline{M}_{(H)} = \pi(M_{(H)}) : m \mapsto \overline{m} \tag{8}$$

is a smooth surjective submersion, whose typical fiber is an orbit of the G action Φ restricted to $G \times M_{(H)}$.

Proof. The orbit map $\pi : M \rightarrow \overline{M} = M/G$ is a surjective smooth map of the smooth manifold M onto the differential space $(M/G, C^\infty(M/G))$. Hence its restriction $\pi|_{M_{(H)}}$ to the connected component $M_{(H)}$ of the orbit type $M_{(H)}$ and the codomain to $\overline{M}_{(H)} = \pi(M_{(H)})$ is a smooth map of the smooth manifold $M_{(H)}$ onto the differential space $(\overline{M}_{(H)}, C^\infty(\overline{M}_{(H)}))$. By Proposition 6, the differential space $(\overline{M}_{(H)}, C^\infty(\overline{M}_{(H)}))$ is a smooth manifold. Hence $\pi|_{M_{(H)}} : M_{(H)} \rightarrow \overline{M}_{(H)}$ is a smooth map of the smooth manifold $M_{(H)}$ onto the smooth manifold $\overline{M}_{(H)}$. At $\overline{m} \in \overline{M}_{(H)}$, the fiber $(\pi|_{M_{(H)}})^{-1}(\overline{m})$ is the G orbit in $M_{(H)}$ through m , which is a smooth submanifold of $M_{(H)}$. We have $T_m M_{(H)} = \ker T_m \pi|_{M_{(H)}} \oplus (\ker T_m \pi|_{M_{(H)}})^\perp$, using the restriction of the G invariant Riemannian metric on M to $\ker T_m \pi|_{M_{(H)}}$, see

Palais [5]. Because the vector space $(\ker T_m \pi_{|M(H)})^\perp$ is isomorphic to $T_{\bar{m}} \bar{M}_{(H)}$, with the same dimension, it follows that the map $T_m \pi_{|M(H)}$ is surjective. Consequently, the map $\pi_{|M(H)}$ is a submersion. \square

Next we construct a connection on the fibration $\pi_{|M(H)} : M_{(H)} \rightarrow \bar{M}_{(H)}$ and then review some geometric facts about geodesics. Because the G action Φ (1) on the smooth manifold M is proper, it has a G invariant Riemannian metric. Let \mathbf{k} be the restriction of this metric to the smooth submanifold $M_{(H)}$. For each $m \in M_{(H)}$ this yields the G invariant decomposition

$$T_m M_{(H)} = \text{ver}_m \oplus \text{hor}_m, \tag{9}$$

where $\text{ver}_m = \ker T_m \pi_{|M(H)}$ and $\text{hor}_m = (\text{ver}_m)^\perp$, using the metric $\mathbf{k}(m)$ on $T_m M_{(H)}$. The distributions $\text{ver} : M_{(H)} \rightarrow TM_{(H)} : m \mapsto \text{ver}_m$ and $\text{hor} : M_{(H)} \rightarrow TM_{(H)} : m \mapsto \text{hor}_m$ are smooth. Moreover, for every $m \in M_{(H)}$ the map

$$(T_m \pi_{|M(H)})|_{\text{hor}_m} : \text{hor}_m \subseteq T_m M_{(H)} \rightarrow T_{\bar{m}} \bar{M}_{(H)} : v_m \mapsto \bar{v}_m, \tag{10}$$

where $\bar{m} = \pi(m)$, is an isomorphism of vector spaces. Thus, Equations (9) and (10) define an Ehresmann connection \mathcal{E} on the fibration $\pi_{|M(H)} : M_{(H)} \rightarrow \bar{M}_{(H)}$. Because

$$\text{ver}_{g \cdot m} \oplus \text{hor}_{g \cdot m} = T_m \Phi_g(T_m M_{(H)}^c) = T_m \Phi_g \text{ver}_m \oplus T_m \Phi_g \text{hor}_m$$

and $T_m \Phi_g \text{ver}_m \subseteq \text{ver}_{g \cdot m}$ imply $\text{ver}_{g \cdot m} = T_m \Phi_g \text{ver}_m$ and $\text{hor}_{g \cdot m} = T_m \Phi_g \text{hor}_m$, the distributions ver and hor are G invariant. Thus, the connection \mathcal{E} is G invariant.

Let $\tau : TM_{(H)} \rightarrow M_{(H)}$ and $\rho : T^*M_{(H)} \rightarrow M_{(H)}$ be the tangent and cotangent bundle projection maps, respectively. The metric \mathbf{k} on $M_{(H)}$ defines a vector bundle isomorphism

$$\mathbf{k}^\sharp : TM_{(H)} \rightarrow T^*M_{(H)} : v_m \mapsto p_m = \mathbf{k}^\sharp(m)(v_m),$$

where $\langle \mathbf{k}^\sharp(m)(v_m) | v_m \rangle = \mathbf{k}(m)(v_m, v_m)$. The inverse of \mathbf{k}^\sharp is \mathbf{k}^\flat . The metric \mathbf{k} determines the Hamiltonian function

$$E : T^*M_{(H)} \rightarrow \mathbb{R} : p_m \mapsto \frac{1}{2} \mathbf{k}(m)(\mathbf{k}^\flat(m)(p_m), \mathbf{k}^\flat(m)(p_m)), \tag{11}$$

which gives rise to the Hamiltonian system $(E, T^*M_{(H)}, \omega)$, where ω is the canonical symplectic form on $T^*M_{(H)}$. The Hamiltonian vector field X_E on $T^*M_{(H)}$ is defined by $X_E \lrcorner \omega = dE$. For $p_m \in T^*M_{(H)}$ let

$$\varphi_t^{X_E} : T^*M_{(H)} \rightarrow T^*M_{(H)} : p_m \mapsto \varphi_t^{X_E}(p_m)$$

be the local flow of the vector field X_E , which is defined for t in an open interval I_m in \mathbb{R} containing 0. For $v_m \in T_m M_{(H)}$ the curve γ_{v_m} given by $t \rightarrow (\rho \circ \varphi_t^{X_E})(\mathbf{k}^\sharp(m)(v_m))$ is a geodesic on $M_{(H)}$, starting at $m \in M_{(H)}$, for the metric \mathbf{k} . There is an open tubular neighborhood U of the zero section of the cotangent bundle $\rho : T^*M_{(H)} \rightarrow M_{(H)}$ such that the local flow $\varphi_t^{X_E}$ is defined for all $t \in [0, 1]$. For each $m \in M_{(H)}$, the exponential map

$$\text{exp}_m : U_{0_m} \subseteq T_m M_{(H)} \rightarrow M_{(H)} : v_m \mapsto \gamma_{v_m}(1) = (\rho \circ \varphi_1^{X_E})(\mathbf{k}^\sharp(m)(v_m)), \tag{12}$$

is a diffeomorphism onto $V_m = \text{exp}_m(U_{0_m})$, where $U_{0_m} \subseteq U$ is a suitable open neighborhood of $0_m \in T_m M_{(H)}$, see Brickell and Clark [15].

Next we reduce the G symmetry of the Hamiltonian system $(E, T^*M_{(H)}, \omega)$. Because the metric \mathbf{k} on $M_{(H)}$ is G invariant, the smooth Hamiltonian E (11) on $M_{(H)}$ is G invariant, it induces a metric $\bar{\mathbf{k}}$ on $\bar{M}_{(H)}$ such that $\pi^* \bar{\mathbf{k}} = \mathbf{k} \circ (\text{perp}, \text{perp})$, where $\text{perp} : TM_{(H)} \rightarrow$

$(TM_{(H)})^\perp$ is orthogonal projection. The smooth Hamiltonian E (11) on $M_{(H)}$ is G invariant, and hence induces a smooth Hamiltonian function

$$\bar{E} : T^*\bar{M}_{(H)} \rightarrow \mathbb{R} : \bar{p}_m \mapsto \frac{1}{2} \bar{\mathbf{k}}(\bar{m}) (\bar{\mathbf{k}}^b(\bar{m})(\bar{p}_m), \bar{\mathbf{k}}^b(m)(\bar{p}_m)).$$

Since the G orbit map $\pi_{|M_{(H)}} : M_{(H)} \rightarrow \bar{M}_{(H)}$ is smooth, symplectic reduction of the Hamiltonian system $(E, T^*M_{(H)}, \omega)$ leads to the G reduced Hamiltonian system $(\bar{E}, T^*\bar{M}_{(H)}, \bar{\omega})$. The reduced system has a Hamiltonian vector field $X_{\bar{E}}$ defined by $X_{\bar{E}} \lrcorner \bar{\omega} = d\bar{E}$. Its local flow

$$\varphi_t^{X_{\bar{E}}} : T^*\bar{M}_{(H)} \rightarrow T^*\bar{M}_{(H)} : \bar{p}_m \mapsto \varphi_t^{X_{\bar{E}}}(\bar{p}_m)$$

is π related to the local flow $\varphi_t^{X_E}$ of X_E , i.e., $\pi \circ \varphi_t^{X_E} = \varphi_t^{X_{\bar{E}}} \circ \pi$. The curve $\bar{\gamma}_{\bar{v}_m}$ given by $t \mapsto (\bar{\rho} \circ \varphi_t^{X_{\bar{E}}})(\bar{\mathbf{k}}^\sharp(\bar{m})(\bar{v}_m))$ is a geodesic on $\bar{M}_{(H)}$ starting at \bar{m} for the reduced metric $\bar{\mathbf{k}}$. Here $\bar{\rho} : T^*\bar{M}_{(H)} \rightarrow \bar{M}_{(H)}$ is the cotangent bundle projection map. Please note that $\pi \circ \gamma_{v_m} = \bar{\gamma}_{\bar{v}_m} \circ \pi$, where $\bar{v}_m = T_m \pi v_m$. There is an open neighborhood $\bar{U} = T\pi(U)$ of the zero section of $\bar{\rho} : T^*\bar{M}_{(H)} \rightarrow \bar{M}_{(H)}$ such that the local flow $\varphi_t^{X_{\bar{E}}}$ of the reduced vector field $X_{\bar{E}}$ is defined for all $t \in [0, 1]$. The reduced exponential map

$$\bar{\text{exp}}_m : \bar{U}_{\bar{0}_m} \subseteq T^*\bar{M}_{(H)} \rightarrow \bar{M}_{(H)} : \bar{v}_m \mapsto \bar{\gamma}_{\bar{v}_m}(1) = (\bar{\rho} \circ \varphi_1^{X_{\bar{E}}})(\bar{\mathbf{k}}^\sharp(\bar{v}_m))$$

is a diffeomorphism onto $\bar{V}_m = \bar{\text{exp}}_m(\bar{U}_{\bar{0}_m})$, where $\bar{U}_{\bar{0}_m} = \pi(U_{0_m})$.

Proposition 8. *The fibration $\pi_{|M_{(H)}} : M_{(H)} \rightarrow \bar{M}_{(H)} = \pi(M_{(H)})$ is locally trivial.*

Proof. For some $b_m > 0$, the open ball $B_m = \{v_m \in \text{hor}_m \mid \mathbf{k}(m)(v_m, v_m) < b_m\}$ is a subset of U_{0_m} . Then $V_m = \text{exp}_m(B_m)$ is a submanifold of $M_{(H)}$ containing m . Look at the geodesic $\gamma_{T_m \Phi_g v_m} = \gamma_{v_{g \cdot m}}$ on $M_{(H)}$ given by

$$t \mapsto (\rho \circ \varphi_t^{X_E})(\mathbf{k}^\sharp(g \cdot m)(T_m \Phi_g v_m))$$

starting at $g \cdot m$. One has $B_{g \cdot m} = T_m \Phi_g(B_m)$. To see this, observe that there is a $v_m \in T_m M_{(H)}$ such that $v_{g \cdot m} = T_m \Phi_g v_m$, since Φ_g is a diffeomorphism. So

$$b_{g \cdot m} = \mathbf{k}(g \cdot m)(v_{g \cdot m}, v_{g \cdot m}) = \mathbf{k}(g \cdot m)(T_m \Phi_g v_m, T_m \Phi_g v_m) = \mathbf{k}(m)(v_m, v_m) = b_m.$$

Thus, $\text{exp}_{g \cdot m} = \Phi_g \circ \text{exp}_m$ is a diffeomorphism of the open ball $B_{g \cdot m}$ of radius $b_{g \cdot m} = b_m$ contained in $U_{g \cdot m}$ onto a submanifold $V_{g \cdot m} = \text{exp}_{g \cdot m}(B_{g \cdot m}) = \Phi_g(V_m)$.

For every $\bar{m}' \in \bar{V}_m$ let $\bar{\gamma}_{\bar{m}, \bar{m}'} : [0, 1] \rightarrow \bar{V}_m : t \mapsto \bar{\text{exp}}_m t \bar{v}'_m$ be the geodesic in \bar{V}_m joining \bar{m} to \bar{m}' , i.e., $\bar{\gamma}_{\bar{m}, \bar{m}'}(1) = \bar{m}'$. Because $\bar{\text{exp}}_m$ is a diffeomorphism, the vector $\bar{v}'_m \in B_m = \pi(B_m)$ is uniquely determined by \bar{m}' . Let

$$\mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}} : (\pi_{|M_{(H)}})^{-1}(\bar{m}) \rightarrow (\pi_{|M_{(H)}})^{-1}(\bar{m}') : n \rightarrow \gamma_{n, n'}(1) = n',$$

where $\gamma_{n, n'} : [0, 1] \rightarrow V_n : t \mapsto \text{exp}_n t v'_n$ is the horizontal lift of the geodesic $\bar{\gamma}_{\bar{m}, \bar{m}'}$ using the connection \mathcal{E} . Here $n' \in B_n$ with $\pi(n') = \bar{m}'$, $n = g \cdot m$, and $v'_n \in B_n$. The map $\mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}}$ is parallel translation of the fiber $(\pi_{|M_{(H)}})^{-1}(\bar{m})$ along the geodesic $\bar{\gamma}_{\bar{m}, \bar{m}'}$ joining \bar{m} to \bar{m}' in \bar{V}_m using the connection \mathcal{E} .

