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# On the Existence of Coupled Fractional Jerk Equations with Multi-Point Boundary Conditions

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**Abstract:** By coincidence degree theory due to Mawhin, some sufficient conditions for the existence of solution for a class of coupled jerk equations with multi-point conditions are established. The new existence results have not yet been reported before. Novel coupled fractional jerk equations with resonant boundary value conditions are discussed in detail for the first time. Our work is interesting and complements known results.

**Keywords:** coupled jerk equations; multi-point boundary conditions; coincidence degree theory; resonance

**MSC:** 34A08; 34B15



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## 1. Introduction

In this paper, we study the following coupled jerk system of the fractional order

$$\begin{cases} D_{0+}^{\alpha}x_1(t) = x_2(t), \\ D_{0+}^{\beta}x_2(t) = x_3(t), \\ D_{0+}^{\gamma}x_3(t) = f(y_1(t), y_2(t), y_3(t)), \\ D_{0+}^{\lambda}y_1(t) = y_2(t), \\ D_{0+}^{\mu}y_2(t) = y_3(t), \\ D_{0+}^{\nu}y_3(t) = g(x_1(t), x_2(t), x_3(t)), \end{cases} \tag{1}$$

with boundary value conditions given as

$$\begin{cases} x_2(0) = x_3(0) = 0, x_1(0) = \sum_{i=1}^{m-1} a_i x_1(\xi_i), \\ y_2(0) = y_3(0) = 0, y_1(0) = \sum_{i=1}^{m-1} b_i y_1(\eta_i), \end{cases} \tag{2}$$

where $t \in (0,1)$, $0 \le \xi_1 \le \xi_2 \le \cdots \le \xi_{m-1} \le 1$, $0 \le \eta_1 \le \eta_2 \le \cdots \le \eta_{m-1} \le 1$, $\sum_{i=1}^{m-1} a_i = \sum_{i=1}^{m-1} b_i = 1$, $\sum_{i=1}^{m-1} a_i\xi_i \ne 0$, $\sum_{i=1}^{m-1} b_i\eta_i \ne 0$, $0 < \varepsilon < 1$, $\varepsilon = \{\alpha, \beta, \gamma, \lambda, \mu, \nu\}$, $D_{0+}^{\varepsilon}$ denotes the Caputo fractional derivative, and $f, g : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions.

In 1978, S. H. Schot [1] presented the definition of a jerk, i.e., the rate of change of acceleration. This involves a third derivative of $x$ and has been found to have numerous applications in many areas of science, such as electrical circuits, laser physics, mechanics, acoustics, and dynamical processes. We refer the reader to [2–11].

As we all know, a three-dimensional dynamical system $\dot{x}(t) = y, \dot{y}(t) = z, \dot{z}(t) = f(x, y, z)$, can be written in the form $\dddot{x}(t) = f(x, \dot{x}, \ddot{x})$. The third order autonomous differential equation is named as a jerk equation. In recent years, the discussion of jerk equations has attracted much attention because it arises in a variety of different scientific fields, such as the theory of chaos, secure communication, electrical engineering, and economic systems. For recent results, we refer the readers to [12–20] and the references therein.

In 2020, Marcelo and Silva [18] used the algebraic criterion to determine a special form for the polynomial jerk function $j$ in order to guarantee the nonchaotic behavior of the following jerk equation:

$$\dddot{x}(t) = j(x, \dot{x}, \ddot{x}).$$

They provided a simpler proof for the nonchaotic behavior. The algebraic criterion proved in their work is quite general and can be used to study the nonchaotic behavior of other types of ordinary differential equations.

In [19], M. Ismail et al. dealt with an initial value problem of a nonlinear third order jerk equation:

$$\begin{cases} \dddot{x} + f(x, \dot{x}, \ddot{x}) = 0, \\ x(0) = 0, \ \dot{x}(0) = A, \ \ddot{x}(0) = 0. \end{cases}$$

The authors succeeded in extending the global error minimization method (GEMM) to obtain analytic approximations. Compared and simulated with the known solutions and the exact numerical ones, their obtained method were proven to be effective and to provide an efficient alternative to the previously known existing methods.

In nature, most nonlinear systems are mutually coupled. Coupled nonlinear systems have rich dynamic behaviors. As an extension of jerk equations, coupled jerk equations also have simple algebraic structures and more complex dynamic characteristics compared with jerk systems.

In [20], Chen et al. considered the following coupled systems of jerk equations:

$$\begin{cases} \dddot{x} = a\ddot{x} - \dot{x} + (|x| - 1) + \epsilon(x - y), \\ \dddot{y} = b\ddot{y} - \dot{y} + y - y^3 + \epsilon(y - x). \end{cases}$$

The authors investigated the dynamical evolution of the above coupled system and obtained bifurcation sets in parameter space by the analysis of the equilibrium points and their stabilities.

