axioms

MDPI

Article

On a Class of Isoperimetric Constrained Controlled Optimization Problems

Savin Treanţă

Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania; savin.treanta@upb.ro

Abstract: In this paper, we investigate the Lagrange dynamics generated by a class of isoperimetric constrained controlled optimization problems involving second-order partial derivatives and boundary conditions. More precisely, we derive necessary optimality conditions for the considered class of variational control problems governed by path-independent curvilinear integral functionals. Moreover, the theoretical results presented in the paper are accompanied by an illustrative example. Furthermore, an algorithm is proposed to emphasize the steps to be followed to solve a control problem such as the one studied in this paper.

Keywords: controlled second-order Lagrangian; Euler–Lagrange equations; isoperimetric constraints; curvilinear integral; differential 1-form

MSC: 49K15; 49K20; 49K21; 65K10

check for updates

Citation: Treanţă, S. On a Class of Isoperimetric Constrained Controlled Optimization Problems. *Axioms* **2021**, *10*, 112. https://doi.org/10.3390/axioms10020112

Academic Editors: Davron Aslonqulovich Juraev, Samad Noeiaghdam and Hsien-Chung Wu

Received: 26 April 2021
Accepted: 1 June 2021
Published: 3 June 2021

1. Introduction

In the last decade, several researchers (see, for instance, Treanţă [1–8], Jayswal et al. [9] and Mititelu and Treanţă [10]) have studied several controlled processes by considering some integral functionals with PDE, PDI, or mixed constraints. More specifically, these researchers have introduced and investigated new classes of optimization problems governed by multiple and path-independent curvilinear integral functionals with mixed constraints involving first-order PDEs of m-flow type, partial differential inequations and boundary conditions. In this regard, quite recently, Treanţă [11] established the optimality conditions for a class of constrained interval-valued optimization problems governed by path-independent curvilinear integral (mechanical work) cost functionals. More exactely, he formulated and proved a minimal criterion of optimality such that a local LU-optimal solution of the considered constrained optimization problem to be its global LU-optimal solution. On the other hand, due to their importance in the applied sciences and engineering, the isoperimetric constrained optimization problems have been introduced, studied and analyzed by many researchers. In this respect, by using the Pontryagin's principle, Schmitendorf [12] established necessary optimality conditions for a class of isoperimetric constrained control problems with inequality constraints at the terminal time. Further, Forster and Long [13] have studied the same isoperimetric constrained optimization problem formulated in Schmitendorf [12] (see, also, Schmitendorf [14]). They have established the associated necessary conditions of optimality by considering an alternative transformation technique. Recently, Benner et al. [15] investigated bang-bang control strategies corresponding to periodic trajectories with isoperimetric constraints for a control problem, with application to nonlinear chemical reactions. For other different but connected ideas on this subject, the reader is directed to the following reasearch works [16–20].

In this paper, motivated and inspired by the research works conducted by Hestenes [21], Lee [22], Schmitendorf [12] and Treanţă [4], we introduce a new class of isoperimetric constrained controlled optimization problems governed by path-independent curvilinear integral functionals which involves second-order partial derivatives and boundary conditions.

Concretely, in comparison with other related research papers, without restrict our analysis to linear systems having convex cost (see Lee [22]), we build a mathematical framework that is more general than in Hestenes [21] and Schmitendorf [12], both by the presence of path-independent curvilinear integrals as isoperimetric constraints but also by the inclusion of second-order partial derivatives and the new proof associated with the main result. Furthermore, besides totally new elements mentioned above, due to the physical meaning of the integral functionals used (as is well-known the path-independent curvilinear integrals represent the mechanical work performed by a variable force in order to move its point of application along a given piecewise smooth curve), this paper becomes a fundamental work for researchers in the field of applied mathematics and ingineering.

The paper is divided as follows. Section 2 introduces the controlled optimization problem under study, and includes the main result of the current paper, namely, Theorem 1. This result establishes the necessary conditions of optimality for the considered isoperimetric constrained variational control problem. Furthermore, an illustrative example is presented in the second part of Section 2. Moreover, to emphasize the steps to be followed to solve a control problem such as the one studied in this paper, an algorithm is presented. Section 3 contains the conclusions of the paper.