Consider the mappings

$$\tau_{\bar{V}_m} : \bar{V}_m \times (\pi_{|M_{(H)}})^{-1}(\bar{m}) \rightarrow (\pi_{|M_{(H)}})^{-1}(\bar{V}_m) : (\bar{m}', n) \mapsto \mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}}(n)$$

and the projection mapping

$$\pi_1 = (\pi_{|M(H)})_{|(\pi_{|M(H)})^{-1}(\bar{V}_m)} : (\pi_{|M(H)})^{-1}(\bar{V}_m) \rightarrow \bar{V}_m : n \mapsto \pi_{|M(H)}(n).$$

Then $\tau_{\bar{V}_m}$ is a local trivialization of the fibration defined by the mapping π_1 because for every $n \in (\pi_{|M(H)})^{-1}(\bar{m})$ and every $\bar{m}' \in \bar{V}_m$

$$(\pi_1 \circ \tau_{\bar{V}_m})(\bar{m}', n) = \pi_1(\mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}}(n)) = \bar{m}'.$$

We now show that $\tau_{\bar{V}_m}$ is a diffeomorphism. Define the smooth maps

$$\rho : \bar{V}_m \times (\pi_{|M(H)})^{-1}(\bar{m}) \rightarrow (\pi_{|M(H)})^{-1}(\bar{V}_m) : (\bar{m}', n) \mapsto \mathcal{P}_{-\bar{\gamma}_{\bar{m}, \bar{m}'}}(n)$$

and

$$\sigma : (\pi_{|M(H)})^{-1}(\bar{V}_m) \rightarrow \bar{V}_m \times (\pi_{|M(H)})^{-1}(\bar{m}) : n \mapsto (\pi_1(n), \rho(\pi_1(n), n)).$$

The following calculation shows that $\sigma \circ \tau_{\bar{V}_m} = \text{id}_{\bar{V}_m \times (\pi_{|M(H)})^{-1}(\bar{m})}$.

$$\begin{aligned} (\sigma \circ \tau_{\bar{V}_m})(\bar{m}', n) &= \sigma(\mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}}(n)) = (\pi_1(\mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}}(n)), \rho(\pi_1(\mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}}(n)), \mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}}(n))) \\ &= (\bar{m}', \rho(\bar{m}', \mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}}(n))) = (\bar{m}', (\mathcal{P}_{-\bar{\gamma}_{\bar{m}, \bar{m}'}} \circ \mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}})(n)) = (\bar{m}', n). \end{aligned}$$

Additionally,

$$(\tau_{\bar{V}_m} \circ \sigma)(n) = \tau_{\bar{V}_m}(\pi_1(n), \rho(\pi_1(n), n)) = (\mathcal{P}_{\bar{\gamma}_{\bar{m}, \bar{m}'}} \circ \mathcal{P}_{-\bar{\gamma}_{\bar{m}, \bar{m}'}})(n) = n,$$

that is, $\tau_{\bar{V}_m} \circ \sigma = \text{id}_{(\pi_{|M(H)})^{-1}(\bar{V}_m)}$. Thus, $\tau_{\bar{V}_m}$ is a diffeomorphism.

The preceding argument can be repeated at each point of $M(H)$. Hence the fibration $\pi_{|M(H)} : M(H) \rightarrow \bar{M}(H)$ is locally trivial. \square

Corollary 1. *The locally trivial fibration defined by $\pi_{|M(H)}$ (8) has a local trivialization*

$$\tau_V : V \subseteq G \cdot M_H = M(H) \rightarrow (U = \pi_{|M(H)}(V)) \times G : \Phi_k(g' \cdot m') \mapsto (\bar{m}', kg'), \quad (13)$$

where V is an open G invariant subset of $M(H)$ with $k, g' \in G$ and $m' \in M_H$.

Proof. Suppose that $m'' = g' \cdot m'$ for some $m', m'' \in M_H$ and some $g' \in G$. Then

$$(\pi_{|M(H)}(m'), e) = (\pi_{|M(H)}(m''), e) = \tau_V(e \cdot m'') = \tau_V(m'') = \tau_V(g' \cdot m') = (\pi_{|M(H)}(m'), g').$$

So $g' = e$. In other words, the G orbit of m' in M_H is $\{m'\}$. Hence $\pi_{|M(H)}(m') = \{m'\}$ for every $m' \in M_H$. Thus, the map τ_V (13) is given by

$$\tau_V : V \subseteq G \cdot M_H \rightarrow U \times G : \Phi_k(g' \cdot m') \mapsto (\{m'\}, kg'). \quad (14)$$

Clearly, τ_V is a diffeomorphism. It intertwines the G action Φ with the G action

$$\phi : G \times (U \times G) \rightarrow G \times U : (k, (u, g)) \mapsto (u, kg) \quad (15)$$

and satisfies $\pi_{|V} = \pi_1 \circ \tau_V$, where $\pi_1 : U \times G \rightarrow U : (u, g) \mapsto u$. Hence τ_V is a local trivialization. \square

Lemma 3. *Every smooth vector field Y_U on $U = \pi_{|M(H)}(V)$ is π_1 related to a smooth G invariant vector field \tilde{Y} on $U \times G$.*

Proof. To see this let $\xi \in \mathfrak{g}$, the Lie algebra of G . Let $\tilde{Y}(u, g) = (Y_U(u), T_e L_g \xi)$, where L_g is left translation on G by g . By construction $\tilde{Y}(u, g) \in T_u U \times T_g G = T_{(u,g)}(U \times G)$. So \tilde{Y} is a vector field on $U \times G$, which is smooth. For every $h \in G$ and every $(u, g) \in U \times G$ one has

$$\tilde{Y}(\phi_h(u, g)) = \tilde{Y}(u, hg) = (Y_U(u), T_e L_{hg} \xi) = T_{(u,g)} \phi_h(Y_U(u), T_e L_g \xi) = T_{(u,g)} \phi_h \tilde{Y}(u, g).$$

So \tilde{Y} is a G invariant vector field on $U \times G$. Moreover,

$$T_{(u,g)} \pi_1 \tilde{Y}(u, g) = Y_U(u) = Y_U(\pi_1(u, g)), \text{ for every } (u, g) \in U \times G.$$

So the vector field \tilde{Y} on $U \times G$ and the vector field Y_U on U are π_1 related. \square

Lemma 4. Every smooth vector field Y_U on U is $\pi|_V$ related to a smooth G invariant vector field X on V .

Proof. Pull the vector field \tilde{Y} on $U \times G$ back by the trivialization τ_V (13) to a vector field X on V . Since τ_V intertwines the G action Φ on V with the G action ϕ on $U \times G$, the vector field X is G invariant. For $m' \in V \cap M_H$ and $g \cdot m' \in V$ one has

$$\begin{aligned} T_{g \cdot m'} \pi|_V X(g \cdot m') &= T_{g \cdot m'} \pi|_V (T_{\tau_V(g \cdot m')} \tau_V^{-1} \tilde{Y}(\tau_V(g \cdot m'))) = T_{g \cdot m'} (\pi|_V \circ \tau_V^{-1}) \tilde{Y}(\pi|_{M_H}(m'), g) \\ &= T_{(\pi|_{M_H}(m'), g)} \pi_1 \tilde{Y}(\pi|_{M_H}(m'), g) = Y_U(\pi|_V(g \cdot m')). \end{aligned}$$

Thus, the G invariant vector field X on V is $\pi|_V$ related to the vector field Y_U on U . \square

Proof of Theorem 1. We just have to piece the local bits together. Cover the orbit type $M_{(H)}$ by $\{(V_i, \tau_{V_i})\}_{i \in I}$, where

$$\tau_{V_i} : V_i \subseteq G \cdot M_H \rightarrow U_i \times G : g' \cdot m' \mapsto (\pi|_{V_i}(m'), g')$$

is a local trivialization of the bundle $\pi|_{M_{(H)}} : M_{(H)} \rightarrow \overline{M}_{(H)}$. Let Y be a smooth vector field on $\overline{M}_{(H)} \subseteq \overline{M}$. Since $\pi|_{M_{(H)}}(V_i \cap M_H) = U_i$ and $\pi|_{M_{(H)}}$ is an open mapping, U_i is an open subset of $\overline{M}_{(H)}$. Because $\{V_i\}_{i \in I}$ covers $M_{(H)}$, it follows that $\{U_i\}_{i \in I}$ is an open covering of $\overline{M}_{(H)}$. Applying Lemma 3 to the smooth vector field $Y_{U_i} = Y|_{U_i}$ and then using Lemma 4, we obtain a G invariant vector field X_{V_i} on V_i , which is $\pi|_{V_i}$ related to the vector field Y_{U_i} on U_i . Since Y is a smooth vector field on $\overline{M}_{(H)}$, on $U_i \cap U_j$, where $i, j \in I$, one has $Y_{U_i} = Y_{U_j}$. So on $V_i \cap V_j$ one has $X_{V_i} = X_{V_j}$. Thus, the G invariant vector fields X_{V_i} piece together to give a smooth G invariant vector field X on $M_{(H)}$. Since X_{V_i} is $\pi|_{V_i}$ related to the vector field Y_{U_i} , the vector field X on $M_{(H)}$ is $\pi|_{M_{(H)}}$ related to the vector field Y on $\overline{M}_{(H)}$. \square

3. Vector Fields on M/G

We start with a local argument in a neighborhood of a point $m \in M$ with compact isotropy group H . By Bochner’s lemma there is a local diffeomorphism $\psi : T_m M \rightarrow M$, which sends $0_m \in T_m M$ to $m \in M$ and intertwines the linear H action

$$\Psi_m : H \times T_m M \rightarrow T_m M : (h, v_m) \mapsto h \star v_m = T_m \Phi_h v_m \tag{16}$$

with the H action $\Phi_H : H \times M \rightarrow M : (h, m) \mapsto \Phi_h(m)$. Because the G action Φ on the smooth manifold M is proper, it has a G invariant Riemannian metric \mathbf{k} . Using the restriction of \mathbf{k} to $T_m M$, we define $E = T_m(G \cdot m)^\perp \subseteq T_m M$. Then there is an H invariant open ball $B \subseteq E$ centered at 0_m with B contained in the domain of the local diffeomorphism ψ such that $\psi(B) = S_m$ is a slice to the G action on M at m .

We now construct a model for the H orbit space B/H of the restriction to B of the linear action Ψ_m on H on $T_m M$. Let $\{v_i\}_{i=1}^n$ be a basis of the vector space E . Hence E

is isomorphic to \mathbb{R}^n . Let $\mathbf{x} = (x_1, \dots, x_n)$ be coordinates on \mathbb{R}^n with respect to the basis $\{v_i\}_{i=1}^n$. Let $\{\sigma_j\}_{j=1}^\ell$ be a set of generators for the algebra of H invariant polynomials on \mathbb{R}^n . Let $\mathbf{y} = (y_1, \dots, y_\ell)$ be coordinates on \mathbb{R}^ℓ . The orbit map of the H action on \mathbb{R}^n is

$$\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n/H \subseteq \mathbb{R}^\ell : \mathbf{x} \mapsto \mathbf{y} = (\sigma_1(\mathbf{x}), \dots, \sigma_\ell(\mathbf{x})). \tag{17}$$

By Schwarz' theorem $f \in C^\infty(\mathbb{R}^n)$ is H invariant, i.e., $f \in C^\infty(\mathbb{R}^n)^H$, if and only if there is a function $F \in C^\infty(\mathbb{R}^\ell)$ such that $f(\mathbf{x}) = F(\sigma_1(\mathbf{x}), \dots, \sigma_\ell(\mathbf{x}))$ for every $\mathbf{x} \in \mathbb{R}^n$. Smooth functions on the orbit space \mathbb{R}^n/H are restrictions to \mathbb{R}^n/H of smooth functions on \mathbb{R}^ℓ . For every $\bar{f} \in C^\infty(\mathbb{R}^n/H)$ the pull back $\rho^*\bar{f}$ by the orbit map ρ is given by

$$\rho^*\bar{f}(\mathbf{x}) = (\bar{f} \circ \rho)(\mathbf{x}) = \bar{f}(\sigma_1(\mathbf{x}), \dots, \sigma_\ell(\mathbf{x})).$$

The H orbit map $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^n/H \subseteq \mathbb{R}^\ell$ is a smooth map of differential spaces. We are interested in B/H , the space of H orbits on the open ball B in \mathbb{R}^n . Restricting ρ to the domain $B \subseteq \mathbb{R}^n$ and the codomain $\Sigma = \rho(B) \subseteq \mathbb{R}^\ell$ gives

$$\rho_B : B \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n/H \subseteq \mathbb{R}^\ell : \mathbf{x} \mapsto \mathbf{y} = (\sigma_1(\mathbf{x}), \dots, \sigma_\ell(\mathbf{x})), \tag{18}$$

which is a surjective smooth map of differential spaces.

Lemma 5. *Let $\rho_B : B \rightarrow B/H \subseteq \mathbb{R}^\ell$ be the orbit mapping of a linear action of a compact Lie group H on an open ball B in \mathbb{R}^n . For every smooth vector field Y on B/H there is a smooth H invariant vector field X on B , which is ρ_B related to Y , i.e., $Y(\bar{f}) = ((\rho_B)_* \circ X \circ (\rho_B)^*)(\bar{f})$ for every $\bar{f} \in C^\infty(B/H)$.*

Proof. Let Y be a smooth vector field on B/H . Since B/H is a differential subspace of \mathbb{R}^ℓ , in coordinates $\mathbf{y} = (y_1, \dots, y_\ell)$ on \mathbb{R}^ℓ we may write $Y(\mathbf{y}) = \sum_{i=1}^\ell g_i(\mathbf{y}) \frac{\partial}{\partial y_i}$, where $g_i \in C^\infty(B/H)$ is the restriction to B/H of a smooth function on \mathbb{R}^ℓ .