During the last two decades, great interest has been devoted to the study of fractional differential equations, which can serve as an excellent tool for the mathematical modeling of systems and processes in the fields of physics, chemistry, biology, electromagnetic, mechanics, economics, dynamical processes, etc. For more details regarding fractional differential equations involving initial or boundary conditions, see for instance, [21–31] and the references therein.

At the same time, the theory of fractional jerk equations has been also analytically investigated by some very interesting and novel papers (see [32–34]). In 2020, Echenausía-Monroy et al. [32] considered a multi-scroll generator system based on fractional jerk equations:

$$\begin{cases} D_{0+}^{q_x} x(t) = y(t), \\ D_{0+}^{q_y} y(t) = z(t), \\ D_{0+}^{q_z} z(t) = g(x, y, z), \end{cases}$$

where $g(x, y, z) = -\alpha[x + y + z - f(x)]$, $D_{0+}^{q_i}$ denotes the Caputo fractional derivatives $i = \{x, y, z\}$, and $\alpha \in \mathbb{R}$ is the control parameter. The authors further studied the effects of the above fractional jerk system and provided a physical interpretation based on statistical analysis. Compared to the integer-order system, their results show that the use of fractional-order can decrease the size of the generated attractor and can modify the long-range correlations in the system. The obtained results can not only be used for applications arising in engineering and the sciences, such as mobile surveillance devices, they can also enrich the analysis and understanding of the implications of fractional integration orders.

From the existing results, we can see a fact: although the solutions for fractional jerk equations have been studied by some authors, to the best of our knowledge, coupled jerk differential equations with the fractional order, have not been investigated until now. The objective of this paper is to fill that gap in the relevant literature.

It is easy to check that the following coupled fractional jerk equations:

$$\begin{cases} \big(D^{\gamma}_{0+}(D^{\beta}_{0+}(D^{\alpha}_{0+}u))\big)(t) = f\big(v(t), D^{\lambda}_{0+}v(t), (D^{\mu}_{0+}(D^{\lambda}_{0+}v))(t)\big), \\ \big(D^{\nu}_{0+}(D^{\mu}_{0+}(D^{\lambda}_{0+}v))\big)(t) = g\big(u(t), D^{\alpha}_{0+}u(t), (D^{\beta}_{0+}(D^{\alpha}_{0+}u))(t)\big), \\ u(0) = \sum_{i=1}^{m-1} a_i u(\xi_i), D^{\alpha}_{0+}u(0) = \big(D^{\beta}_{0+}(D^{\alpha}_{0+}u)\big)(0) = 0, \\ v(0) = \sum_{i=1}^{m-1} b_i v(\eta_i), D^{\lambda}_{0+}v(0) = \big(D^{\mu}_{0+}(D^{\lambda}_{0+}v)\big)(0) = 0, \end{cases} \tag{3}$$

are equivalent to (1) and (2). Due to the conditions (2), BVP (3) happens to be at resonance in the sense that the associated linear homogeneous equations:

$$\begin{cases} \big(D^{\gamma}_{0+}(D^{\beta}_{0+}(D^{\alpha}_{0+}u))\big)(t) = 0, \big(D^{\nu}_{0+}(D^{\mu}_{0+}(D^{\lambda}_{0+}v))\big)(t) = 0, \\ u(0) = \sum_{i=1}^{m-1} a_i u(\xi_i), D^{\alpha}_{0+}u(0) = \big(D^{\beta}_{0+}(D^{\alpha}_{0+}u)\big)(0) = 0, \\ v(0) = \sum_{i=1}^{m-1} b_i v(\eta_i), D^{\lambda}_{0+}v(0) = \big(D^{\mu}_{0+}(D^{\lambda}_{0+}v)\big)(0) = 0, \end{cases}$$

have $(u, v) = (c, d), c, d \in \mathbb{R}$ as nontrivial solutions.

In addition, we remark that if all the fractional orders $\alpha, \beta, \gamma, \lambda, \mu$, and $\nu$ are equal to 1, then Equation (3) can be rewritten as

$$\begin{cases} \dddot{u}(t) = f(v, \dot{v}, \ddot{v}), \ \dddot{v}(t) = g(u, \dot{u}, \ddot{u}), \\ u(0) = \sum_{i=1}^{m-1} a_i u(\xi_i), \ \dot{u}(0) = \ddot{u}(0) = 0, \\ v(0) = \sum_{i=1}^{m-1} b_i v(\eta_i), \ \dot{v}(0) = \ddot{v}(0) = 0, \end{cases}$$

which is a standard nonlinear coupled jerk system and is a complement of [20]. Compared with [20], we can see the nonhomogeneous terms $f$ and $g$ in the above system are more general. Furthermore, if $f = g$ and $u = v$, then the above problem can be written by $\dddot{u}(t) = f(u, \dot{u}, \ddot{u}), \ u(0) = \sum_{i=1}^{m-1} a_i u(\xi_i), \ \dot{u}(0) = \ddot{u}(0) = 0$. Compared with previous research works, such as [19] whose conditions include $x(0) = 0$, we find that the boundary value conditions, $u(0) = \sum_{i=1}^{m-1} a_i u(\xi_i)$, are nonlocal initial conditions, which were first used by Byszewski [35] and are more appropriate to describe natural phenomena for they contain more additional information.