2. Isoperimetric Constrained Controlled Optimization Problem

In the following, let $\mathcal{L}_\zeta\big(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t\big)$, $\zeta = \overline{1,m}$, be C^3-class functions, called *multi-time controlled second-order Lagrangians*, where $t = (t^\alpha) = (t^1, \cdots, t^m) \in \Lambda_{t_0,t_1} \subset \mathbb{R}^m_+$, $s = (s^i) = \Big(s^1, \cdots, s^n\Big) : \Lambda_{t_0,t_1} \to \mathbb{R}^n$ is a C^4-class function (called the *state variable*) and $u = (u^\vartheta) = \Big(u^1, \cdots, u^k\Big) : \Lambda_{t_0,t_1} \to \mathbb{R}^k$ is a piecewise continuous function (called the *control variable*). Furthermore, denote $s_\alpha(t) := \dfrac{\partial s}{\partial t^\alpha}(t)$, $s_{\alpha\beta}(t) := \dfrac{\partial^2 s}{\partial t^\alpha \partial t^\beta}(t)$, $\alpha, \beta \in \{1, ..., m\}$, and consider $\Lambda_{t_0,t_1} = [t_0, t_1]$ (*multi-time interval* in $\mathbb{R}^m_+$) is a hyper-parallelepiped determined by the diagonally opposite points t_0, $t_1 \in \mathbb{R}^m_+$. Moreover, we assume that the previous multi-time controlled second-order Lagrangians determine a controlled closed (complete integrable) Lagrange 1-form

$$\mathcal{L}_\zeta\big(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t\big)dt^\zeta$$

(see summation over the repeated indices, Einstein summation), which generates the following controlled path-independent curvilinear integral functional

$$J(s(\cdot), u(\cdot)) = \int_{\Upsilon_{t_0,t_1}} \mathcal{L}_\zeta\big(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t\big)dt^\zeta, \tag{1}$$

where Υ_{t_0,t_1} is a smooth curve, included in Λ_{t_0,t_1}, joining the points t_0, $t_1 \in \mathbb{R}^m_+$.

Isoperimetric constrained controlled optimization problem. *Find the pair (s^*, u^*) that minimizes the above controlled path-independent curvilinear integral functional (1), among all the pair functions (s, u) satisfying*

$$s(t_0) = s_0, \quad s(t_1) = s_1, \quad s_\gamma(t_0) = \tilde{s}_{\gamma 0}, \quad s_\gamma(t_1) = \tilde{s}_{\gamma 1}$$

and the isoperimetric constraints (constant level sets of some controlled curvilinear integral functionals) defined as follows:

$$\int_{\Upsilon_{t_0,t_1}} g^a_\zeta\big(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t\big)dt^\zeta = l^a, \quad a = 1, 2, \cdots, r \le n,$$

where

$$g^a_\zeta\big(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t\big)dt^\zeta, \quad a = 1, 2, \cdots, r$$

are (C^1-class functions) complete integrable differential 1-forms, that is, $D_\gamma g_\zeta = D_\zeta g_\gamma,\ \gamma, \zeta \in \{1, \cdots, m\},\ \gamma \neq \zeta$, *where* $D_\gamma := \dfrac{\partial}{\partial t^\gamma},\ \gamma \in \{1, \cdots, m\}$.

In order to formulate the necessary optimality conditions of the above controlled optimization problem (1), associated with the aforementioned isoperimetric constraints, we introduce the curve $\Upsilon_{t_0,t} \subset \Upsilon_{t_0,t_1}$ and the auxiliary variables

$$y^a(t) = \int_{\Upsilon_{t_0,t}} g^a_\zeta(s(\tau), s_\gamma(\tau), s_{\alpha\beta}(\tau), u(\tau), \tau)d\tau^\zeta, \quad a = 1, 2, \cdots, r,$$

which satisfy $y^a(t_0) = 0,\ y^a(t_1) = l^a$. It results that the functions y^a fulfil the following controlled complete integrable first-order PDEs

$$\frac{\partial y^a}{\partial t^\zeta}(t) = g^a_\zeta(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t), \quad y^a(t_1) = l^a.$$

Now, under the Abadie constraint qualifications, considering the Lagrange multiplier $p = (p_a(t))$ and by denoting $y = (y^a(t))$, we build new multi-time controlled second-order Lagrangians

$$\mathcal{L}_{1\zeta}(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), y(t), y_\zeta(t), p(t), t) = \mathcal{L}_\zeta(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t)$$
$$+p_a(t)\left(g^a_\zeta(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t) - \frac{\partial y^a}{\partial t^\zeta}(t)\right), \quad \zeta = \overline{1, m},$$

which change the initial controlled optimization problem (with isoperimetric constraints defined by controlled path-independent curvilinear integral functionals) into an unconstrained controlled optimization problem

$$\min_{(s(\cdot), u(\cdot), y(\cdot), p(\cdot))} \int_{\Upsilon_{t_0,t_1}} \mathcal{L}_{1\zeta}(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), y(t), y_\zeta(t), p(t), t)dt^\zeta \tag{2}$$
$$s(t_q) = s_q, \quad s_\gamma(t_q) = \tilde{s}_{\gamma q}, \quad q = 0, 1$$
$$y(t_0) = 0, \quad y(t_1) = l.$$

According to Lagrange theory (Treanţă [4]), a minimum point of (1) is found among the minimum points of (2).