We begin the proof by showing that the orbit space B/H is connected. Observe that the open ball $B \subseteq \mathbb{R}^n$ is centered at the origin and the action of H on B is the restriction of the linear action of H on T_mM , see Equation (16). The linearity of the action of H on B implies that it commutes with scalar multiplication. Moreover, the origin $\mathbf{0} \in \mathbb{R}^n$ is H invariant so that it is an orbit of H . Hence $\bar{\mathbf{0}} = \rho_B(\mathbf{0}) \in B/H$. Let $\mathbf{x} \in B$. For each $t \in [0, 1]$ and every $h \in H$ we have $h \star (t\mathbf{x}) = t(h \star \mathbf{x})$. Therefore the line segment $[0, 1] \rightarrow B : t \mapsto t\mathbf{x}$ joins H orbits through the points $\mathbf{0}$ and \mathbf{x} . Thus, the H orbit through $\mathbf{0}$ and the H orbit through \mathbf{x} belong to the same connected component of B/H . This implies that B/H is connected.

The connectedness of B/H ensures that there is a unique principal type (K) whose corresponding orbit type $B_{(K)}$ is open and dense in B , see Duistermaat and Kolk ([13], p.118). Moreover, the orbit space $B_{(K)}/H$ is a connected smooth manifold, and

$$\rho|_{B_{(K)}} : B_{(K)} \rightarrow B_{(K)}/H : \mathbf{x} \mapsto H \star \mathbf{x}$$

is a locally trivial fibration, whose fiber over $\rho|_{B_{(K)}}(\mathbf{x})$ is the H orbit $H \star \mathbf{x}$. Hence for every $\mathbf{y} \in B_{(K)}/H \subseteq B/H$ there exists an open neighborhood V of \mathbf{y} in $B_{(K)}/H$ such that $W = \rho^{-1}(V)$ is trivial. In other words, there is a diffeomorphism $\tau : W \rightarrow H \times V$ such that $\rho|_W = \text{pr}_2 \circ \tau$, where $\text{pr}_2 : H \times V \rightarrow V$ is projection on the second factor. This implies that there is a smooth H invariant vector field X_W on $W \subseteq B_{(K)}$, which is $\rho|_W$ related to the restriction $Y|_V$ of Y to $V = \rho(W)$.

Repeating the above argument at each point $\mathbf{y} \in B_{(K)}/H$ leads to a covering $\{W_\alpha\}_{\alpha \in I}$ of $B_{(K)}$ by H invariant open subsets W_α of $B_{(K)}$ on which there exists an H invariant vector field X_{W_α} , which is ρ related to the restriction of the vector field Y to $V_\alpha = W_\alpha/H$. Using

an H invariant partition of unity on $B_{(K)}$, we obtain a vector field $X_{B_{(K)}}$ on $B_{(K)}$, which is $\rho|_{B_{(K)}}$ related to $Y|_{B_{(K)}/H}$, i.e., $Y|_{B_{(K)}/H} = (\rho|_{B_{(K)}})_* \circ X_{B_{(K)}} \circ (\rho|_{B_{(K)}})^*$.

The module $\mathfrak{X}(B)^H$ of H invariant smooth vector fields on B is finitely generated by polynomial vector fields, see [16], and we denote a generating set by $\{X_j\}_{j=1}^N$. Hence, every H invariant smooth vector field X_B on B is of the form $X_B = \sum_{j=1}^N h_j X_j$ for some $h_1, \dots, h_n \in C^\infty(B)$. Similarly, every K invariant smooth vector field on $B_{(K)}$ can be written as $\sum_{j=1}^N f_j X_j$, where $f_j \in C^\infty(B_{(K)})$. Since $B_{(K)}$ is open and dense in B , a generic $f \in C^\infty(B_{(K)})$ need not extend to a smooth function on B . Therefore a generic vector field on $B_{(K)}$ need not extend to a smooth vector field on B . On the other hand, the H invariant vector field $X_{B_{(K)}}$ on $B_{(K)}$, is obtained above from a smooth bounded vector field Y on B/H . Therefore

$$X_{B_{(K)}}(\mathbf{x}) = \sum_{j=1}^N k_{j|B_{(K)}}(\mathbf{x}) X_j(\mathbf{x}), \tag{19}$$

for each $\mathbf{x} \in B_{(K)} \subseteq B \subseteq \mathbb{R}^n$, where every $k_j \in C^\infty(\mathbb{R}^n)$, and $k_{j|B_{(K)}}$ is the restriction of k_j to $B_{(K)}$.

Since $B_{(K)}$ is open and dense in B , we may define

$$X(\mathbf{x}) = \begin{cases} X_{B_{(K)}}(\mathbf{x}) & \text{if } \mathbf{x} \in B_{(K)} \\ \sum_{i=1}^N \lim_{k \rightarrow \infty} k_j(\mathbf{x}_k) X_j(\mathbf{x}_k), & \text{where } \{\mathbf{x}_k\} \subseteq B_{(K)} \text{ and} \\ & \mathbf{x} = \lim_{k \rightarrow \infty} \mathbf{x}_k \in B \setminus B_{(K)}, \end{cases} \tag{20}$$

provided that $\lim_{k \rightarrow \infty} k_j(\mathbf{x}_k)$ exists and is unique. Since the vector fields $X_j(\mathbf{x}_k)$ are smooth on B ,

$$\lim_{k \rightarrow \infty} k_j(\mathbf{x}_k) X_j(\mathbf{x}_k) = (\lim_{k \rightarrow \infty} k_j(\mathbf{x}_k)) X_j(\lim_{k \rightarrow \infty} \mathbf{x}_k) \text{ for every } j = 1, \dots, N.$$

Moreover, since $B_{(K)}$ is open and dense in B , it is open and dense in \bar{B} , the closure of B . In Equation (19), each function $k_{j|B_{(K)}}$ is the restriction to $B_{(K)} \subseteq B \subseteq \mathbb{R}^n$ of a smooth function k_j on \mathbb{R}^n . Moreover, the choice of polynomial basis $\{X_j\}_{j=1}^N$ ensures that the right-hand side of Equation (19) extends to the closure \bar{B} of B . Hence all the the limits $\lim_{k \rightarrow \infty} k_j(\mathbf{x}_k)$ in Equation (20) exist, and $X(\mathbf{x})$ is defined for all $\mathbf{x} \in B$.

We need to show that this definition of X on B depends only on $X_{B_{(K)}}$. Since each k_j is continuous on B and its first partial derivatives are bounded on \bar{B} , it follows that k_j are uniformly continuous on \bar{B} . In particular, if $c : [0, 1] \rightarrow \bar{B}$ is a smooth curve, such that $c([0, 1]) \subseteq B_{(K)}$ and $c(1) \in B$, then

$$k_{j|B}(c(1)) = k_{j|B}(c(0)) + \int_0^1 \frac{\partial k_{j|B}}{\partial t}(c(t)) dt = k_{j|B_{(K)}}(c(0)) + \int_0^1 \frac{\partial k_{j|B_{(K)}}}{\partial t}(c(t)) dt.$$

Thus, the values of k_j on B are uniquely determined by $k_{j|B_{(K)}}$. Repeating this argument for all the first-order partial derivatives of k_j , we deduce that the first-order partial derivatives of k_j on B are uniquely determined by $k_{j|B_{(K)}}$ and its first partial derivatives. Continuing this process for every partial derivative of every order shows that the restriction of k_j to B is uniquely determined by $k_{j|B_{(K)}}$.

The above argument applies to each of the functions f_i for $i = 1, \dots, n$ in Equation (19) and ensures that the H invariant vector field $X_{B_{(K)}}$ (19), thought of as the smooth section $B_{(K)} \rightarrow TB_{(K)} = B_{(K)} \times \mathbb{R}^n$, extends to a smooth H invariant map $X : B \rightarrow B \times \mathbb{R}^n$, which is ρ related to the section $Y : B/H \rightarrow T(B/H)$. It remains to show that X is a vector field on B .

By construction $X_{B_{(K)}}$ is an H invariant vector field on an open dense subset $B_{(K)}$ of B , which is $\rho|_{B_{(K)}}$ related to the vector field $Y|_{B_{(K)}/H}$. The closure of $B_{(K)}$ in B is the union of

orbit types $B_{(J)}$, where $(J) \leq (K)$. Suppose that $\mathbf{x} = \lim_{k \rightarrow \infty} \mathbf{x}_k \in B_{(J)}$, where $\mathbf{x}_k \in B_{(K)}$ for all $k \in \mathbb{Z}_{\geq 1}$. Then $\mathbf{y} = \rho_B(\mathbf{x}) \in B_{(J)}/H$ and

$$\begin{aligned} T_{\mathbf{x}}\rho_B(X(x)) &= T_{\mathbf{x}}\rho_B(X(\lim_{k \rightarrow \infty} \mathbf{x}_k)) = \lim_{k \rightarrow \infty} T_{\mathbf{x}}\rho_B X(\mathbf{x}_k) \\ &= \lim_{k \rightarrow \infty} Y(\rho_B(\mathbf{x}_k)) = Y(\rho_B(\mathbf{x})) = Y(\mathbf{y}), \end{aligned}$$

because $Y|_{B_{(K)}/H}$ is the restriction to $B_{(K)}/H$ of a smooth, and hence continuous, vector field on B/H . By Proposition 7, for every orbit type $B_{(J)}$, the manifold $B_{(J)}/H$ is an invariant manifold of the vector field Y . So $Y|_{B_{(J)}/H}$ is a vector field on $B_{(J)}/H$. Hence for every $\mathbf{x} \in B$,

$$X(\mathbf{x}) \in (T_{\mathbf{x}}\rho_B)^{-1}(Y(\mathbf{y})) \subseteq (T_{\mathbf{x}}\rho_B)^{-1}(T_{\mathbf{y}}(B_{(J)}/H)) \subseteq T_{\mathbf{x}}B.$$

Therefore X is a smooth vector field on B , which is ρ_B related to the vector field Y on B/H . \square

The aim of the rest of this section is to prove.

Theorem 2. Let $\Phi : G \times M \rightarrow M$ be a proper action of a Lie group G on a connected smooth manifold M with orbit map $\pi : M \rightarrow M/G$. Every smooth vector field on the locally closed subcartesian differential space $(M/G, C^\infty(M/G))$ is π related to a smooth G invariant vector field on M .

First we prove.

Lemma 6. Let S_m be a slice to the G action Φ at $m \in M$ and suppose that \widehat{X} is a smooth H invariant vector field on some H invariant open neighborhood U_m of m in S_m . Here H is the isotropy group G_m at m . Then the vector field \widehat{X} extends to a smooth G invariant vector field X on M .

Proof. Let $U_m \subseteq S_m$ be an H invariant open subset of S_m containing m . Because S_m is a slice, $G \cdot U_m = \{\Phi_g(U_m) \in M \mid g \in G\}$ is a G invariant open subset of M , which contains the G orbit $G \cdot m$. On $G \cdot U_m$ define the vector field $X = \{(\Phi_g)_* \widehat{X} \mid g \in G\}$. We check that X is well defined. Suppose that $g \cdot s_m = g' \cdot s'_m$, where $g, g' \in G$ and $s_m, s'_m \in S_m$. Since S_m is a slice, it follows that $g^{-1}g' = h \in H$. Hence

$$(\Phi_{g'})_* \widehat{X} = (\Phi_{gh})_* \widehat{X} = (\Phi_g)_*((\Phi_h)_* \widehat{X}) = (\Phi_g)_* \widehat{X},$$

where the last equality above follows because the vector field \widehat{X} is H invariant. So the vector field X on $G \cdot U_m$ is well defined and by definition is G invariant.

Next we show that X is smooth. Let L be a complement to the Lie algebra \mathfrak{h} of the Lie group H in the Lie algebra \mathfrak{g} of the Lie group G . For every $\zeta \in L$ and $\eta \in \mathfrak{h}$ consider the map $\mu : G \rightarrow (\exp L)H : \exp(\zeta + \eta) \mapsto \exp \zeta \exp \eta$, which sends the identity element e_G of G to $e_G \cdot e_H = e_G$. It is a local diffeomorphism, since its tangent $T_{e_G} \mu : \mathfrak{g} \rightarrow L \oplus \mathfrak{h} = \mathfrak{g}$ is the identity map. Thus, there are open subsets V_G, V_L , and V_H of $e_G, 0_L$, and e_H , respectively, such that $\mu(V_G) = \exp V_L \cdot V_H$. Hence every $g \in V_G$ may be written uniquely as $g = (\exp \zeta)h$ for some $\zeta \in V_L$ and some $h \in V_H$. For every $s_m \in U_m \subseteq S_m$ we have

$$\begin{aligned} (\Phi_g)_* X(s_m) &= (\Phi_{(\exp \zeta)h})_* X(s_m) = (\Phi_{\exp \zeta})_* ((\Phi_h)_* X(s_m)) \\ &= (\Phi_{\exp \zeta})_* X(s_m), \text{ since } X \text{ is } H \text{ invariant on } S_m \\ &= T_{s_m} \Phi_{s_m} X(\Phi_{\exp -\zeta}(s_m)). \end{aligned} \tag{21}$$

Consider the local diffeomorphism

$$\varphi : V_L \times S_m \rightarrow G \cdot S_m : (\zeta, s) \mapsto \Phi_{\exp \zeta}(s) = \Phi(\exp \zeta, s).$$

Then $\Phi_{\exp \xi} = \varphi_{|\{\xi\} \times S_m}$. So if $q = \varphi(\xi, s) = \Phi_{\exp \xi}(s)$, then $\Phi_{\exp \xi}^{-1}(q) = s$. Let W be a neighborhood of $\{0\} \times S_m \subseteq V_L \times S_m$ such that φ restricted to W yields a diffeomorphism $\varphi|_W : W \subseteq V_L \times S_m \rightarrow U = \varphi(W) \subseteq G \cdot S_m$. For $s \in S_m$ let $e^{t\widehat{X}_m}(s)$ be the integral curve of the vector field \widehat{X}_m starting at s . Since X_m is a G invariant extension of \widehat{X}_m to a vector field on $G \cdot S_m$ (whose smoothness we want to prove) of a smooth H invariant vector field \widehat{X}_m on S_m , it follows that $X_m|_{S_m} = \widehat{X}_m$ is a smooth vector field on S_m . Therefore

$$\varphi_t^{\widehat{X}_m}(s) = \varphi_t^{X_m|_{S_m}}(s) = \varphi_t^{X_m}(s), \tag{22}$$

for all $s \in S_m$. Consider a curve c_q in $U \subseteq G \cdot S_m$ starting at $q = \Phi_{\exp \xi}(s)$ defined by $c_q(t) = \Phi_{\exp \xi}(\varphi_t^{\widehat{X}_m}(\Phi_{\exp \xi}^{-1}(q)))$. Using Equation (22) we obtain $\dot{c}_q(0) = T_q \Phi_{\exp \xi}(\widehat{X}_m(s)) = X(q)$ for all $q \in U$. Since the family of curves $t \rightarrow c_q(t)$ depends smoothly on $q \in U$ and U is an open subset of $G \cdot S_m$ containing S_m , it follows that X_U is a smooth vector field on U . For any $m' \in G \cdot S_m$ there exists a $g \in G$ such that the open set $\Phi_g(U)$ contains m' . Since X is G invariant, smoothness of X on U ensures that X is smooth on $\Phi_g(U)$. Hence X is a smooth vector field on $G \cdot S_m$.