As far as nonlinear jerk equations are concerned, most of the efforts on this topic are related to the initial value problem. For example, see [15,16] and the references therein. Among the existing results, no one result can be applied to our problem. Therefore, our results enrich the existing literature. This is another reason why we study problem (1).

This paper is organized as follows. In Section 2, we give some notations and lemmas. In Section 3, we establish a theorem of existence of a solution for the problem (1) and (2) by applying the coincidence degree theory due to Mawhin [36]. In Section 4, we give an example to demonstrate our results.

## 2. Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory that can be found in the recent literature [30].

**Definition 1** ([30])**.** *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds,$$

*provided that the right-hand side is pointwise defined on $(0, \infty)$.*

**Definition 2** ([30]). *The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C^n[0,+\infty)$ is given by*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad n-1 < \alpha < n,$$

*where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of $\alpha$.*

**Lemma 1** ([30]). *Let $n-1 < \alpha \le n$, $D_{0+}^{\alpha} u \in L^1(0,1)$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

*where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$.*

**Lemma 2** ([30]). *If $\beta > 0$, $\alpha + \beta > 0$, then the equation*

$$I_{0+}^{\alpha} I_{0+}^{\beta} f(x) = I_{0+}^{\alpha+\beta} f(x),$$

*is satisfied for a continuous function $f$.*

Now, let us recall notation about the coincidence degree continuation theorem [36].

**Definition 3** ([36]). *Let $Y$ and $Z$ be normed spaces. A linear operator $L : dom(L) \subset Y \to Z$ is said to be a Fredholm operator of index zero provided that*

*(i)* $\operatorname{Im} L$ *is a closed subset of $Z$;*
*(ii)* $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$.

It follows from Definition 3, if $L$ is a Fredholm operator of index zero, then there exist continuous projectors $P : Y \to Y$, $Q : Z \to Z$ as continuous projectors such that $\ker L = \operatorname{Im} P$, $\operatorname{Im} L = \ker Q$ and $Y = \ker L \oplus \ker P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible. We denote the inverse of this map by $K_P$. If $\Omega$ is an open bounded subset of $Y$, the map $N$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N = K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

**Theorem 1** ([36]). *Let $L$ be a Fredholm operator of index zero and $N$ be $L$-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:*

*(1)* $Lx \neq \lambda Nx$ *for each* $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;
*(2)* $Nx \notin \operatorname{Im} L$ *for each* $x \in \ker L \cap \partial\Omega$; *and*
*(3)* $\deg(JQN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, *where $Q : Z \to Z$ is a continuous projection as above with $\operatorname{Im} L = \ker Q$ and $J : \operatorname{Im} Q \to \ker L$ is any isomorphism.*
*Then, the equation $Lx = Nx$ has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

## 3. Main Results

In this section, we will prove the existence and uniqueness results for (1) and (2).

We use the Banach space $E = C[0,1]$ with the norm $\|u\|_\infty = \max_{0 \le t \le 1} |u(t)|$. Define two linear spaces

$$X = \{u(t) |\ u(t) \in E, D_{0+}^{\alpha} u(t) \in E, (D_{0+}^{\beta}(D_{0+}^{\alpha} u))(t) \in E\},$$
$$Y = \{v(t) |\ v(t) \in E, D_{0+}^{\lambda} u(t) \in E, (D_{0+}^{\mu}(D_{0+}^{\lambda} u))(t) \in E\}.$$

Clearly, $X$ and $Y$ are Banach spaces with the norm

$$\|u\|_X = \max\{\|u\|_\infty, \|D_{0+}^{\alpha} u(t)\|_\infty, \|(D_{0+}^{\beta}(D_{0+}^{\alpha} u))(t)\|_\infty\},$$
$$\|v\|_Y = \max\{\|v\|_\infty, \|D_{0+}^{\lambda} v(t)\|_\infty, \|(D_{0+}^{\mu}(D_{0+}^{\lambda} v))(t)\|_\infty\}.$$

We consider the Banach space $X \times Y$ endowed with the norm defined by $\|(u,v)\|_{X\times Y} = \max\{\|u\|_X, \|v\|_Y\}$ and $Z = E \times E$ is a Banach space with the norm defined by $\|(x,y)\|_Z = \max\{\|x\|_\infty, \|y\|_\infty\}$.