A *multi-index* (see Saunders [23]) is an m-tuple U of natural numbers. The components of U are denoted $U(\alpha)$, where α is an ordinary index, $1 \leq \alpha \leq m$. The multi-index 1_α is defined by $1_\alpha(\alpha) = 1,\ 1_\alpha(\beta) = 0$ for $\alpha \neq \beta$. The addition and the substraction of the multi-indexes are defined componentwise (although the result of a substraction might not be a multi-index): $(U \pm V)(\alpha) = U(\alpha) \pm V(\alpha)$. The length of a multi-index is $\mid U \mid = \sum_{\alpha=1}^{m} U(\alpha)$, and its factorial is $U! = \prod_{\alpha=1}^{m}(U(\alpha))!$. The number of distinct indices represented by $\{\alpha_1, \alpha_2, ..., \alpha_k\},\ \alpha_j \in \{1, 2, ..., m\},\ j = \overline{1, k}$, is

$$\mu(\alpha_1, \alpha_2, ..., \alpha_k) = \frac{\mid 1_{\alpha_1} + 1_{\alpha_2} + ... + 1_{\alpha_k} \mid !}{(1_{\alpha_1} + 1_{\alpha_2} + ... + 1_{\alpha_k})!}.$$

The following theorem represents the main result of this paper. It establishes the necessary conditions of optimality associated with the considered isoperimetric constrained controlled optimization problem.

Theorem 1. *If* $(s^*(\cdot),\ u^*(\cdot),\ y^*(\cdot),\ p^*(\cdot))$ *is solution for (2), then*

$$(s^*(\cdot),\ u^*(\cdot),\ y^*(\cdot),\ p^*(\cdot))$$

is solution of the following Euler–Lagrange system of PDEs

$$\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^i} - D_\gamma \frac{\partial \mathcal{L}_{1\zeta}}{\partial s^i_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{L}_{1\zeta}}{\partial s^i_{\alpha\beta}} = 0, \quad i = \overline{1,n},\ \zeta = \overline{1,m}$$

$$\frac{\partial \mathcal{L}_{1\zeta}}{\partial u^\vartheta} - D_\gamma \frac{\partial \mathcal{L}_{1\zeta}}{\partial u^\vartheta_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{L}_{1\zeta}}{\partial u^\vartheta_{\alpha\beta}} = 0, \quad \vartheta = \overline{1,k},\ \zeta = \overline{1,m}$$

$$\frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a} - D_\zeta \frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_\zeta} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_{\alpha\beta}} = 0, \quad a = \overline{1,r},\ \zeta = \overline{1,m}$$

$$\frac{\partial \mathcal{L}_{1\zeta}}{\partial p_a} - D_\gamma \frac{\partial \mathcal{L}_{1\zeta}}{\partial p_{a,\gamma}} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{L}_{1\zeta}}{\partial p_{a,\alpha\beta}} = 0, \quad a = \overline{1,r},\ \zeta = \overline{1,m}$$

where $p_{a,\gamma} := \dfrac{\partial p_a}{\partial t^\gamma}$, $p_{a,\alpha\beta} := \dfrac{\partial^2 p_a}{\partial t^\alpha \partial t^\beta}$, $u^\vartheta_{\alpha\beta} := \dfrac{\partial^2 u^\vartheta}{\partial t^\alpha \partial t^\beta}$, $y^a_{\alpha\beta} := \dfrac{\partial^2 y^a}{\partial t^\alpha \partial t^\beta}$, $\alpha,\ \beta,\ \gamma,\ \zeta \in \{1,2,...,m\}$.

Proof. Let $(s(t), u(t), y(t), p(t))$ be a solution for (2) and $s(t) + \varepsilon \mathsf{h}(t)$ is a variation of $s(t)$, with $\mathsf{h}(t_0) = \mathsf{h}(t_1) = 0$, $\mathsf{h}_\eta(t_0) = \mathsf{h}_\eta(t_1) = 0$, $\eta \in \{1,2,...,m\}$ (see $\mathsf{h}_\eta := \dfrac{\partial \mathsf{h}}{\partial t^\eta}$). Furthermore, let $p(t) + \varepsilon \mathsf{f}(t)$ be a variation of $p(t)$, with $\mathsf{f}(t_0) = \mathsf{f}(t_1) = 0$. In the same manner, consider $u(t) + \varepsilon \mathsf{m}(t)$, $y(t) + \varepsilon \mathsf{n}(t)$ be a variation of $u(t)$ and $y(t)$, respectively, with $\mathsf{m}(t_0) = \mathsf{m}(t_1) = \mathsf{n}(t_0) = \mathsf{n}(t_1) = 0$. The functions $\mathsf{h}, \mathsf{f}, \mathsf{m}, \mathsf{n}$ represent some "small" variations and ε is a "small" parameter used in our variational arguments. By considering the aforementioned variations, the controlled curvilinear integral functional becomes a function depending by ε, that is, a controlled curvilinear integral with parameter

$$I(\varepsilon) = \int_{\Upsilon_{t_0,t_1}} \mathcal{L}_{1\zeta}(s(t) + \varepsilon \mathsf{h}(t), s_\gamma(t) + \varepsilon \mathsf{h}_\gamma(t), s_{\alpha\beta}(t) + \varepsilon \mathsf{h}_{\alpha\beta}(t), u(t) + \varepsilon \mathsf{m}(t),$$

$$y(t) + \varepsilon \mathsf{n}(t), y_\zeta(t) + \varepsilon \mathsf{n}_\zeta(t), p(t) + \varepsilon \mathsf{f}(t), t) dt^\zeta.$$