The above argument can be repeated at each point $m \in M$. This leads to a covering $\{G \cdot S_{m_\alpha}\}_{\alpha \in I}$ of M by open G invariant subsets $G \cdot S_{m_\alpha}$, where S_{m_α} is a slice at m_α for the action of G on M and I is an index set. If Y is a vector field on M , then for each $\alpha \in I$ there exists a G invariant vector field X_{m_α} on $G \cdot S_{m_\alpha}$ that is π related to the restriction of Y to $(G \cdot S_{m_\alpha})/G \subseteq M/G$. Using a G invariant partition of unity on M subordinate to the covering $\{G \cdot S_{m_\alpha}\}_{\alpha \in I}$, we can glue the pieces X_{m_α} together to obtain a smooth G invariant vector field X on M , which is π related to the vector field Y on M/G . \square

Proof of Theorem 2. Applying Lemma 6 to the push forward by the local diffeomorphism $\psi|_B : B \subseteq T_m M \rightarrow U_m \subseteq S_m \subseteq M$, given by the Bochner lemma, of the vector field on B constructed in Lemma 5, proves Theorem 2. \square

Proposition 9. *If Y is a derivation of $C^\infty(M/G)$, which is π related to a derivation X of $C^\infty(M)^G$, then Y is a smooth vector field on M/G .*

Proof. Since M is a smooth manifold, X is a smooth G invariant vector field on M , which is π related to derivation Y of $C^\infty(M/G)$. Thus, the image under π of a maximal integral curve of X on M , is a maximal integral curve of Y on M/G . Hence Y is a smooth vector field on the locally closed subcartesian differential space $(M/G, C^\infty(M/G))$. \square

4. Differential 1-Forms on the Orbit Space

In this section we define the notion of a differential 1-form on the orbit space M/G of a proper group action $\Phi : G \times M \rightarrow M : (m, g) \mapsto g \cdot m$ on a smooth manifold M with orbit map $\pi : M \rightarrow M/G : m \mapsto \bar{m} = G \cdot m$. We show that the differential 1-forms on M/G together with the exterior derivative generate a differential exterior algebra.

Theorem 2 and Proposition 9 show that Y is a vector field on M/G if and only if there is a G invariant vector field X on M , which is π related to Y , i.e., every integral curve of Y is the image under the map π of an integral curve of X . Let $\Lambda^1(M/G)$ be the set of differential 1-forms on M/G , i.e., the set of linear mappings

$$\theta : \mathfrak{X}(M/G) \rightarrow C^\infty(M/G) : Y \mapsto \theta(Y) = \langle \theta | Y \rangle,$$

which are linear over the ring $C^\infty(M/G)$, i.e., $\langle \theta | \bar{f} Y \rangle = \bar{f} \langle \theta | Y \rangle$ for every $\bar{f} \in C^\infty(M/G)$ and every $Y \in \mathfrak{X}(M/G)$.

In order to prove some basic properties of differential 1-forms on M/G , we need to prove some properties of the G orbit map π (2).

The map

$$T_m\pi : T_mM \rightarrow T_{\bar{m}}(M/G) = \text{span}_{\mathbb{R}}\{Y(\bar{m}) \mid Y \in \mathfrak{X}(M/G)\} : v_m = X(m) \mapsto Y(\bar{m}),$$

where $X \in \mathfrak{X}(M)^G$ and Y is the vector field on M/G constructed in Proposition 2, is the tangent of the map π at $m \in M$. To show that $T_m\pi$ is well defined we argue as follows. Suppose that $v_m = X'(m)$, where $X' \in \mathfrak{X}(M)^G$. Then

$$T_m\pi(X(m) - X'(m)) = T_m\pi(v_m - v_m) = 0,$$

since $T_m\pi$ is a linear map.

Lemma 7. For each $m \in M$

$$\ker T_m\pi = \text{span}_{\mathbb{R}}\{X_{\xi}(m) \in T_mM \mid \xi \in \mathfrak{g}\}, \tag{23}$$

where \mathfrak{g} is the Lie algebra of G .

Proof. By definition $\pi^{-1}(\bar{m}) = G \cdot m$. Thus,

$$T_m(\pi^{-1}(\bar{m})) = T_m(G \cdot m) = \text{span}_{\mathbb{R}}\{X_{\xi} \in T_mM \mid \xi \in \mathfrak{g}\}. \tag{24}$$

The curve $\gamma_m : I_m \subseteq \mathbb{R} \rightarrow M : t \mapsto \exp t\xi \cdot m$ is an integral curve of X_{ξ} starting at m . So $\pi(\gamma_m(t)) = \pi(m) = \bar{m}$ for every $t \in I_m$. Thus, $T_m\pi X_{\xi}(m) = \frac{d}{dt}\bigg|_{t=0} \pi(\gamma_m(t)) = 0_{\bar{m}}$, i.e., $X_{\xi}(m) \in \ker T_m\pi$. Consequently, $\text{span}_{\mathbb{R}}\{X_{\xi} \in T_mM \mid \xi \in \mathfrak{g}\} \subseteq \ker T_m\pi$.

To prove the reverse inclusion, we argue as follows. Since $m \in M$, it follows that $m \in M_{(H)}$, where $H = G_m$. Let $M_{(K)}$ be the maximal orbit type of the proper G action on M . The maximal orbit type $M_{(K)}$ is a dense open subset of M , whose boundary $\partial M_{(K)} = \text{cl}(M_{(K)}) \setminus M_{(K)}$ contains $M_{(H)}$, since the orbit types of the G action stratify M . Suppose that v_m is a nonzero vector in $\ker T_m\pi$. There is a vector field X on M with $X(m) = v_m$ with an integral curve $\gamma_p : I_p \subseteq \mathbb{R} \rightarrow M : t \mapsto \varphi_t^X(p)$ starting at $p \in M_{(K)}$ such that $\gamma_p(\tau) = m$ for some $\tau \in I_p \cap \mathbb{R}_{>0}$. We may suppose that $\gamma_p([0, \tau]) \subseteq M_{(K)}$. Since $\bar{M}_{(K)} = \pi(M_{(K)})$ is a smooth submanifold of the differential space $(M/G, C^\infty(M/G))$, the curve $\Gamma_m : (0, \tau] \rightarrow \bar{M}_{(K)} : t \mapsto \gamma_p(\tau - t)$ is a smooth integral curve of the vector field $-X$ such that $\Gamma_m((0, \tau]) \subseteq M_{(K)}$. Hence on $(0, \tau]$ the curve $\pi \circ \Gamma$ on the smooth manifold $\bar{M}_{(K)}$ is smooth. Thus,

$$\begin{aligned} \frac{d}{dt}(\pi \circ \Gamma_m)(t) &= \frac{d}{dt} \pi(\varphi_t^{-X}(m)) = \frac{d}{dt}\bigg|_{s=0} \pi(\varphi_{t+s}^{-X}(m)) = \frac{d}{dt}\bigg|_{s=0} \pi(\varphi_t^{-X}(\varphi_s^{-X}(m))) \\ &= T_{\varphi_t^{-X}(m)} \pi(T_m\pi(-X(m))) = 0_{\pi(\varphi_t^{-X}(m))}, \end{aligned}$$

since $X(m) = v_m \in \ker T_m\pi$. Thus, the curve $\pi \circ \Gamma_m$ is constant, since $\bar{M}_{(K)}$ is a smooth manifold. Because the curve $\pi \circ \Gamma_m$ is continuous on $[0, \tau]$, we obtain $\pi(\Gamma_m(t)) = \pi(\Gamma_m(0)) = \pi(m) = \bar{m}$. Hence $\Gamma_m(t) \subseteq \pi^{-1}(\bar{m})$ for all $t \in [0, \tau]$. But $\lim_{t \searrow 0} \Gamma'_m(t) = v_m$. So $v_m \in T_m\pi^{-1}(\bar{m})$. Hence

$$\ker T_m\pi \subseteq T_m\pi^{-1}(\bar{m}) = \text{span}_{\mathbb{R}}\{X_{\xi}(m) \in T_mM \mid \xi \in \mathfrak{g}\},$$

where the equality follows from Equation (24). This verifies Equation (23). \square

A differential 1-form ω on M is semi-basic with respect to the G action Φ if and only if $X_{\xi} \lrcorner \omega = 0$ for every $\xi \in \mathfrak{g}$, the Lie algebra of G .

Proposition 10. For every $\theta \in \Lambda^1(M/G)$, the differential 1-form $\pi^*\theta$ on M is G invariant and semi-basic.

Proof. By definition of π^* the map

$$\pi^*\theta : \mathfrak{X}(M)^G \rightarrow C^\infty(M)^G : X \mapsto \langle \pi^*\theta|X \rangle \tag{25}$$

is linear, since the map $\mathfrak{X}(M)^G \rightarrow \mathfrak{X}(M/G) : X \mapsto Y$ is linear. Moreover, for any $f \in C^\infty(M)^G$

$$\begin{aligned} \langle \pi^*\theta|fX \rangle &= \pi^*(\langle \theta|\bar{f}Y \rangle), \text{ since the map } X \mapsto Y \text{ is a module} \\ &\quad \text{homomorphism} \\ &= \pi^*(\bar{f} \langle \theta|Y \rangle), \text{ because } \theta \in \Lambda^1(M/G) \\ &= \pi^*(\bar{f}) \pi^*(\langle \theta|Y \rangle) = f \langle \pi^*\theta|X \rangle. \end{aligned}$$

Thus, $\pi^*\theta \in \Lambda^1(M/G)$. For every $\zeta \in \mathfrak{g}$ one has $L_{X_\zeta}(\langle \pi^*\theta|X \rangle) = L_{X_\zeta}(\pi^*(\langle \theta|Y \rangle)) = 0$, because $\pi^*(\langle \theta|Y \rangle) \in C^\infty(M)^G$. So $\pi^*\theta$ is a semi-basic 1-form on M . \square

Proposition 11. *Let ϑ be a G invariant semi-basic differential 1-form on M . Then there is a 1-form θ on M/G such that $\vartheta = \pi^*\theta$.*

Proof. Given $Y \in \mathfrak{X}(M/G)$, there is an $X \in \mathfrak{X}(M)^G$, which is π related to Y , i.e., $T_m\pi X(m) = Y(\pi(m))$ for every $m \in M$. It is clear that the definition of θ needs to be

$$\pi^*(\langle \theta|Y \rangle) = \langle \vartheta|X \rangle. \tag{26}$$

It remains to show that θ is well defined. Since the 1-form ϑ and the vector field X are G invariant, we obtain

$$\Phi_g^*(\langle \vartheta|X \rangle)(m) = \langle \Phi_g^*\vartheta|\Phi_g^*X \rangle(\Phi_g(m)) = \langle \vartheta|X \rangle(m),$$

for every $(g, m) \in G \times M$. Thus, the function $M \rightarrow \mathbb{R} : m \mapsto \langle \vartheta|X \rangle(m)$ is smooth and G invariant. We now show that the mapping $\theta : \mathfrak{X}(M/G) \rightarrow C^\infty(M/G)$, where θ is given in Equation (26), is well defined. Suppose that $X' \in \mathfrak{X}(M)^G$ such that X' is π related to Y . Then $T_m\pi(X(m) - X'(m)) = Y(m) - Y(m) = 0$ for every $m \in M$. So $(X(m) - X'(m)) \in \text{span}_{\mathbb{R}}\{X_\zeta(m) \in T_mM \mid \zeta \in \mathfrak{g}\}$, by Proposition 10. Thus,

$$\langle \vartheta|X \rangle = \langle \vartheta|(X - X') \rangle + \langle \vartheta|X' \rangle = \langle \vartheta|X' \rangle,$$

since the 1-form ϑ on M is semi-basic. This shows that the map $\theta : \mathfrak{X}(M/G) \rightarrow C^\infty(M/G)$ is well defined. From Equation (26) it follows that θ is a linear mapping and that $\langle \theta|\bar{f}Y \rangle = \bar{f} \langle \theta|Y \rangle$ for every $\bar{f} \in C^\infty(M/G)$. Hence θ is a differential 1-form on M/G , i.e., $\theta \in \Lambda^1(M/G)$. Every $X \in \mathfrak{X}(M)^G$ is π related to a $Y \in \mathfrak{X}(M/G)$. Thus, $\langle \pi^*\theta|X \rangle = \pi^*(\langle \theta|Y \rangle) = \langle \vartheta|X \rangle$, that is, $\vartheta = \pi^*\theta$. \square

5. De Rham’s Theorem

In this section we construct an exterior algebra of differential forms on the orbit space M/G with an exterior derivative and show that de Rham’s theorem holds for the sheaf of differential exterior algebras.