We define the linear operator $L_1$ from $\mathrm{dom}L_1 \cap X$ to $E$ by

$$L_1 u = \big(D^{\gamma}_{0+}(D^{\beta}_{0+}(D^{\alpha}_{0+}u))\big)(t),$$

where $\mathrm{dom}L_1 = \{u \in X | u(0) = \sum_{i=1}^{m-1} a_i u(\xi_i), D^{\alpha}_{0+}u(0) = \big(D^{\beta}_{0+}(D^{\alpha}_{0+}u)\big)(0) = 0\}$.

We define the linear operator $L_2$ from $\mathrm{dom}L_2 \cap Y$ to $E$ by

$$L_2 v = \big(D^{\nu}_{0+}(D^{\mu}_{0+}(D^{\lambda}_{0+}v))\big)(t),$$

where $\mathrm{dom}L_2 = \{v \in Y | v(0) = \sum_{i=1}^{m-1} b_i v(\eta_i), D^{\lambda}_{0+}v(0) = \big(D^{\mu}_{0+}(D^{\lambda}_{0+}v)\big)(0) = 0\}$.

We define the operator $L : \mathrm{dom}L \cap (X \times Y) \to Z$ by

$$L(u,v) = (L_1 u, L_2 v), \tag{4}$$

where $\mathrm{dom}L = \{(u,v) \in X \times Y | u \in \mathrm{dom}L_1, v \in \mathrm{dom}L_2\}$, and we define $N : X \times Y \to Z$ by setting

$$N(u,v) = (N_1 v, N_2 u),$$

where $N_1 : Y \to E$ is defined by

$$N_1 v(t) = f\big(v(t), D^{\lambda}_{0+}v(t), (D^{\mu}_{0+}(D^{\lambda}_{0+}v))(t)\big)$$

and $N_2 : X \to E$ is defined by

$$N_2 u(t) = g\big(u(t), D^{\alpha}_{0+}u(t), (D^{\beta}_{0+}(D^{\alpha}_{0+}u))(t)\big).$$

Then, the problem (1) and (2) can be written by $L(u,v) = N(u,v)$.

**Lemma 3.** *Let the mapping $L$ be defined by* (4), *then*

$$\mathrm{Ker}L = \big\{(u,v) \in X \times Y : (u,v) = (c,d), c,d \in \mathbb{R}\big\}, \tag{5}$$

$$\mathrm{Im}L = \big\{(x,y) \in E : \sum_{i=1}^{m-1} a_i I^{\alpha+\beta+\gamma}_{0+} x(\xi_i) = \sum_{i=1}^{m-1} b_i I^{\lambda+\mu+\nu}_{0+} y(\eta_i) = 0\big\}. \tag{6}$$

**Proof.** Clearly, $L_1$ and $L_2$ are linear operators. By $L_1 u = 0$, i.e., $\big(D^{\gamma}_{0+}(D^{\beta}_{0+}(D^{\alpha}_{0+}u))\big)(t) = 0$ and Lemma 1, we have

$$(D^{\beta}_{0+}(D^{\alpha}_{0+}u))(t) = c_0, \;\; c_0 \in \mathbb{R}.$$

Applying the operator of $I^{\beta}_{0+}$ on both sides of the above equation, we have

$$D^{\alpha}_{0+}u(t) = I^{\beta}_{0+}c_0 + c_1 \;\; c_0, c_1 \in \mathbb{R}.$$

By Lemma 2, we obtain

$$u(t) = I^{\beta+\alpha}_{0+}c_0 + I^{\alpha}_{0+}c_1 + c, \; c_0, c_1, c \in \mathbb{R}.$$

In view of $D^{\alpha}_{0+}u(0) = \big(D^{\beta}_{0+}(D^{\alpha}_{0+}u)\big)(0) = 0$, we obtain $c_0 = c_1 = 0$. Then, we have $u(t) = c$. By similar proof, if $L_2 v = 0$, then we have $v(t) = d, d \in \mathbb{R}$. Hence, one has that (5) holds. It is clear that $\mathrm{Ker}L = (c,d) \cong \mathbb{R}^2$.