By hypothesis, we must have the following relation

$$0 = \frac{d}{d\varepsilon} I(\varepsilon)|_{\varepsilon=0} = \int_{\Upsilon_{t_0,t_1}} \Big(\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j} \mathsf{h}^j + \frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_\gamma} \mathsf{h}^j_\gamma + \frac{1}{\mu(\alpha,\beta)} \frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}} \mathsf{h}^j_{\alpha\beta} + \frac{\partial \mathcal{L}_{1\zeta}}{\partial u^\vartheta} \mathsf{m}^\vartheta$$

$$+ \frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a} \mathsf{n}^a + \frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_\zeta} \mathsf{n}^a_\zeta + \frac{\partial \mathcal{L}_{1\zeta}}{\partial p_a} \mathsf{f}^a \Big) dt^\zeta$$

$$= BT + \int_{\Upsilon_{t_0,t_1}} \Big(\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j} - D_\gamma \frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}} \Big) \mathsf{h}^j dt^\zeta$$

$$+ \int_{\Upsilon_{t_0,t_1}} \Big(\frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a} - D_\zeta \frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_\zeta} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_{\alpha\beta}} \Big) \mathsf{n}^a dt^\zeta$$

$$+ \int_{\Upsilon_{t_0,t_1}} \Big(\frac{\partial \mathcal{L}_{1\zeta}}{\partial u^\vartheta} - D_\gamma \frac{\partial \mathcal{L}_{1\zeta}}{\partial u^\vartheta_\gamma} + \frac{1}{\mu(\alpha,\beta)} D^2_{\alpha\beta} \frac{\partial \mathcal{L}_{1\zeta}}{\partial u^\vartheta_{\alpha\beta}} \Big) \mathsf{m}^\vartheta dt^\zeta$$

$$+\int_{\Upsilon_{t_0,t_1}}\Big(\frac{\partial \mathcal{L}_{1\zeta}}{\partial p_a}-D_\gamma\frac{\partial \mathcal{L}_{1\zeta}}{\partial p_{a,\gamma}}+\frac{1}{\mu(\alpha,\beta)}D^2_{\alpha\beta}\frac{\partial \mathcal{L}_{1\zeta}}{\partial p_{a,\alpha\beta}}\Big)\mathsf{f}^a dt^\zeta.$$

Taking into account the formula of integration by parts, we find the following equalities

$$\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_\gamma}\mathsf{h}^j_\gamma=-\mathsf{h}^j D_\gamma\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_\gamma}+D_\gamma\left(\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_\gamma}\mathsf{h}^j\right),$$

$$\frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_\zeta}\mathsf{n}^a_\zeta=-\mathsf{n}^a D_\zeta\frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_\zeta}+D_\zeta\left(\frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_\zeta}\mathsf{n}^a\right),$$

$$\frac{1}{\mu(\alpha,\beta)}\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}}\mathsf{h}^j_{\alpha\beta}=\frac{1}{\mu(\alpha,\beta)}\left[\mathsf{h}^j D^2_{\alpha\beta}\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}}-D_\alpha\left(\mathsf{h}^j D_\beta\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}}\right)+D_\beta\left(\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}}\mathsf{h}^j_\alpha\right)\right].$$

The boundary terms BT vanish (see, also, $\mathsf{h}(t_q)=\mathsf{m}(t_q)=\mathsf{n}(t_q)=\mathsf{f}(t_q)=0,\ \mathsf{h}_\eta(t_q)=0,\ q=0,1$), by considering the following equalities

$$D_\gamma\left(\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_\gamma}\mathsf{h}^j\right)=D_\zeta\left(\frac{\partial \mathcal{L}_{1\gamma}}{\partial s^j_\gamma}\mathsf{h}^j\right),$$

$$D_\alpha\left(\mathsf{h}^j D_\beta\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}}\right)=D_\zeta\left(\mathsf{h}^j D_\beta\frac{\partial \mathcal{L}_{1\alpha}}{\partial s^j_{\alpha\beta}}\right),$$

$$D_\beta\left(\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}}\mathsf{h}^j_\alpha\right)=D_\zeta\left(\frac{\partial \mathcal{L}_{1\beta}}{\partial s^j_{\alpha\beta}}\mathsf{h}^j_\alpha\right).$$