Let $\ell \in \mathbb{Z}_{\geq 1}$. A differential ℓ -form θ on M/G is an element of $L_{\text{alt}}^\ell(T(M/G), \mathbb{R})$, the set of alternating ℓ multilinear real valued mappings on $T(M/G) = \mathfrak{X}(M/G)$, namely

$$\begin{aligned} \theta : \overbrace{\mathfrak{X}(M/G) \times \cdots \times \mathfrak{X}(M/G)}^\ell &\rightarrow C^\infty(M/G) : \\ (Y_1, \dots, Y_\ell) &\mapsto Y_\ell \lrcorner (\cdots (Y_1 \lrcorner \theta) \cdots) = \langle \theta|(Y_1, \dots, Y_\ell) \rangle, \end{aligned}$$

which is linear over $C^\infty(M/G)$, that is, $\langle \theta|(Y_1, \dots, \bar{f}Y_i, \dots, Y_\ell) \rangle = \bar{f} \langle \theta|(Y_1, \dots, Y_\ell) \rangle$ for every $1 \leq i \leq \ell$ and every $\bar{f} \in C^\infty(M/G)$. A differential 0-form on M/G is a smooth

function on M/G . Let $\Lambda^\ell(M/G)$ be the real vector space of differential ℓ -forms on M/G . For each $\bar{m} \in M/G$ let $\Lambda_{\bar{m}}^\ell(M/G) = \text{span}_{\mathbb{R}}\{\theta(\bar{m}) \in L_{\text{alt}}^\ell(T_{\bar{m}}(M/G), \mathbb{R}) \mid \theta \in \Lambda^\ell(M/G)\}$.

Proposition 12. Let $\theta \in \Lambda^\ell(M/G)$ with $\ell \in \mathbb{Z}_{\geq 1}$. Then the ℓ -form $\vartheta = \pi^*\theta \in \Lambda_{\text{sb}}^\ell(M)^G$, the set of semi-basic G invariant ℓ -forms on M . Here

$$(\pi^*\theta)(m)(X_1(m), \dots, X_\ell(m)) = \theta(\pi(m))(T_m\pi X_1(m), \dots, T_m\pi X_\ell(m)),$$

for every $m \in M$ and every $X_j \in \mathfrak{X}(M)^G$ for $1 \leq j \leq \ell$.

Proof. The proof is analogous to the proof of Proposition 10 for 1-forms on M/G and is omitted. \square

Proposition 13. Let $\vartheta \in \Lambda_{\text{sb}}^\ell(M)^G$, where $\ell \in \mathbb{Z}_{\geq 1}$. Then there is an ℓ -form $\theta \in \Lambda^\ell(M/G)$ such that $\vartheta = \pi^*(\theta)$.

Proof. The proof is analogous to the proof of Proposition 11 for G invariant semi-basic 1-forms on M and is omitted. \square

We now define the exterior algebra $\Lambda(M/G)$ of differential forms on M/G . Let $\theta \in \Lambda^h(M/G)$ and $\phi \in \Lambda^k(M/G)$. The exterior product is the $h+k$ form $\theta \wedge \phi$ on M/G corresponding to the G invariant semi-basic $h+k$ -form $\pi^*\theta \wedge \pi^*\phi$ on M . Then $(\Lambda(M/G) = \sum_{\ell} \oplus \Lambda^\ell(M/G), \wedge)$ is an exterior algebra of differential forms on M/G .

The exterior derivative operator don on $\Lambda(M/G)$ is defined in terms of the Lie bracket of vector fields on M/G . If $Y, Y' \in \mathfrak{X}(M/G)$, then there are $X_Y, X_{Y'} \in \mathfrak{X}(M)^G$, each of which is π related to Y and Y' , respectively. Their Lie bracket $[X_Y, X_{Y'}] \in \mathfrak{X}(M)^G$. Then there is a vector field $Y_{[X_Y, X_{Y'}]}$ on M/G , which is π related to $[X_Y, X_{Y'}]$. Define $[Y, Y'] = Y_{[X_Y, X_{Y'}]}$. The following lemma shows that this Lie bracket is well defined.

Lemma 8. For every $\bar{f} \in C^\infty(M/G)$ and every $Y, Y' \in \mathfrak{X}(M/G)$

$$[Y, Y'](\bar{f}) = Y'(Y(\bar{f})) - Y(Y'(\bar{f})). \tag{27}$$

Proof. We compute.

$$\begin{aligned} \pi^*([Y, Y'](\bar{f})) &= [X_Y, X_{Y'}](\pi^*(\bar{f})), \text{ by definition of Lie bracket} \\ &= X_{Y'}(X_Y(\pi^*(\bar{f}))) - X_Y(X_{Y'}(\pi^*(\bar{f}))), \text{ because } X_Y, X_{Y'} \in \mathfrak{X}(M)^G \\ &= X_{Y'}(\pi^*(Y(\bar{f}))) - X_Y(\pi^*(Y'(\bar{f}))), \\ &\quad \text{since } X_Y(\pi^*(\bar{f})) = \pi^*(Y(\bar{f})) \text{ and } X_{Y'}(\pi^*(\bar{f})) = \pi^*(Y'(\bar{f})) \\ &= \pi^*(Y'(Y(\bar{f}))) - \pi^*(Y(Y'(\bar{f}))) \\ &= \pi^*(Y'(Y(\bar{f})) - Y(Y'(\bar{f}))), \end{aligned}$$

from which Equation (27) follows, because the orbit map π is surjective. \square

Corollary 2. $[\cdot, \cdot]$ is a Lie bracket on $\mathfrak{X}(M/G)$.

Proof. The corollary follows from a computation using Equation (27). We give another argument. Bilinearity of the Lie bracket is straightforward to verify. We need only show that the Jacobi identity holds. We compute.

$$\begin{aligned} [Y'', [Y, Y']] &= [Y_{X''}, Y_{[X, X']}] = Y_{[X_{Y''}, [X_Y, X_{Y'}]} \\ &= Y_{[[X_{Y''}, X_Y], X_{Y'}] + [X_{Y'}, [X_{Y''}, X_{Y'}]}] \text{ by the Jacobi identity on } \mathfrak{X}(M)^G \\ &= Y_{[[X_{Y''}, X_Y], X_{Y'}]} + Y_{[X_{Y'}, [X_{Y''}, X_{Y'}]} = [[Y'', Y], Y'] + [Y, [Y'', Y']], \end{aligned}$$

which is the Jacobi identity on $\mathfrak{X}(M/G)$. \square

Let φ be an ℓ -form on M/G . Inductively define the exterior derivative $d\varphi$ as the $(\ell + 1)$ -form given by

$$d\varphi(Y_0, Y_1, \dots, Y_\ell) = \sum_{i=0}^{\ell} (-1)^i d(Y_i \lrcorner \varphi)(Y_0, \dots, \widehat{Y}_i, \dots, Y_\ell) + \sum_{0 \leq i < j \leq \ell} (-1)^{i+j} ([Y_i, Y_j] \lrcorner \varphi)(Y_0, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_\ell). \tag{28}$$

Here $Y_i \lrcorner \varphi$ is the $\ell - 1$ form on M/G defined by

$$\langle (Y_i \lrcorner \varphi) | (Y_0, \dots, \widehat{Y}_i, \dots, Y_{\ell-1}) \rangle = \langle \varphi | (Y_i, Y_1, \dots, \widehat{Y}_i, \dots, Y_{\ell-1}) \rangle,$$

for $Y_0, \dots, \widehat{Y}_i, \dots, Y_{\ell-1} \in \mathfrak{X}(M/G)$. To complete the definition of exterior derivative, we define d on 0-forms. This we do as follows. Let $\bar{f} \in \Lambda^0(M/G) = C^\infty(M/G)$. Define the 1-form $d\bar{f}$ by $d\bar{f}(\bar{m})(Y(\bar{m})) = Y(\bar{f})(\bar{m})$, for every $\bar{m} \in M/G$, every $\bar{f} \in C^\infty(M/G)$, and every $Y \in \mathfrak{X}(M/G)$.

Lemma 9. Let $\theta \in \Lambda^\ell(M/G)$. Then

$$d(\pi^*\theta) = \pi^*(d\theta). \tag{29}$$

Proof. Suppose that θ is an ℓ -form on M/G . Pulling back the forms on both sides of Equation (28) by the orbit map π gives

$$\pi^*(d\theta)(X_0, \dots, X_\ell) = \sum_{i=0}^{\ell} \pi^*(d(Y_i \lrcorner \theta))(X_0, \dots, \widehat{X}_i, \dots, X_\ell) + \sum_{0 \leq i < j \leq \ell} (-1)^{i+j} \pi^*([Y_i, Y_j] \lrcorner \theta)(X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_\ell). \tag{30}$$

By induction, assume that Equation (29) holds for all forms of degree strictly less than ℓ . Then

$$\pi^*(d(Y_i \lrcorner \theta)) = d(\pi^*(Y_i \lrcorner \theta)). \tag{31}$$

Now $\pi^*(Y_i \lrcorner \theta) = X_i \lrcorner \pi^*\theta$, where $T\pi \circ X_i = Y_i \circ \pi$, since

$$\begin{aligned} (X_i \lrcorner \pi^*\theta)(X_0, \dots, \widehat{X}_i, \dots, X_\ell) &= (-1)^i (\pi^*\theta)(X_0, \dots, X_i, \dots, X_\ell) \\ &= (-1)^i \theta(T\pi X_0, \dots, T\pi X_i, \dots, T\pi X_\ell) = (-1)^i \theta(Y_0, \dots, Y_i, \dots, Y_\ell) \\ &= (Y_i \lrcorner \theta)(Y_0, \dots, \widehat{Y}_i, \dots, Y_\ell) = \pi^*(Y_i \lrcorner \theta)(X_0, \dots, \widehat{X}_i, \dots, X_\ell). \end{aligned}$$

Additionally,

$$\begin{aligned} \pi^*([Y_i, Y_j] \lrcorner \theta)(X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_\ell) &= \\ &= ([Y_i, Y_j] \lrcorner \theta)(T\pi X_0, \dots, \widehat{T\pi X_i}, \dots, \widehat{T\pi X_j}, \dots, T\pi X_\ell) \\ &= ([X_i, X_j] \lrcorner \pi^*\theta)(X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_\ell), \\ &\text{since } [Y_i, Y_j] = [T\pi X_i, T\pi X_j] = T\pi[X_i, X_j]. \end{aligned}$$

Thus, Equation (30) reads

$$\begin{aligned} \pi^*(d\theta)(X_0, \dots, X_\ell) &= \sum_{i=0}^{\ell} (-1)^i d(X_i \lrcorner \pi^*\theta)(X_0, \dots, \widehat{X}_i, \dots, X_\ell) \\ &\quad + \sum_{0 \leq i < j \leq \ell} (-1)^{i+j} ([X_i, X_j] \lrcorner \pi^*\theta)(X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_\ell) \end{aligned}$$

$$= d(\pi^*\theta)(X_0, \dots, X_\ell).$$

□

Lemma 10. *If $\theta \in \Lambda^k(M/G)$ and $\phi \in \Lambda^h(M/G)$, then*

$$d(\theta \wedge \phi) = d\theta \wedge \phi + (-1)^k \theta \wedge d\phi. \tag{32}$$

Proof. On M we have

$$\begin{aligned} \pi^*(d(\theta \wedge \phi)) &= d(\pi^*(\theta \wedge \phi)) = d(\pi^*\theta \wedge \pi^*\phi) \\ &= d(\pi^*\theta) \wedge \pi^*\phi + (-1)^k \pi^*\theta \wedge d(\pi^*\phi) \\ &= \pi^*(d\theta \wedge \phi + (-1)^k \theta \wedge d\phi), \end{aligned}$$

which implies that Equation (32) holds, since the orbit map π is surjective. □

Lemma 11. $d^2\theta = 0$ for every $\theta \in \Lambda^\ell(M/G)$.

Proof. Suppose that $\ell \geq 1$. Then $\pi^*\theta$ is an ℓ -form on M . Because M is a smooth manifold, one has $d^2(\pi^*\theta) = 0$. By Lemma 8 $\pi^*(d\theta) = d(\pi^*\theta)$. So

$$\pi^*(d^2\theta) = \pi^*(d(d\theta)) = d(\pi^*(d\theta)) = d^2(\pi^*\theta) = 0.$$

Since the G orbit map π is surjective, $\pi^*(d^2\theta) = 0$ implies $d^2\theta = 0$.

We now treat the case when $\ell = 0$. Let $\bar{f} \in C^\infty(M/G)$ and let $Y_0, Y_1 \in \mathfrak{X}(M/G)$. Then

$$\begin{aligned} d(d\bar{f})(Y_0, Y_1) &= d(Y_0 \lrcorner d\bar{f})Y_1 - d(Y_1 \lrcorner d\bar{f})Y_0 - [Y_0, Y_1] \lrcorner d\bar{f} \\ &= d(Y_0(\bar{f}))Y_1 - d(Y_1(\bar{f}))Y_0 - [Y_0, Y_1](\bar{f}) \\ &= Y_1(Y_0(\bar{f})) - Y_0(Y_1(\bar{f})) - Y_1(Y_0(\bar{f})) + Y_0(Y_1(\bar{f})) = 0. \end{aligned} \tag{33}$$

□

We prove an equivariant version of the Poincaré lemma in \mathbb{R}^n .