Next, we prove that (6) holds. Let $x \in \mathrm{Im}L_1$, and thus there exists $u \in \mathrm{dom}L_1$ such that $x(t) = \big(D^{\gamma}_{0+}(D^{\beta}_{0+}(D^{\alpha}_{0+}u))\big)(t)$. By Lemma 1 and the definition of $\mathrm{dom}L_1$, we have

$$u(t) = I^{\alpha+\beta+\gamma}_{0+}x(t) + I^{\beta+\alpha}_{0+}c_0 + I^{\alpha}_{0+}c_1 + c.$$

In view of $D^{\alpha}_{0+}u(0) = \big(D^{\beta}_{0+}(D^{\alpha}_{0+}u)\big)(0) = 0$, we have $c_0 = c_1 = 0$. Hence,

$$u(t) = I^{\alpha+\beta+\gamma}_{0+}x(t) + c.$$

According to $u(0) = \sum_{i=1}^{m-1} a_i u(\xi_i)$ and $\sum_{i=1}^{m-1} a_i = 1$, we have

$$\sum_{i=1}^{m-1} a_i I^{\alpha+\beta+\gamma}_{0+}x(\xi_i) = 0.$$

On the other hand, suppose $x$ satisfies the above equation. Letting $u(t) = I^{\alpha+\beta+\gamma}_{0+}x(t)$, we can prove $u(t) \in \mathrm{dom}L_1$ and $L_1u(t) = x$.

Similarly, we can obtain

$$\sum_{i=1}^{m-1} b_i I^{\lambda+\mu+\nu}_{0+}y(\eta_i) = 0.$$

Thus, (6) holds. The proof is complete. □

**Lemma 4.** *Let $L$ be defined by (4), then $L$ is a Fredholm operator of index zero. Furthermore, the operator $K_P : \mathrm{Im}\, L \to \mathrm{dom}\, L \cap \mathrm{Ker}P$ can be written by*

$$K_P(x,y) = \big(I^{\alpha+\beta+\gamma}_{0+}x,\ I^{\lambda+\mu+\nu}_{0+}y\big).$$

**Proof.** Define the operators $P_1 : X \to X$, $P_2 : Y \to Y$ and $P : (u,v) \to (P_1u, P_2v)$, as

$$P_1u = u(0),\ \ P_2v = v(0).$$

Clearly, $\mathrm{Ker}L = \mathrm{Im}\, P$ and $P(u,v) = P^2(u,v)$.

Note that

$$\mathrm{Ker}P = \big\{(u,v)|u(0) = 0,\ v(0) = 0\big\}.$$

Since $(u,v) = (u,v) - P(u,v) + P(u,v)$, it is clear that $X \times Y = \mathrm{Ker}P + \mathrm{Ker}L$. By a simple calculation, we have $\mathrm{Ker}L \cap \mathrm{Ker}P = \{(0,0)\}$. Thus, we obtain

$$X \times Y = \mathrm{Ker}L \oplus \mathrm{Ker}P.$$

Consider the linear operators $Q_1, Q_2 : E \to E$ defined by

$$Q_1x(t) = \frac{\Gamma(1+\alpha+\beta+\gamma)}{\sum_{i=1}^{m-1} a_i\xi_i}\sum_{i=1}^{m-1} a_i I^{\alpha+\beta+\gamma}_{0+}x(\xi_i),\ \ Q_2y(t) = \frac{\Gamma(1+\lambda+\mu+\nu)}{\sum_{i=1}^{m-1} b_i\eta_i}\sum_{i=1}^{m-1} b_i I^{\lambda+\mu+\nu}_{0+}y(\eta_i).$$

Clearly, $Q(x,y) = \big(Q_1x(t), Q_2y(t)\big) \cong \mathbb{R}^2$. Take $x(t) \in E$, by a direct computation, we have that

$$Q_1(Q_1x(t)) = Q_1x(t) \cdot \frac{\Gamma(1+\alpha+\beta+\gamma)}{\sum_{i=1}^{m-1} a_i\xi_i}\sum_{i=1}^{m-1} a_i(I^{\alpha+\beta+\gamma}_{0+}1)(t=\xi_i) = Q_1x(t).$$

Similarly, $Q_2^2 = Q_2$. This gives that $Q^2(x,y) = Q(x,y)$. It is easy to check from $(x,y) = (x,y) - Q(x,y) + Q(x,y)$ that $Z = \mathrm{Im}L + \mathrm{Im}Q$. Take $(u,v) \in \mathrm{Im}L \cap \mathrm{Im}Q$. According to the definitions of $\mathrm{Im}L$ and $\mathrm{Im}Q$, we obtain $(u,v) = (0,0)$. This implies that

$$Z = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Now, $\mathrm{Ind}\, L = \dim\, \ker L - \mathrm{codim}\, \mathrm{Im}\, L = 0$. This means that $L$ is a Fredholm mapping of index zero.