In addition, we assume that the solution $(s(t),u(t),y(t),p(t))$ in (2) fulfils the following complete integrability conditions (closeness conditions) of Lagrange 1-form $L_{1\zeta}$, that is,

$$\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^i}\frac{\partial s^i}{\partial t^\alpha}+\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^i_\gamma}\frac{\partial s^i_\gamma}{\partial t^\alpha}+\frac{1}{\mu(\alpha,\beta)}\frac{\partial \mathcal{L}_{1\zeta}}{\partial s^i_{\alpha\beta}}\frac{\partial s^i_{\alpha\beta}}{\partial t^\alpha}+\frac{\partial \mathcal{L}_{1\zeta}}{\partial p_a}\frac{\partial p_a}{\partial t^\alpha}+\frac{\partial \mathcal{L}_{1\zeta}}{\partial t^\alpha}$$
$$+\frac{\partial \mathcal{L}_{1\zeta}}{\partial u^\vartheta}\frac{\partial u^\vartheta}{\partial t^\alpha}+\frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a}\frac{\partial y^a}{\partial t^\alpha}+\frac{\partial \mathcal{L}_{1\zeta}}{\partial y^a_\zeta}\frac{\partial y^a_\zeta}{\partial t^\alpha}$$
$$=\frac{\partial \mathcal{L}_{1\alpha}}{\partial s^i}\frac{\partial s^i}{\partial t^\zeta}+\frac{\partial \mathcal{L}_{1\alpha}}{\partial s^i_\gamma}\frac{\partial s^i_\gamma}{\partial t^\zeta}+\frac{1}{\mu(\alpha,\beta)}\frac{\partial \mathcal{L}_{1\alpha}}{\partial s^i_{\alpha\beta}}\frac{\partial s^i_{\alpha\beta}}{\partial t^\zeta}+\frac{\partial \mathcal{L}_{1\alpha}}{\partial p_a}\frac{\partial p_a}{\partial t^\zeta}+\frac{\partial \mathcal{L}_{1\alpha}}{\partial t^\zeta}$$
$$+\frac{\partial \mathcal{L}_{1\alpha}}{\partial u^\vartheta}\frac{\partial u^\vartheta}{\partial t^\zeta}+\frac{\partial \mathcal{L}_{1\alpha}}{\partial y^a}\frac{\partial y^a}{\partial t^\zeta}+\frac{\partial \mathcal{L}_{1\alpha}}{\partial y^a_\zeta}\frac{\partial y^a_\zeta}{\partial t^\zeta}.$$

Furthermore, we assume that the variation functions $\mathsf{h}, \mathsf{f}, \mathsf{m}, \mathsf{n}$ satisfy the closeness conditions of the 1-form

$$\mathcal{L}_{1\zeta}(s(t)+\varepsilon\mathsf{h}(t), s_\gamma(t)+\varepsilon\mathsf{h}_\gamma(t), s_{\alpha\beta}(t)+\varepsilon\mathsf{h}_{\alpha\beta}(t), u(t)+\varepsilon\mathsf{m}(t),$$

$$y(t)+\varepsilon\mathsf{n}(t), y_\zeta(t)+\varepsilon\mathsf{n}_\zeta(t), p(t)+\varepsilon\mathsf{f}(t), t)dt^\zeta.$$

This condition adds the following PDEs

$$\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^i}\frac{\partial\mathsf{h}^i}{\partial t^\alpha}+\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^i_\gamma}\frac{\partial\mathsf{h}^i_\gamma}{\partial t^\alpha}+\frac{1}{\mu(\alpha,\beta)}\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^i_{\alpha\beta}}\frac{\partial\mathsf{h}^i_{\alpha\beta}}{\partial t^\alpha}+\frac{\partial\mathcal{L}_{1\zeta}}{\partial p_a}\frac{\partial\mathsf{f}^a}{\partial t^\alpha}$$

$$+\frac{\partial\mathcal{L}_{1\zeta}}{\partial u^\vartheta}\frac{\partial\mathsf{m}^\vartheta}{\partial t^\alpha}+\frac{\partial\mathcal{L}_{1\zeta}}{\partial y^a}\frac{\partial\mathsf{n}^a}{\partial t^\alpha}+\frac{\partial\mathcal{L}_{1\zeta}}{\partial y^a_\zeta}\frac{\partial\mathsf{n}^a_\zeta}{\partial t^\alpha}$$

$$=\frac{\partial\mathcal{L}_{1\alpha}}{\partial s^i}\frac{\partial\mathsf{h}^i}{\partial t^\zeta}+\frac{\partial\mathcal{L}_{1\alpha}}{\partial s^i_\gamma}\frac{\partial\mathsf{h}^i_\gamma}{\partial t^\zeta}+\frac{1}{\mu(\alpha,\beta)}\frac{\partial\mathcal{L}_{1\alpha}}{\partial s^i_{\alpha\beta}}\frac{\partial\mathsf{h}^i_{\alpha\beta}}{\partial t^\zeta}+\frac{\partial\mathcal{L}_{1\alpha}}{\partial p_a}\frac{\partial\mathsf{f}^a}{\partial t^\zeta}$$