Lemma 12. *Let G be a Lie group, which acts linearly on \mathbb{R}^n by $\Phi : G \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let H be a compact subgroup of G . Let β be an H invariant closed ℓ -form with $\ell \geq 1$ on an open H invariant ball B centered at the origin of \mathbb{R}^n , whose closure is compact. Suppose that β is semi-basic with respect to the G action Φ , i.e., $X_\xi \lrcorner \beta = 0$ for every $\xi \in \mathfrak{g}$, the Lie algebra of G . Here $X_\xi(x) = T_e\Phi_m\xi$. Then there is an H invariant $(\ell - 1)$ -form α on B , which is semi-basic with respect to the G action Φ , such that $\beta = d\alpha$.*

Proof. Let X be a linear vector field on \mathbb{R}^n all of whose eigenvalues are negative real numbers. By averaging over the compact group H , we may assume that X is H invariant. Let φ_t be the flow of X , which maps B into itself. Moreover, $\varphi_\infty = 0$. On B one has

$$\begin{aligned} \beta &= -(\varphi_\infty^*\beta - \varphi_0^*\beta) = -\int_0^\infty \frac{d}{dt}(\varphi_t^*\beta) dt \\ &= -\int_0^\infty \varphi_t^*(L_X\beta) dt = -\int_0^\infty \varphi_t^*(d(X \lrcorner \beta) + X \lrcorner d\beta) dt \\ &= -\int_0^\infty \varphi_t^*(d(X \lrcorner \beta)) dt, \text{ since } \beta \text{ is closed} \\ &= -d\left(\int_0^\infty \varphi_t^*(X \lrcorner \beta) dt\right), \text{ since } d\varphi_t^* = \varphi_t^* d. \end{aligned}$$

The $(\ell - 1)$ -form $\alpha = \int_0^\infty \varphi_t^*(X \lrcorner \beta) dt$ on B is H invariant, since φ_t commutes with the H action on B , and $X \lrcorner \beta$ is an H invariant $(\ell - 1)$ -form on B , because the vector field X and the ℓ -form β are both H invariant. Thus, $\beta = d\alpha$ on B . Moreover, α is G semi-basic, since

$$L_{X_\xi}(\varphi_t^*(X \lrcorner \beta)) = \varphi_t^*(L_{X_\xi}(X \lrcorner \beta)) = \varphi_t^*(L_{X_\xi}X \lrcorner L_{X_\xi}\beta) = 0.$$

The last equality above follows because the ℓ -form β is G semi-basic. \square

Since M/G is a locally contractible space, we have

Proposition 14 (Poincaré Lemma). *Let $\bar{\theta}$ be a closed ℓ -form on M/G with $\ell \geq 1$. For each $\bar{m} \in M/G$ there is a contractible open neighborhood $\mathcal{U}_{\bar{m}}$ of \bar{m} and an $(\ell - 1)$ -form ϕ on $\mathcal{U}_{\bar{m}}$ such that $\bar{\theta} = d\bar{\phi}$ on $\mathcal{U}_{\bar{m}}$.*

Proof. Since the G action Φ on M is proper, it has a slice S_m at m , where $\pi(m) = \bar{m}$. Using Bochner’s lemma there is an open neighborhood U_m of m in S_m , which is the image of an $H = G_m$ invariant open ball $B \subseteq T_mM$, centered at the origin 0_m whose closure is compact, under a diffeomorphism $\psi : B \subseteq T_mM \rightarrow U_m \subseteq M$. The diffeomorphism ψ intertwines the linear H action

$$H \times T_mM \rightarrow T_mM : (h, v_m) \mapsto T_m\Phi_g v_m$$

with the $H \subseteq G$ action Φ on U_m . Let ϑ be the semi-basic G invariant form on $G \cdot U_m$ such that $(\pi|_{G \cdot U_m})^*\bar{\theta} = \vartheta$. Since $\bar{\theta}$ is closed by hypothesis, it follows that the semi-basic ℓ -form ϑ on $G \cdot U_m$ is closed. Let $\phi = \vartheta|_{U_m}$. Then ϕ is a semi-basic H invariant closed ℓ -form on U_m . Under the map ψ the ℓ -form ϕ pulls back to a G semi-basic H invariant ℓ -form $\psi^*\phi$ on $B \subseteq T_mM$. By Lemma 12 there is a G semi-basic H invariant $(\ell - 1)$ -form γ on B such that $\psi^*\phi = d\gamma$. Hence $\alpha = \psi_*\gamma$ is a semi-basic H invariant $(\ell - 1)$ -form on U_m . The $(\ell - 1)$ -form α on U_m extends to a G invariant $(\ell - 1)$ -form σ on $G \cdot U_m$ defined by

$$\sigma(\Phi_g(m))(T_s\Phi_g v_s) = \alpha(s)v_s,$$

for every $s \in U_m$ and every $v_s \in T_sS_m$. Arguing as in the proof of Lemma 6, it follows that σ is a smooth G invariant $(\ell - 1)$ -form on $G \cdot U_m$. The form σ is semi-basic. Moreover, $d\sigma = \vartheta$ on $G \cdot U_m$, since for every $g \in G$ one has

$$d\sigma = d(\Phi_g^*\alpha) = \Phi_g^*(d\alpha) = \Phi_g^*(\vartheta) = \vartheta.$$

Let $\mathcal{U}_{\bar{m}} = \pi(U_m)$. Since U_m is contractible and the G orbit map π is continuous and open, it follows that the open neighborhood $\mathcal{U}_{\bar{m}}$ of $\bar{m} \in M/G$ is contractible. Since the ℓ -form σ is semi-basic, there is an ℓ -form $\bar{\phi}$ on $\mathcal{U}_{\bar{m}}$ such that $\pi^*\bar{\phi} = \sigma$ on $G \cdot m$. On $G \cdot U_m$ we have

$$\pi^*\bar{\theta} = \vartheta = d\sigma = d(\pi^*\bar{\phi}) = \pi^*(d\bar{\phi}).$$

Because the orbit map π is surjective, it follows that $\bar{\theta} = d\bar{\phi}$ on $\mathcal{U}_{\bar{m}}$, which proves the proposition. \square

Lemma 13. *Let $\bar{f} \in C^\infty(M/G)$ and suppose that $\mathcal{U}_{\bar{m}}$ is a connected open neighborhood of $\bar{m} \in M/G$ such that $d\bar{f} = 0$, then \bar{f} is constant on $\mathcal{U}_{\bar{m}}$.*

Proof. It follows from our hypotheses that $f = \pi^*\bar{f}$ is a smooth G invariant function on the open connected component U_m of $\pi^{-1}(\mathcal{U}_{\bar{m}})$ containing m . Moreover, on U_m we have $df = d(\pi^*\bar{f}) = \pi^*(d\bar{f}) = 0$. Since M is a smooth manifold, it follows that f is constant on U_m . Hence \bar{f} is constant on the connected open set $\pi(U_m) = \mathcal{U}_{\bar{m}}$ because π is a continuous open map. \square

To prove de Rham’s theorem, we will need some sheaf theory, which can be found in appendix C of Lukina, Takens, and Broer [17]. Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in I}$ be an open covering

of M/G . Because M/G is locally contractible, the open covering \mathcal{U} has a good refinement \mathcal{U}' , that is, every $\mathcal{U}_\beta \in \mathcal{U}'$ with $\beta \in J \subseteq I$ is locally contractible and $\mathcal{U}_{\beta_1} \cap \dots \cap \mathcal{U}_{\beta_n}$ is either contractible or empty. In addition, because M/G is paracompact, every open covering has a locally finite subcovering. Since the G action on M is proper, the orbit space M/G has a $C^\infty(M/G)$ partition of unity subordinate to the covering \mathcal{U} .

Define the differential exterior algebra valued sheaf Λ over M/G by

$$\Lambda : \mathcal{U}_\alpha \mapsto (\Lambda(\mathcal{U}_\alpha), \wedge, d|_{\mathcal{U}_\alpha}),$$

whose sections are differential forms on \mathcal{U}_α . The sheaf Λ induces the subsheaves

$$\Lambda^\ell : \mathcal{U}_\alpha \rightarrow (\Lambda^\ell(M/G), \wedge, d|_{\mathcal{U}_\alpha}),$$

whose sections are differential ℓ -forms on \mathcal{U}_α . Please note that

$$\Lambda \rightarrow M/G : \Lambda_{\bar{m}} = \sum_{\ell} \Lambda_{\bar{m}}^\ell \rightarrow \bar{m}$$

is a smooth vector bundle, as is

$$\Lambda^\ell \rightarrow M/G : \Lambda_{\bar{m}}^\ell \rightarrow \bar{m}.$$

Let \mathcal{R} be the sheaf of locally constant \mathbb{R} -valued functions on M/G . The two exact sequence of sheaves

$$0 \rightarrow \mathcal{R} \rightarrow \Lambda \rightarrow \dots \text{ and } 0 \rightarrow \mathcal{R} \rightarrow \Lambda^\ell \rightarrow \dots$$

are exact.

We say that the sheaf Λ is fine if for every open subset \mathcal{U} of M/G , every smooth function \bar{f} on M/G and every smooth section $s : \mathcal{U} \subseteq M/G \rightarrow \Lambda(\mathcal{U})$ of the sheaf Λ , then $\bar{f}|_{\mathcal{U}} s \in \Lambda(\mathcal{U})$.

Theorem 3. *The sheaves Λ and Λ^ℓ of sections of the vector bundles Λ and Λ^ℓ are fine.*

Proof. We treat the case of the sheaf Λ . The proof for the sheaf Λ^ℓ is similar and is omitted. The definition of fineness holds by definition of differential form. \square

Corollary 3. *Λ and Λ^ℓ are fine sheaves of sections over M/G , which is paracompact. Let \mathcal{U} be an open covering of M/G . Then $H^q(\mathcal{U}, \Lambda)$, the sheaf of q^{th} cohomology group of \mathcal{U} with values in the sheaf Λ , vanishes for all $q \in \mathbb{Z}_{\geq 1}$. Similarly, $H^q(\mathcal{U}, \Lambda^\ell) = 0$ for all $q \in \mathbb{Z}_{\geq 1}$.*

We are now in position to formulate de Rham’s theorem. Let Λ^ℓ be the sheaf of differential ℓ -forms on M/G and let $d : \Lambda^\ell \rightarrow \Lambda^{\ell+1}$ be the sheaf homomorphism induced by exterior differentiation. For each $\ell \in \mathbb{Z}_{\geq 0}$ let $\mathcal{Z}^\ell = \ker d$, whose elements are closed ℓ -forms on M/G . By Lemma 13 $\mathcal{Z}^0 = \mathcal{R}$. Define the ℓ^{th} de Rham cohomology group $H_{\text{dR}}^\ell(M/G) = \Gamma(M/G, \mathcal{Z}^\ell) / d\Gamma(M/G, \Lambda^{\ell-1})$ when $\ell \in \mathbb{Z}_{\geq 1}$ and $H_{\text{dR}}^0(M/G) = \Gamma(M/G, \mathcal{Z}^0)$. Here $\Gamma(M/G, \mathcal{G})$ is the set of sections of the sheaf $M/G \rightarrow \mathcal{G}$.

Theorem 4 (de Rham’s Theorem). *The sheaf cohomology of Λ^ℓ with coefficients in \mathcal{R} does not depend on the good covering \mathcal{U} of M/G . Thus, for every $\ell \in \mathbb{Z}_{\geq 0}$ the ℓ^{th} de Rham cohomology group $H_{\text{dR}}^\ell(M/G)$ is isomorphic to the ℓ^{th} sheaf cohomology group $H^\ell(\mathcal{U}, \mathcal{R})$ of the good covering \mathcal{U} with values in the sheaf \mathcal{R} of locally constant real valued functions.*

Proof. We give a sketch, leaving out the homological algebra, which is standard. For more details, see [17] or [18]. Let \mathcal{U} be a good covering of M/G . The Poincaré lemma holds on

any finite intersection of contractible open sets in \mathcal{U} , so the following sequence of sheaves is exact

$$0 \rightarrow \mathcal{Z}^\ell \xrightarrow{\iota} \Lambda^\ell \xrightarrow{d} \mathcal{Z}^{\ell+1} \rightarrow 0,$$

where $\iota : \mathcal{Z}^\ell \rightarrow \Lambda^\ell$ is the inclusion mapping. This exact sequence gives rise to the long exact sequence of cohomology groups

$$0 \rightarrow H^0(\mathcal{U}, \mathcal{Z}^\ell) \xrightarrow{\iota_*} H^0(\mathcal{U}, \Lambda^\ell) \xrightarrow{d_*} H^0(\mathcal{U}, \mathcal{Z}^{\ell+1}) \xrightarrow{\delta_*} H^1(\mathcal{U}, \mathcal{Z}^\ell) \rightarrow \dots,$$

where ι_* , d_* , and δ_* are homomorphisms on cohomology induced by the inclusion, exterior differentiation and coboundary homomorphisms, respectively. Since Λ^ℓ is a fine sheaf, its cohomology vanishes for $\ell \geq 1$ and the above sequence falls apart into the exact sequence

$$0 \rightarrow H^0(\mathcal{U}, \mathcal{Z}^\ell) \xrightarrow{\iota_*} H^0(\mathcal{U}, \Lambda^\ell) \xrightarrow{d_*} H^0(\mathcal{U}, \mathcal{Z}^{\ell+1}) \xrightarrow{\delta_*} H^1(\mathcal{U}, \mathcal{Z}^\ell) \rightarrow 0, \tag{34}$$

and for every $k \geq 1$ the exact sequence

$$0 \rightarrow H^k(\mathcal{U}, \mathcal{Z}^{\ell+1}) \xrightarrow{\delta_*} H^{k+1}(\mathcal{U}, \mathcal{Z}^\ell) \rightarrow 0. \tag{35}$$

Now $H^0(M/G, \mathcal{R}) = \Gamma(M/G, \mathcal{Z}^0) = H^0_{\text{dR}}(M/G)$. Applying the sequence (34) consecutively gives

$$H^\ell(\mathcal{U}, \mathcal{R}) \simeq H^1(\mathcal{U}, \mathcal{Z}^{\ell-1}).$$

Exactness of the sequence (35) gives

$$H^1(\mathcal{U}, \mathcal{Z}^{\ell-1}) \simeq H^0(\mathcal{U}, \mathcal{Z}^\ell) / d_*(H^0(\mathcal{U}, \Lambda^{\ell-1})).$$

Here \simeq means is isomorphic to. \square

Corollary 4. *For the zeroth cohomology we have*

$$H^\ell(\mathcal{U}, \mathcal{R}) \simeq \Gamma(M/G, \mathcal{Z}^\ell) / d(\Gamma(M/G, \Lambda^{\ell-1})) = H^\ell_{\text{dR}}(M/G) \text{ for } \ell \in \mathbb{Z}_{\geq 1}. \tag{36}$$

Our version of de Rham’s theorem is not the same as Smith’s version, since the \mathbb{Z}_2 invariant semi-basic 1-form $x_2 dx_1 - x_1 dx_2$ in Section 6 is not a Smith 1-form, see also Smith ([7], p. 133). However, his cohomology and ours agree. Our results extend those of Koszul [10], who hypothesized that M/G was a smooth manifold and that the group G was compact.