Next, we will prove that $K_P$ is the inverse of $L_{\mathrm{dom}L \bigcap \mathrm{Ker}P}$.
For $(x,y) \in \mathrm{Im}L$, we have

$$LK_P(x,y) = L(I_{0+}^{\alpha+\beta+\gamma}x,\ I_{0+}^{\lambda+\mu+\nu}y) = (D_{0+}^{\alpha+\beta+\gamma} I_{0+}^{\alpha+\beta+\gamma}x,\ D_{0+}^{\lambda+\mu+\nu} I_{0+}^{\lambda+\mu+\nu}y) = (x,y).$$

For $(u,v) \in \mathrm{dom}L \cap \mathrm{Ker}P$, according to the definitions of $\mathrm{dom}L$ and $\mathrm{Ker}P$, it is easy to verify that the constants $c_i, d_i,\ i = 0,1,2$ in the following equations

$$u(t) = I_{0+}^{\alpha+\beta+\gamma}x(t) + I_{0+}^{\beta+\alpha}c_0 + I_{0+}^{\alpha}c_1 + c_2, \quad v(t) = I_{0+}^{\lambda+\mu+\nu}y(t) + I_{0+}^{\mu+\lambda}d_0 + I_{0+}^{\lambda}d_1 + d_2,$$

are all equal to zero. Thus, we obtain

$$K_pL(x,y) = \big(I_{0+}^{\alpha+\beta+\gamma} D_{0+}^{\gamma}(D_{0+}^{\beta}(D_{0^+}^{\alpha}x)), I_{0+}^{\lambda+\mu+\nu} D_{0+}^{\nu}(D_{0+}^{\mu}(D_{0^+}^{\lambda}y))\big) = (x,y).$$

That shows that $K_P = (L_{\mathrm{dom}L \bigcap \mathrm{Ker}P})^{-1}$. The proof is complete. □

To simplify our statement, we write

$$a = \frac{1}{\Gamma(\gamma+1)}, b = \frac{1}{\Gamma(\nu+1)}.$$

For every $(u,v) \in X \times Y$,

$$\begin{aligned} \|P(u,v)\|_{X\times Y} &= \|(P_1u, P_2v)\|_{X\times Y} = \max\{\|P_1u\|_X; \|P_2v\|_Y\} \\ &= \max\{\|u(0)\|_X;\ \|v(0)\|_Y\} \\ &= \max\{|u(0)|;\ |v(0)|\}. \end{aligned} \tag{7}$$

For each $(x,y) \in \mathrm{Im}L$, we have

$$\begin{aligned} \|K_P(x,y)\|_{X\times Y} &= \|(I_{0+}^{\alpha+\beta+\gamma}x, I_{0+}^{\lambda+\mu+\nu}y)\|_{X\times Y} \\ &= \max\{\|I_{0+}^{\alpha+\beta+\gamma}x\|_X;\ \|I_{0+}^{\lambda+\mu+\nu}y\|_Y\} \\ &\le \max\Big\{\max\{\|I_{0+}^{\alpha+\beta+\gamma}x\|_\infty, \|D_{0^+}^{\alpha} I_{0+}^{\alpha+\beta+\gamma}x\|_\infty, \|(D_{0+}^{\beta}(D_{0^+}^{\alpha} I_{0+}^{\alpha+\beta+\gamma}x))\|_\infty\}; \\ &\qquad \max\{\|I_{0+}^{\lambda+\mu+\nu}y\|_\infty,\ \|D_{0^+}^{\lambda} I_{0+}^{\lambda+\mu+\nu}y\|_\infty, \|(D_{0+}^{\mu}(D_{0^+}^{\lambda} I_{0+}^{\lambda+\mu+\nu}y))\|_\infty\}\Big\} \\ &= \max\Big\{\max\{\|I_{0+}^{\alpha+\beta+\gamma}x\|_\infty, \|I_{0+}^{\beta+\gamma}x\|_\infty, \|I_{0+}^{\gamma}x\|_\infty\}; \\ &\qquad \max\{\|I_{0+}^{\lambda+\mu+\nu}y\|_\infty, \|I_{0+}^{\mu+\nu}y\|_\infty, \|I_{0+}^{\nu}y\|_\infty\}\Big\} \\ &= \max\left\{\frac{1}{\Gamma(\gamma+1)}\|x\|_\infty;\ \frac{1}{\Gamma(\nu+1)}\|y\|_\infty\right\} \\ &= \max\{a\|x\|_\infty;\ b\|y\|_\infty\}. \end{aligned} \tag{8}$$

With the similar arguments to [21], we obtain the following lemma.

**Lemma 5.** *$K_P(I-Q)N : Y \to Y$ is completely continuous.*

To obtain our main results, we need the following conditions.