$$+\frac{\partial\mathcal{L}_{1\alpha}}{\partial u^\vartheta}\frac{\partial\mathsf{m}^\vartheta}{\partial t^\zeta}+\frac{\partial\mathcal{L}_{1\alpha}}{\partial y^a}\frac{\partial\mathsf{n}^a}{\partial t^\zeta}+\frac{\partial\mathcal{L}_{1\alpha}}{\partial y^a_\zeta}\frac{\partial\mathsf{n}^a_\zeta}{\partial t^\zeta}.$$

Finally, we get

$$0=\int_{\Upsilon_{t_0,t_1}}\Big(\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^j}-D_\gamma\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^j_\gamma}+\frac{1}{\mu(\alpha,\beta)}D^2_{\alpha\beta}\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^j_{\alpha\beta}}\Big)\mathsf{h}^j dt^\zeta$$

$$+\int_{\Upsilon_{t_0,t_1}}\Big(\frac{\partial\mathcal{L}_{1\zeta}}{\partial y^a}-D_\zeta\frac{\partial\mathcal{L}_{1\zeta}}{\partial y^a_\zeta}+\frac{1}{\mu(\alpha,\beta)}D^2_{\alpha\beta}\frac{\partial\mathcal{L}_{1\zeta}}{\partial y^a_{\alpha\beta}}\Big)\mathsf{n}^a dt^\zeta$$

$$+\int_{\Upsilon_{t_0,t_1}}\Big(\frac{\partial\mathcal{L}_{1\zeta}}{\partial u^\vartheta}-D_\gamma\frac{\partial\mathcal{L}_{1\zeta}}{\partial u^\vartheta_\gamma}+\frac{1}{\mu(\alpha,\beta)}D^2_{\alpha\beta}\frac{\partial\mathcal{L}_{1\zeta}}{\partial u^\vartheta_{\alpha\beta}}\Big)\mathsf{m}^\vartheta dt^\zeta$$

$$+\int_{\Upsilon_{t_0,t_1}}\Big(\frac{\partial\mathcal{L}_{1\zeta}}{\partial p_a}-D_\gamma\frac{\partial\mathcal{L}_{1\zeta}}{\partial p_{a,\gamma}}+\frac{1}{\mu(\alpha,\beta)}D^2_{\alpha\beta}\frac{\partial\mathcal{L}_{1\zeta}}{\partial p_{a,\alpha\beta}}\Big)\mathsf{f}^a dt^\zeta$$

and, since the smooth curve Υ_{t_0,t_1} is arbitrary, we obtain the Euler–Lagrange system of PDEs formulated in theorem. □

Remark 1. *The Euler–Lagrange system of PDEs in Theorem 1 can be rewritten as follows*

$$\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^i}-D_\gamma\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^i_\gamma}+\frac{1}{\mu(\alpha,\beta)}D^2_{\alpha\beta}\frac{\partial\mathcal{L}_{1\zeta}}{\partial s^i_{\alpha\beta}}=0,\quad i=\overline{1,n},\ \zeta=\overline{1,m}$$

$$\frac{\partial\mathcal{L}_{1\zeta}}{\partial u^\vartheta}-D_\gamma\frac{\partial\mathcal{L}_{1\zeta}}{\partial u^\vartheta_\gamma}+\frac{1}{\mu(\alpha,\beta)}D^2_{\alpha\beta}\frac{\partial\mathcal{L}_{1\zeta}}{\partial u^\vartheta_{\alpha\beta}}=0,\quad \vartheta=\overline{1,k},\ \zeta=\overline{1,m}$$

$$\frac{\partial p_a}{\partial t^\zeta}=0,\quad a=\overline{1,r},\ \zeta=\overline{1,m}$$

$$\frac{\partial y^a}{\partial t^\zeta}(t)=g^a_\zeta(s(t),s_\gamma(t),s_{\alpha\beta}(t),u(t),t),\quad a=\overline{1,r},\ \zeta=\overline{1,m}.$$

In consequence, the Lagrange multiplier p is constant. Moreover, it is well determined only if the optimal solution is not an extrem for at least one of the following controlled path-independent curvilinear integral functionals

$$\int_{\Upsilon_{t_0,t_1}} g^a_\zeta(s(t),s_\gamma(t),s_{\alpha\beta}(t),u(t),t)dt^\zeta,\quad a=\overline{1,r}.$$