6. An Example

In this section we give an example, which illustrates Theorem 2 and the construction of differential 1-forms on the orbit space of a proper group action on a smooth manifold.

Consider the \mathbb{Z}_2 action on \mathbb{R}^2 generated by

$$\zeta : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : x = (x_1, x_2) \mapsto (-x_1, -x_2) = -x.$$

The algebra of \mathbb{Z}_2 invariant polynomials on \mathbb{R}^2 is generated by the polynomials $\sigma_1(x) = x_1^2$, $\sigma_2(x) = x_2^2$, and $\sigma_3(x) = x_1x_2$, which are subject to the relation

$$\sigma_3^2(x) = \sigma_1(x)\sigma_2(x), \sigma_1(x) \geq 0 \ \& \ \sigma_2(x) \geq 0, \text{ for all } x \in \mathbb{R}^2. \tag{37}$$

Let

$$\sigma : \mathbb{R}^2 \rightarrow \Sigma \subseteq \mathbb{R}^3 : x \mapsto (\sigma_1(x), \sigma_2(x), \sigma_3(x)) \tag{38}$$

be the Hilbert map of the \mathbb{Z}_2 action associated to the polynomial generators $\sigma_1(x)$, $\sigma_2(x)$, and $\sigma_3(x)$. The map σ (38) is the orbit map of the \mathbb{Z}_2 action on \mathbb{R}^2 . The relation

$$\sigma_3^2 = \sigma_1\sigma_2, \quad \sigma_1 \geq 0 \ \& \ \sigma_2 \geq 0 \tag{39}$$

defines the orbit space $\mathbb{R}^2/\mathbb{Z}_2$ as a closed semialgebraic subset Σ of \mathbb{R}^3 with coordinates $(\sigma_1, \sigma_2, \sigma_3)$. Geometrically Σ is a cone in \mathbb{R}^3 with vertex $(0, 0, 0)$.

Because \mathbb{Z}_2 is a compact Lie group, which acts linearly on \mathbb{R}^2 , Schwarz' theorem [19] implies that the space $C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ of \mathbb{Z}_2 invariant smooth functions on \mathbb{R}^2 is equal to $C^\infty(\Sigma)$, where $\bar{f} \in C^\infty(\Sigma)$ if and only if there is an $\bar{F} \in C^\infty(\mathbb{R}^3)$ such that $\bar{f} = \bar{F}|_\Sigma$.

Lemma 14. *Let $f \in C^\infty(\mathbb{R}^2)$ satisfy $f(x_1, x_2) = -f(-x_1, -x_2)$ for every $(x_1, x_2) \in \mathbb{R}^2$. Then there are $f_1, f_2 \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ such that $f(x_1, x_2) = x_1f_1(x_1, x_2) + x_2f_2(x_1, x_2)$ for every $(x_1, x_2) \in \mathbb{R}^2$.*

Proof. Since $f(x_1, x_2) = -f(-x_1, -x_2)$, it follows that $f(0, 0) = 0$. Suppose that there is an integer $k \geq 1$ such that $D^j f(0, 0) = (0, 0)$ for $0 \leq j \leq k - 1$ and $D^k f(0, 0) \neq (0, 0)$. Then by Taylor's theorem with integral remainder we have $f(x, y) = \sum_{\ell=0}^k g_\ell(x_1, x_2)x_1^{k-\ell}x_2^\ell$ for every $(x_1, x_2) \in \mathbb{R}^2$, where $g_\ell \in C^\infty(\mathbb{R}^2)$ and $g_\ell(0, 0) = \frac{\partial^k f}{\partial x_1^{k-\ell} \partial x_2^\ell}(0, 0)$ for $0 \leq \ell \leq k$. By hypothesis

$$\sum_{\ell=0}^k g_\ell(x_1, x_2)x_1^{k-\ell}x_2^\ell = f(x_1, x_2) = -f(-x_1, -x_2) = (-1)^{k+1} \sum_{\ell=0}^k g_\ell(-x_1, -x_2)x_1^{k-\ell}x_2^\ell.$$

So

$$g_\ell(x_1, x_2) = (-1)^{k+1}g_\ell(-x_1, -x_2), \quad \text{for all } 0 \leq \ell \leq k. \tag{40}$$

If k is odd Equation (40) implies $g_\ell \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ for $0 \leq \ell \leq k$. Consequently,

$$f(x_1, x_2) = x_1(g_0(x_1, x_2)x_1^{k-1}) + x_2\left(\sum_{\ell=1}^k g_\ell(x_1, x_2)x_1^{k-\ell}x_2^{\ell-1}\right),$$

which proves the lemma when k is odd. When k is even, Equation (40) reads $g_\ell(x_1, x_2) = -g_\ell(-x_1, -x_2)$ for $0 \leq \ell \leq k$, which implies $g_\ell(0, 0) = 0$ for $0 \leq \ell \leq k$. In particular, $D^k f(0, 0) = 0$, which contradicts our hypothesis.

Now suppose that f is flat at $(0, 0)$, i.e., $D^k f(0, 0) = 0$ for every $k > 0$. Then x_1 and x_2 divide f , i.e., $f_1 = f/(2x_1)$ and $f_2 = f/(2x_2)$ are smooth functions on \mathbb{R}^2 . To see this note that f_1 and f_2 are smooth for all $(x_1, x_2) \neq (0, 0)$. Since f is flat at $(0, 0)$, so are f_1 and f_2 . Clearly $f(x_1, x_2) = x_1f_1(x_1, x_2) + x_2f_2(x_1, x_2)$. From

$$f_1(-x_1, -x_2) = f(-x_1, -x_2)/(2(-x_1)) = f(x_1, x_2)/(2x_1) = f_1(x_1, x_2)$$

it follows that $f_1 \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$. Similarly, $f_2 \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$. \square

Proposition 15. *The $C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ module $\mathfrak{X}(\mathbb{R}^2)^{\mathbb{Z}_2}$ of \mathbb{Z}_2 invariant smooth vector fields on \mathbb{R}^2 is generated by*

$$X_1 = x_1 \frac{\partial}{\partial x_1}, \quad X_2 = x_2 \frac{\partial}{\partial x_1}, \quad X_3 = x_1 \frac{\partial}{\partial x_2}, \quad \text{and} \quad X_4 = x_2 \frac{\partial}{\partial x_2}. \tag{41}$$

Proof. A smooth vector field X on \mathbb{R}^2 may be written as $X(x_1, x_2) = f(x_1, x_2) \frac{\partial}{\partial x_1} + g(x_1, x_2) \frac{\partial}{\partial x_2}$, where f and g are smooth. $X \in \mathfrak{X}(\mathbb{R}^2)^{\mathbb{Z}_2}$ if and only if

$$f(x_1, x_2) \frac{\partial}{\partial x_1} + g(x_1, x_2) \frac{\partial}{\partial x_2} = X(x_1, x_2) = \zeta^* X(x_1, x_2)$$

$$= -f(-x_1, -x_2) \frac{\partial}{\partial x_1} - g(-x_1, -x_2) \frac{\partial}{\partial x_2},$$

that is, $f(x_1, x_2) = -f(-x_1, -x_2)$ and $g(x_1, x_2) = -g(-x_1, -x_2)$ for every $(x_1, x_2) \in \mathbb{R}^2$. Using lemma 38 write $f(x_1, x_2) = x_1g_1(x_1, x_2) + x_2g_2(x_1, x_2)$ and $g(x_1, x_2) = x_1h_1(x_1, x_2) + x_2h_2(x_1, x_2)$, where g_1, g_2, h_1 , and $h_2 \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$. Hence for every $(x_1, x_2) \in \mathbb{R}^2$ we have

$$\begin{aligned} X(x_1, x_2) &= (x_1g_1(x_1, x_2) + x_2g_2(x_1, x_2)) \frac{\partial}{\partial x_1} \\ &\quad + (x_1h_1(x_1, x_2) + x_2h_2(x_1, x_2)) \frac{\partial}{\partial x_2} \\ &= (g_1X_1 + g_2X_2 + h_1X_3 + h_2X_4)(x_1, x_2), \end{aligned}$$

where g_1, g_2, h_1 , and $h_2 \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$. \square

Lemma 15. *The vector fields*

$$\begin{aligned} Y_1 &= 2\bar{\sigma}_1 \frac{\partial}{\partial \sigma_1} + \bar{\sigma}_3 \frac{\partial}{\partial \sigma_3}, & Y_2 &= 2\bar{\sigma}_3 \frac{\partial}{\partial \sigma_1} + \bar{\sigma}_2 \frac{\partial}{\partial \sigma_3} \\ Y_3 &= 2\bar{\sigma}_3 \frac{\partial}{\partial \sigma_2} + \bar{\sigma}_1 \frac{\partial}{\partial \sigma_3}, & Y_4 &= 2\bar{\sigma}_2 \frac{\partial}{\partial \sigma_2} + \bar{\sigma}_3 \frac{\partial}{\partial \sigma_3} \end{aligned} \tag{42}$$

on $\Sigma \subseteq \mathbb{R}^3$, where $\bar{\sigma}_i = (\sigma_i)|_\Sigma$ for $i = 1, 2, 3$, are σ related to the \mathbb{Z}_2 invariant vector fields X_i (41) for $i = 1, 2, 3$.

Proof. The calculation

$$\begin{aligned} L_{X_1}\sigma_1 &= x_1 \frac{\partial x_1^2}{\partial x_1} = 2\sigma_1, & L_{X_1}\sigma_2 &= x_1 \frac{\partial x_2^2}{\partial x_1} = 0, & L_{X_1}\sigma_3 &= x_1 \frac{\partial(x_1x_2)}{\partial x_1} = \sigma_3 \\ L_{X_2}\sigma_1 &= x_2 \frac{\partial x_1^2}{\partial x_1} = 2\sigma_3, & L_{X_2}\sigma_2 &= x_2 \frac{\partial x_2^2}{\partial x_1} = 0, & L_{X_2}\sigma_3 &= x_2 \frac{\partial(x_1x_2)}{\partial x_1} = \sigma_2 \\ L_{X_3}\sigma_1 &= x_1 \frac{\partial x_1^2}{\partial x_2} = 0, & L_{X_3}\sigma_2 &= x_1 \frac{\partial x_2^2}{\partial x_2} = 2\sigma_3, & L_{X_3}\sigma_3 &= x_1 \frac{\partial(x_1x_2)}{\partial x_2} = \sigma_1 \\ L_{X_4}\sigma_1 &= x_2 \frac{\partial x_1^2}{\partial x_2} = 0, & L_{X_4}\sigma_2 &= x_2 \frac{\partial x_2^2}{\partial x_2} = 2\sigma_2, & L_{X_4}\sigma_3 &= x_2 \frac{\partial(x_1x_2)}{\partial x_2} = \sigma_3 \end{aligned}$$

gives the vector fields

$$\begin{aligned} \tilde{Y}_1(\sigma_1, \sigma_2, \sigma_3) &= 2\sigma_1 \frac{\partial}{\partial \sigma_1} + \sigma_3 \frac{\partial}{\partial \sigma_3}, & \tilde{Y}_2(\sigma_1, \sigma_2, \sigma_3) &= 2\sigma_3 \frac{\partial}{\partial \sigma_1} + \sigma_2 \frac{\partial}{\partial \sigma_3} \\ \tilde{Y}_3(\sigma_1, \sigma_2, \sigma_3) &= 2\sigma_3 \frac{\partial}{\partial \sigma_2} + \sigma_1 \frac{\partial}{\partial \sigma_3}, & \tilde{Y}_4(\sigma_1, \sigma_2, \sigma_3) &= 2\sigma_2 \frac{\partial}{\partial \sigma_2} + \sigma_3 \frac{\partial}{\partial \sigma_3} \end{aligned} \tag{43}$$

on \mathbb{R}^3 . Since $L_{X_i}(\sigma_3^2 - \sigma_1\sigma_2) = 0$ for $i = 1, 2, 3, 4$, the vector fields \tilde{Y}_i on \mathbb{R}^3 given by (43) leave invariant the ideal I of $C^\infty(\mathbb{R}^3)$ generated by $\sigma_3^2 - \sigma_1\sigma_2$. Hence for each $i = 1, 2, 3, 4$ the vector field \tilde{Y}_i define the vector field $Y_i = \tilde{Y}_i|_\Sigma$ on Σ , which is given in Equation (42). The vector fields Y_i are σ related to the \mathbb{Z}_2 invariant vector fields X_i (41) for $i = 1, 2, 3, 4$, because $Y_i(\sigma(x)) = (\tilde{Y}_i|_\Sigma)(\sigma_1, \sigma_2, \sigma_3) = T_x\sigma X_i(x)$. \square

Since the tangent to the Hilbert mapping σ (38) is defined and is surjective, the tangent bundle $T\Sigma$ of the semialgebraic variety Σ (37) is the semialgebraic subset of \mathbb{R}^7 with coordinates $(\sigma_1, \sigma_2, \sigma_3, \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4)$ defined by Equation (37) and

$$\sigma_3\tilde{Y}_1 - \sigma_1\tilde{Y}_2 = 0 \text{ and } \sigma_3\tilde{Y}_3 - \sigma_1\tilde{Y}_4 = 0.$$

By Theorem 2 every smooth vector field on Σ is σ related to a smooth \mathbb{Z}_2 invariant vector field on \mathbb{R}^2 . Because the $C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ module $\mathfrak{X}(\mathbb{R}^2)^{\mathbb{Z}_2}$ of smooth \mathbb{Z}_2 invariant vector fields on \mathbb{R}^2 is generated by the vector fields X_i for $1 \leq i \leq 4$ given by Equation (41), it follows that the σ related vector fields Y_i for $1 \leq i \leq 4$ given by Equation (42) generate the $C^\infty(\Sigma)$ module $\mathfrak{X}(\Sigma)$ of smooth vector fields on Σ .