$(\mathrm{H}_1)$ There exist positive continuous functions $a_i(t), b_i(t)$, $i = 1,2,3$ such that for all $(x_1, x_2, x_3) \in \mathbb{R}^3$, one has

$$|f(x_1,x_2,x_3)| \le a_1(t)|x_1| + a_2(t)|x_2| + a_3(t)|x_3|,$$
$$|g(x_1,x_2,x_3)| \le b_1(t)|x_1| + b_2(t)|x_2| + b_3(t)|x_3|.$$

$(\mathrm{H}_2)$ There exists a constant $A > 0$ such that, if for any $t \in [0,1]$, $|u(t)| > A$ or $|v(t)| > A$, then

$$u(t) \cdot Q_1(N_1 v) > 0, \qquad v(t) \cdot Q_2(N_2 u) > 0$$

or

$$u(t) \cdot Q_1(N_1 v) < 0, \qquad v(t) \cdot Q_2(N_2 u) < 0.$$

$(\mathrm{H}_3)$ $\max\left\{ p + a\sum_{i=1}^{3} p_i,\ \ q + b\sum_{i=1}^{3} q_i,\ \ p + b\sum_{i=1}^{3} q_i,\ \ q + a\sum_{i=1}^{3} p_i \right\} < 1,$
where $p_i = \|a_i\|_\infty, q_i = \|b_i\|_\infty,\ i = 1,2,3,\ p = \frac{p_1+p_2+p_3}{\Gamma\left(\alpha+\beta+\gamma+1\right)}, q = \frac{q_1+q_2+q_3}{\Gamma\left(\lambda+\mu+\nu+1\right)}.$

**Lemma 6.** $\Omega_1 = \left\{(u,v) \in \mathrm{dom}L \setminus \mathrm{Ker}L :\ L(u,v) = \lambda N(u,v), \lambda \in [0,1]\right\}$ *is bounded.*

**Proof.** For $(u,v) \in \Omega_1$, thus $\lambda \ne 0$. And $L(u,v) = \lambda N(u,v) \in \mathrm{Im}L = \mathrm{Ker}Q$, according to the definition of $\mathrm{Ker}Q$, we have

$$Q_1(N_1 v) = Q_2(N_2 u) = 0.$$

By $(\mathrm{H}_2)$, there exit $t_0, t_1 \in [0,1]$ such that

$$|u(t_0)| \le A,\ |v(t_1)| \le A.$$

Again for $(u,v) \in \Omega_1$, $(u,v) \in \mathrm{dom}(L) \setminus \mathrm{Ker}(L)$, then $(I-P)(u,v) \in \mathrm{dom}L \cap \mathrm{Ker}P$ and $LP(u,v) = (0,0)$; thus, from (8), we have

$$\begin{aligned}
\|(I-P)(u,v)\|_{X\times Y} &= \|K_P L(I-P)(u,v)\|_{X\times Y} \\
&= \|K_P(L_1 u, L_2 v)\|_{X\times Y} \\
&\le \max\{a\|L_1 v\|_\infty;\ b\|L_2 u\|_\infty\} \\
&= \max\{a\|N_1 v\|_\infty;\ b\|N_2 u\|_\infty\}.
\end{aligned} \tag{9}$$

By $Lu = \lambda Nu$ and $u \in \mathrm{dom}L$, we have $(L_1 u, L_2 v) = (\lambda N_1 v, \lambda N_2 u)$, i.e., $L_1 u = \lambda N_1 v$ and $L_2 v = \lambda N_2 u$. Thus,

$$u(t) = \lambda I_{0+}^{\alpha+\beta+\gamma} N_1 v(t) + I_{0+}^{\beta+\alpha} c_0 + I_{0+}^{\alpha} c_1 + c_2, \quad v(t) = \lambda I_{0+}^{\lambda+\mu+\nu} N_2 u(t) + I_{0+}^{\mu+\lambda} d_0 + I_{0+}^{\lambda} d_1 + d_2.$$

By the definition of $\mathrm{dom}L$, we have $c_i = d_i = 0,\ i = 0,1$. Then,

$$u(t) = \lambda I_{0+}^{\alpha+\beta+\gamma} N_1 v(t) + c_2, \quad v(t) = \lambda I_{0+}^{\lambda+\mu+\nu} N_2 u(t) + d_2.$$

Furthermore, we have

$$u(t_0) - u(0) = \lambda I_{0+}^{\alpha+\beta+\gamma} N_1 v(t_0) + c_2 - \left[\lambda I_{0+}^{\alpha+\beta+\gamma} N_1 v(0) + c_2\right] = I_{0+}^{\alpha+\beta+\gamma} N_1 v(t_0),$$
$$v(t_1) - v(0) = \lambda I_{0+}^{\lambda+\mu+\nu} N_2 u(t_1) + d_2 - \left[\lambda I_{0+}^{\lambda+\mu+\nu} N_2 u(0) + d_2\right] = I_{0+}^{\lambda+\mu+\nu} N_2 u(t_1),$$

that are,

$$u(0) = u(t_0) - \lambda I_{0+}^{\alpha+\beta+\gamma} N_1 v(t_0), \qquad v(0) = v(t_1) - \lambda I_{0+}^{\lambda+\mu+\nu} N_2 u(t_1).$$