Illustrative example. Let us find the minimum for the following controlled curvilinear integral functional

$$J(s(\cdot), u(\cdot)) = \int_{\Upsilon_{0,1}} \Big(s^2(t) + u^2(t)\Big)dt^1 + \Big(s^2(t) + u^2(t)\Big)dt^2$$

subject to: $\displaystyle\int_{\Upsilon_{0,1}} s_{t^1}(t)dt^1 + s_{t^2}(t)dt^2 = 0$ *(path-independent curvilinear integral) and the boundary conditions* $s(0,0) = 0,\ s(1,1) = 0$*, where* $\Upsilon_{0,1}$ *is a* C^1*-class curve, included in* $[0,1]^2$*, joining the points* $(0,0),\ (1,1)$*.*

Solution. The path-independence associated with the cost functional $J(s(\cdot), u(\cdot))$ *gives the relation*

$$s\Big(\frac{\partial s}{\partial t^2} - \frac{\partial s}{\partial t^1}\Big) = u\Big(\frac{\partial u}{\partial t^1} - \frac{\partial u}{\partial t^2}\Big).$$

Furthermore, the associated Lagrange 1-form has the following components

$$\mathcal{L}_{11} = s^2(t) + u^2(t) + p(y_{t^1}(t) - s_{t^1}(t)),$$

$$\mathcal{L}_{12} = s^2(t) + u^2(t) + p(y_{t^2}(t) - s_{t^2}(t))$$

and the extremals are described by the following system of Euler–Lagrange PDEs

$$2s + \frac{\partial p}{\partial t^1} = 0, \quad 2s + \frac{\partial p}{\partial t^2} = 0,$$

$$2u = 0,$$

$$y_{t^1}(t) - s_{t^1}(t) = 0, \quad y_{t^2}(t) - s_{t^2}(t) = 0,$$

implying that $(s^*, u^*) = (0,0)$ *is the optimal solution of the considered isoperimetric constrained controlled optimization problem.*

Further, taking into account the above illustrative example and the theory developed in the paper, we formulate an algorithm. The main intention of the next algorithm is to synthesize the concrete steps to be followed to solve a control problem such as those studied in the paper. In particular, for a controlled path-independent curvilinear integral cost functional and a set of mixed (isoperimetric and boundary conditions) restrictions and self or normal data, the main goal is to find $(s^\star, u^\star)$ (satisfying the set of mixed constraints and normal data) such that $J(s^\star, u^\star) \leq J(s, u)$, for all feasible points (s, u). For this purpose, we start with a feasible point (s, u). If the pair (s, u) fulfils the necessary optimality conditions formulated in Theorem 1, then the "Generating Stage" (see below) is satisfied and we go to the next step, namely "Detecting Stage"; else, the algorithm stops. If the set of self or normal data is fulfilled, then the "Detecting Stage" is satisfied and we go to the next step, namely "Deciding Stage" (see below); else, the algorithm stops. For $(s^\star, u^\star)$ derived in "Detecting Stage", if $J(s^\star, u^\star) \leq J(s, u)$ holds for all feasible points (s, u), then $(s^\star, u^\star)$ is an optimal solution; else, the Algorithm 1 stops.

Algorithm 1:

DATA:

• controlled path-independent curvilinear integral cost functional

$$\min_{(s,u)} J(s,u) = \int_{\Upsilon_{t_0,t_1}} \mathcal{L}_\zeta\big(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t\big) dt^\zeta;$$

• set of mixed constraints

$$\int_{\Upsilon_{t_0,t_1}} g^a_\zeta\big(s(t), s_\gamma(t), s_{\alpha\beta}(t), u(t), t\big) dt^\zeta = l^a, \quad a = 1,2,\cdots,r \leq n$$

and

$$s(t_q) = s_q, \quad s_\gamma(t_q) = \tilde{s}_{\gamma q}, \quad q = 0,1;$$

• set of self or normal data

- the differential 1-form $g = \left(g^a_\zeta\right)$ satisfies the closeness conditions;

RESULT:

$$\mathsf{S} = \{(s^\star, u^\star) | J(s^\star, u^\star) \leq J(s,u),$$
$$\text{with } (s^\star, u^\star) \text{ satisfying the set}$$
$$\text{of ; mixed ; constraints ; and ; normal ; data}\};$$

BEGIN

• Generating Stage: consider (s,u) a feasible point
if the necessary optimality conditions (see Theorem 1)
are not compatible with respect to (s,u)
then STOP
else GO to the next step

• Detecting Stage: monitoring of Lagrange multipliers
if the set of self or normal data is not fulfilled
then STOP
else GO to the next step

• Deciding Stage: let $(s^\star, u^\star)$ be derived in Detecting Stage
if $J(s,u) \geq J(s^\star, u^\star)$ holds for all feasible points (s,u)
then $(s^\star, u^\star)$ is an optimal solution
else STOP

END

3. Conclusions

In this paper, we have studied a new class of isoperimetric constrained controlled optimization problems. In accordance with Lagrange Theory, necessary optimality conditions have been formulated and proved for the considered class of variational control problems governed by path-independent curvilinear integrals and second-order partial derivatives. The theoretical mathematical results developed in the paper have been highlighted by an illustrative example and an algorithm.