Lemma 16. *The differential 1-forms*

$$\tilde{\vartheta}_1 = x_1 dx_1, \tilde{\vartheta}_2 = x_1 dx_2, \tilde{\vartheta}_3 = x_2 dx_1, \tilde{\vartheta}_4 = x_2 dx_2. \tag{44}$$

generate the $C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ module $\Lambda^1(\mathbb{R}^2)^{\mathbb{Z}_2}$ of \mathbb{Z}_2 invariant 1-forms on \mathbb{R}^2 .

Proof. We use the differential forms

$$\begin{aligned} \vartheta_1 &= dx_1^2 = 2x_1 dx_1, & \vartheta_2 &= dx_2^2 = 2x_2 dx_2, \\ \vartheta_3 &= d(x_1x_2) = x_1 dx_2 + x_2 dx_1, & \vartheta_4 &= x_1 dx_2 - x_2 dx_1 \end{aligned} \tag{45}$$

instead of those given in (44), because we then obtain $\vartheta_k = d\sigma_k$ for $k = 1, 2, 3$. Suppose that the 1-form $\vartheta(x_1, x_2) = f_1(x_1, x_2) dx_1 + f_2(x_1, x_2) dx_2$ on \mathbb{R}^2 , where $f_i \in C^\infty(\mathbb{R}^2)$ for $i = 1, 2$, is invariant under the \mathbb{Z}_2 action generated by $\zeta : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : (x_1, x_2) \mapsto (-x_1, -x_2)$. Then for every $(x_1, x_2) \in \mathbb{R}^2$

$$\begin{aligned} f_1(x_1, x_2) dx_1 + f_2(x_1, x_2) dx_2 &= \vartheta(x_1, x_2) = (\zeta^*\vartheta)(x_1, x_2) \\ &= f_1(-x_1, -x_2) d(-x_1) + f_2(-x_1, -x_2) d(-x_2) \\ &= -f_1(-x_1, -x_2) dx_1 - f_2(-x_1, -x_2) dx_2. \end{aligned}$$

So $\zeta^*\vartheta = \vartheta$ if and only if for $i = 1, 2$ one has $f_i(x_1, x_2) = -f_i(-x_1, -x_2)$ for every $(x_1, x_2) \in \mathbb{R}^2$. By Lemma 14 if $g(x_1, x_2) = -g(-x_1, -x_2)$ for some $g \in C^\infty(\mathbb{R}^2)$, then there are $g_i \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ for $i = 1, 2$ such that $g(x_1, x_2) = x_1g_1(x_1, x_2) + x_2g_2(x_1, x_2)$ for every $(x_1, x_2) \in \mathbb{R}^2$. Consequently, for some $h_1, h_2, k_1, k_2 \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$

$$\begin{aligned} \vartheta(x_1, x_2) &= (x_1h_1(x_1, x_2) + x_2h_2(x_1, x_2)) dx_1 + (x_1k_1(x_1, x_2) + x_2k_2(x_1, x_2)) dx_2 \\ &= h_1(x_1, x_2) x_1 dx_1 + h_2(x_1, x_2) x_2 dx_1 \\ &\quad + k_1(x_1, x_2) x_1 dx_2 + k_2(x_1, x_2) x_2 dx_2 \\ &= \tilde{h}_1(x_1, x_2) \vartheta_1 + \tilde{h}_2(x_1, x_2) \vartheta_2 + \tilde{k}_1(x_1, x_2) \vartheta_3 + \tilde{k}_2(x_1, x_2) \vartheta_4, \end{aligned}$$

for every $(x_1, x_2) \in \mathbb{R}^2$. Here $\tilde{h}_1 = 2h_1, \tilde{h}_2 = 2k_2, \tilde{k}_1 = k_1 + h_2$, and $\tilde{k}_2 = k_1 - h_2$. This proves the lemma. \square

For $i = 1, \dots, 4$ define the 1-forms θ_i on Σ by

$$\sigma^*((Y_i|\Sigma) \lrcorner \theta_i) = X_i \lrcorner \vartheta_i, \tag{46}$$

see the proof of Proposition 10. The 1-forms θ_i generate the $C^\infty(\mathbb{R}^2/\mathbb{Z}_2)$ module of 1-forms on Σ , since the \mathbb{Z}_2 invariant 1-forms ϑ_i for $i = 1, \dots, 4$ generate the $C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$ module $\Lambda(\mathbb{R}^2)^{\mathbb{Z}_2}$ of \mathbb{Z}_2 invariant 1-forms on \mathbb{R}^2 , see proposition 23. Every \mathbb{Z}_2 invariant 1-form on \mathbb{R}^2 is semi-basic, since the Lie algebra of \mathbb{Z}_2 is $\{0\}$.

Fact 1. *On Σ we have*

$$\theta_1 = d\bar{\sigma}_1, \theta_2 = d\bar{\sigma}_2, \text{ and } \theta_3 = d\bar{\sigma}_3. \tag{47}$$

Let θ_4 be the 1-form on Σ defined by its values

$$\begin{aligned} Y_1|\Sigma \lrcorner \theta_4 &= -\bar{\sigma}_3, & Y_2|\Sigma \lrcorner \theta_4 &= \bar{\sigma}_1, \\ Y_3|\Sigma \lrcorner \theta_4 &= -\bar{\sigma}_2, & Y_4|\Sigma \lrcorner \theta_4 &= -\bar{\sigma}_3. \end{aligned} \tag{48}$$

Here $\bar{\sigma}_i = \sigma_i|_\Sigma$ for $i = 1, 2, 3$. The 1-form θ_4 is not the restriction of a 1-form on \mathbb{R}^3 to Σ .

Proof. Equation (47) follows immediately from the definition of θ_i given in Equation (46). We give three proofs of the assertion about θ_4 .

1. Consider the 1 form $\theta = \frac{\sigma_1}{2\sigma_3} d\sigma_2 - \frac{\sigma_2}{2\sigma_3} d\sigma_1$ on \mathbb{R}^3 . Then

$$\sigma^*(\theta|_\Sigma) = \frac{x_1^2 dx_2^2 - x_2^2 dx_1^2}{2x_1x_2} = x_1 dx_2 - x_2 dx_1 = \vartheta_4.$$

The following argument shows that the 1-form $\theta|_\Sigma = \frac{\sigma_1}{2\sigma_3}|_\Sigma (d\sigma_2)|_\Sigma - \frac{\sigma_2}{2\sigma_3}|_\Sigma (d\sigma_1)|_\Sigma$ is not smooth, because its coefficients are not smooth functions on Σ . First we need some geometric information about the \mathbb{Z}_2 orbit space $\Sigma \subseteq \mathbb{R}^3$ defined by $\sigma_3^2 = \sigma_1\sigma_2$ with $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$. The only subgroups of \mathbb{Z}_2 are the identity $\{e\}$ and \mathbb{Z}_2 . The isotropy group $(\mathbb{Z}_2)_x$ at $x \in \mathbb{R}^2$ is \mathbb{Z}_2 if $x = 0$ and $\{e\}$ if $x \neq 0$. The corresponding orbit types are $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$, whose image under the orbit map σ is $O = \{(0, 0, 0)\}$, the vertex of the cone Σ , and $\Sigma \setminus O$, which is a smooth manifold. Thus, $\theta|_{(\Sigma \setminus O)}$ is a smooth 1-form, whose pull back under σ is the smooth 1-form ϑ_4 on $\mathbb{R}^2 \setminus \{(0, 0)\}$. The 1-form $\theta|_{(\Sigma \setminus O)}$ does not extend to a smooth 1-form $\theta|_\Sigma$ because the functions $\frac{\sigma_1}{2\sigma_3}|_\Sigma$ and $\frac{\sigma_2}{2\sigma_3}|_\Sigma$ are not smooth at $(0, 0, 0)$, the vertex of the cone Σ . To see this let $\sigma^0 = (\sigma_1^0, \sigma_2^0, \sigma_3^0) \in \Sigma \setminus O$. The closed line segment $\ell_{\sigma^0}([0, 1])$, where $\ell_{\sigma^0} : [0, 1] \rightarrow \Sigma : t \mapsto t\sigma^0 = (t\sigma_1^0, t\sigma_2^0, t\sigma_3^0)$, lies in Σ and joins $(0, 0, 0)$ to σ^0 . Now $\frac{\sigma_1}{2\sigma_3}(\ell_{\sigma^0}(t)) = \frac{t\sigma_1^0}{2t\sigma_3^0} = \frac{\sigma_1^0}{2\sigma_3^0}$. So $\frac{\sigma_1}{2\sigma_3}(0, 0, 0) = \frac{\sigma_1^0}{2\sigma_3^0}$. Hence the function $\frac{\sigma_1}{2\sigma_3}|_\Sigma$ is not continuous at $(0, 0, 0)$. A similar argument shows that the function $\frac{\sigma_2}{2\sigma_3}|_\Sigma$ is not continuous at $(0, 0, 0)$.

2. The following argument shows that the 1-form θ_4 on Σ defined in Equation (48) is not the restriction to Σ of any smooth 1-form on \mathbb{R}^3 . Suppose it is. Then $\theta_4 = \sum_{j=1}^3 \bar{A}_j d\sigma_j$, for some $\bar{A}_j \in C^\infty(\Sigma) = C^\infty(\mathbb{R}^3)/I$, where I is the ideal of $C^\infty(\mathbb{R}^3)$ generated by $\sigma_3^2 - \sigma_1\sigma_2$. Using (48) we obtain

$$-\sigma_3 + I = \bar{\sigma}_3 = Y_1|_\Sigma \lrcorner \theta_4 = (2\sigma_1 \frac{\partial}{\partial \sigma_1} + \sigma_3 \frac{\partial}{\partial \sigma_3})|_\Sigma \lrcorner \theta_4 = 2\bar{\sigma}_1 \bar{A}_1 + \bar{\sigma}_3 \bar{A}_3,$$

which implies

$$-\sigma_3 = 2\sigma_1 A_1 + \sigma_3 A_3 + I. \tag{49a}$$

Similarly,

$$\sigma_1 = 2\sigma_3 A_1 + \sigma_2 A_3 + I \tag{49b}$$

$$-\sigma_2 = 2\sigma_3 A_2 + \sigma_1 A_3 + I \tag{49c}$$

$$-\sigma_3 = 2\sigma_2 A_2 + \sigma_3 A_3 + I. \tag{49d}$$

Set $A_1 = -\sigma_2$ and $A_3 = -1 + 2\sigma_3$. Then

$$2\sigma_1 A_1 + \sigma_3 A_3 = -2\sigma_1\sigma_2 - \sigma_3 + 2\sigma_3^2 = -\sigma_3 + I.$$

So Equation (49a) holds. Multiplying (49b) by σ_1 and (49c) by σ_2 and adding gives

$$\begin{aligned} \sigma_1^2 - \sigma_2^2 &= 2(\sigma_1 A_1 + \sigma_2 A_2)\sigma_3 + 2\sigma_1\sigma_2 A_3 + I \\ &= 2\sigma_3[\sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3] + I = 2\sigma_3[-\sigma_1\sigma_2 + \sigma_2 A_2 - \sigma_3 + \sigma_3^2] + I = 2\sigma_3[\sigma_2 A_2 - \sigma_3 + \sigma_3^2] + I. \end{aligned} \tag{50}$$

But σ_3 does not divide $\sigma_1^2 - \sigma_2^2$, which does not lie in I . Thus, Equation (50) does not hold for any choice of $A_2 \in C^\infty(\mathbb{R}^3)$. Hence our hypothesis is false, i.e., the 1-form θ_4 on $\mathbb{R}^2/\mathbb{Z}_2 = \Sigma$ is not the restriction to Σ of a 1-form on \mathbb{R}^3 .

3. Our third proof is more analytic. The 1-form θ_4 (48) on the orbit space $\Sigma \subseteq \mathbb{R}^3$ is not the restriction to Σ of a 1 form $\theta = \sum_{j=1}^3 A_j d\sigma_j$ on \mathbb{R}^3 , where $A_j \in C^\infty(\mathbb{R}^3)$. Suppose that $\theta_4 = \theta|_\Sigma$, then

$$\sigma^*(d\theta_4) = d(\sigma^*\theta_4) = d\theta_4 = d(x_1 dx_2 - x_2 dx_1) = 2 dx_1 \wedge dx_2,$$

which does not vanish at $(0, 0) \in \mathbb{R}^2$. However, the 2-form

$$\sigma^*((d\theta)|_\Sigma) = \sum_{j=1}^3 d(\sigma^* A_j) \wedge \sigma^*(d\bar{\sigma}_j) = \sum_{j=1}^3 d(\sigma^* A_j) \wedge d(\sigma^*(\bar{\sigma}_j)) = \sum_{j=1}^3 d(\sigma^* A_j) \wedge \vartheta_j$$

vanishes at $(0, 0)$, since the 1-forms ϑ_j (44) for $j = 1, 2, 3$ vanish at $(0, 0)$. This is a contradiction, since $d\theta_4 = d(\theta|_\Sigma)$. \square

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