As a new research direction on the class of problems introduced in this paper, we mention, for example, the study of well-posedness.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Treanţă, S. A necessary and sufficient condition of optimality for a class of multidimensional control problems. *Optim. Control Appl. Methods* **2020**, *41*, 2137–2148. [CrossRef]
2. Treanţă, S. On a global efficiency criterion in multiobjective variational control problems with path-independent curvilinear integral cost functionals. *Ann. Oper. Res.* **2020**, 1–9. [CrossRef]
3. Treanţă, S. Saddle-point optimality criteria in modified variational control problems with PDE constraints. *Optim. Control Appl. Methods* **2020**, *41*, 1160–1175. [CrossRef]
4. Treanţă, S. Constrained variational problems governed by second-order Lagrangians. *Appl. Anal.* **2020**, *99*, 1467–1484. [CrossRef]
5. Treanţă, S.; Mititelu, Ş. Efficiency for variational control problems on Riemann manifolds with geodesic quasiinvex curvilinear integral functionals. *Rev. Real Acad. Cienc. Exactas FíSicas Nat. Ser. Matemáticas* **2020**, *114*, 113. [CrossRef]
6. Treanţă, S.; Arana-Jiménez, M.; Antczak, T. A necessary and sufficient condition on the equivalence between local and global optimal solutions in variational control problems. *Nonlinear Anal.* **2020**, *191*, 111640.
7. Treanţă, S. On a modified optimal control problem with first-order PDE constraints and the associated saddle-point optimality criterion. *Eur. J. Control* **2020**, *51*, 1–9. [CrossRef]
8. Treanţă, S. Efficiency in generalized V-KT-pseudoinvex control problems. *Int. J. Control* **2020**, *93*, 611–618. [CrossRef]
9. Jayswal, A.; Antczak, T.; Jha, S. Modified objective function approach for multitime variational problems. *Turk. J. Math.* **2018**, *42*, 1111–1129.
10. Mititelu, Ş.; Treanţă, S. Efficiency conditions in vector control problems governed by multiple integrals. *J. Appl. Math. Comput.* **2018**, *57*, 647–665. [CrossRef]
11. Treanţă, S. On a class of constrained interval-valued optimization problems governed by mechanical work cost functionals. *J. Optim. Theory Appl.* **2021**, *188*, 913–924. [CrossRef]
12. Schmitendorf, W.E. Pontryagin's principle for problems with isoperimetric constraints and for problems with inequality terminal constraints. *J. Optim. Theory Appl.* **1976**, *18*, 561–567. [CrossRef]
13. Forster, B.A.; Long, N.V. Pontryagin's principle for problems with isoperimetric constraints and for problems with inequality terminal constraints: Comment. *J. Optim. Theory Appl.* **1978**, *25*, 317–322. [CrossRef]
14. Schmitendorf, W.E. Pontryagin's principle for problems with isoperimetric constraints and for problems with inequality terminal constraints: Reply. *J. Optim. Theory Appl.* **1978**, *25*, 323. [CrossRef]
15. Benner, P.; Seidel-Morgenstern, A.; Zuyev, A. Periodic switching strategies for an isoperimetric control problem with application to nonlinear chemical reactions. *Appl. Math. Model.* **2019**, *69*, 287–300. [CrossRef]
16. Bildhauer, M.; Fuchs, M.; Muller, J. A reciprocity principle for constrained isoperimetric problems and existence of isoperimetric subregions in convex sets. *Calc. Var. Partial. Differ. Equ.* **2018**, *57*, 60. [CrossRef]
17. Curtis, J.P. Complementary extremum principles for isoperimetric optimization problems. *Optim. Eng.* **2004**, *5*,417–430. [CrossRef]
18. Demyanov, V.F.; Tamasyan, G.S. Exact penalty functions in isoperimetric problems. *Optimization* **2011**, *60*, 153–177. [CrossRef]
19. Harper, L.H. *Global Methods for Combinatorial Isoperimetric Problems*; Cambridge University Press: Cambridge, UK, 2010.
20. Urziceanu, S.A. Necessary optimality conditions in isoperimetric constrained optimal control problems. *Symmetry* **2019**, *11*, 1380. [CrossRef]
21. Hestenes, M. *Calculus of Variations and Optimal Control Theory*; John Wiley and Sons: New York, NY, USA, 1966.
22. Lee, E.B. Linear Optimal Control Problems with lsoperimetric Constraints. *IEEE Trans. Autom. Control* **1967**, *12*, 87–90. [CrossRef]
23. Saunders, D.J. *The Geometry of Jet Bundles*; London Mathematical Society Lecture Notes Series 142; Cambridge University Press: Cambridge, UK, 1989.