Together with $|u(t_0)| \leq A$, we can derive that

$$
\begin{aligned}
&|u(0)| \leq |u(t_0)| + \left|\lambda I_{0+}^{\alpha+\beta+\gamma} N_1 v(t_0)\right| \\
&\leq A + \left|\frac{\lambda}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t_0} (t_0-s)^{\alpha+\beta+\gamma-1} f\big(v(s), D_{0^+}^{\lambda} v(s), (D_{0+}^{\mu}(D_{0^+}^{\lambda} v))(s)\big) ds\right| \\
&\leq A + \frac{1}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t_0} (t_0-s)^{\alpha+\beta+\gamma-1} \left| f\big(v(s), D_{0^+}^{\lambda} v(s), (D_{0+}^{\mu}(D_{0^+}^{\lambda} v))(s)\big)\right| ds \\
&\leq A + \frac{1}{\Gamma(\alpha+\beta+\gamma)} \int_0^{t_0} (t_0-s)^{\alpha+\beta+\gamma-1} \cdot \\
&\qquad \big(a_1(s)|v(s)| + a_2(s)|D_{0^+}^{\lambda} v(s)| + a_3(s)|(D_{0+}^{\mu}(D_{0^+}^{\lambda} v))(s)|\big) ds \\
&\leq A + \frac{1}{\Gamma(\alpha+\beta+\gamma)} \big(p_1\|v(t)\|_\infty + p_2\|D_{0^+}^{\lambda} v(t)\|_\infty + p_3\|(D_{0+}^{\mu}(D_{0^+}^{\lambda} v))(t)\|_\infty\big) \cdot \\
&\qquad \int_0^{t_0} (t_0-s)^{\alpha+\beta+\gamma-1} ds \\
&\leq A + \frac{p_1+p_2+p_3}{\Gamma\big(\alpha+\beta+\gamma\big)} \int_0^{t_0} (t_0-s)^{\alpha+\beta+\gamma-1} ds \cdot \|v(t)\|_Y \\
&= A + p\|v(t)\|_Y.
\end{aligned} \tag{10}
$$

With the similar arguments, we obtain

$$
|v(0)| \leq A + \frac{q_1+q_2+q_3}{\Gamma\big(\lambda+\mu+\nu+1\big)} \|u(t)\|_X = A + q\|u(t)\|_X. \tag{11}
$$

From (7) and (9), we have

$$
\begin{aligned}
\|(u,v)\|_{X\times Y} = \|P(u,v) + (I-P)(u,v)\|_{X\times Y} &\leq \|P(u,v)\|_{X\times Y} + \|(I-P)(u,v)\|_{X\times Y} \\
&\leq \max\big\{|u(0)| + a\|N_1 v\|_\infty, \quad |u(0)| + b\|N_2 u\|_\infty, \\
&\qquad |v(0)| + a\|N_1 v\|_\infty, \quad |v(0)| + b\|N_2 u\|_\infty\big\}.
\end{aligned}
$$

In what follows, the proof can be divided into four cases.
**Case 1**. $\|(u,v)\|_{X\times Y} \leq |u(0)| + a\|N_1 v\|_\infty$.

By (10) and ($H_1$), we have

$$
\begin{aligned}
&\|(u,v)\|_{X\times Y} \leq \big|u(0)\big| + a\|N_1 v\|_\infty \\
&\leq A + p\|v(t)\|_Y + a\|N_1 v\|_\infty \\
&= A + p\|v(t)\|_Y + a\big\|f\big(v(t), D_{0^+}^{\lambda} v(t), (D_{0+}^{\mu}(D_{0^+}^{\lambda} v))(t)\big)\big\|_\infty \\
&\leq A + p\|v(t)\|_Y + a\big[a_1(t)|v(t)| + a_2(t)|D_{0^+}^{\lambda} v(t)| + a_3(t)|(D_{0+}^{\mu}(D_{0^+}^{\lambda} v))(t)|\big] \\
&\leq A + p\|v(t)\|_Y + a\big[p_1\|v(t)\|_\infty + p_2\|D_{0^+}^{\lambda} v(t)\|_\infty + p_3\|(D_{0+}^{\mu}(D_{0^+}^{\lambda} v))(t)\|_\infty\big] \\
&= A + p\|v(t)\|_Y + a(p_1+p_2+p_3)\|v(t)\|_Y \\
&= A + \big(p + a\Sigma_{i=1}^3 p_i\big) \cdot \|v(t)\|_Y \\
&\leq A + \big(p + a\Sigma_{i=1}^3 p_i\big) \cdot \|(u,v)\|_{X\times Y}.
\end{aligned}
$$

By ($H_4$), we have

$$
\|(u,v)\|_{X\times Y} \leq \frac{A}{1 - p - a\Sigma_{i=1}^3 p_i} := M.
$$

Therefore, $\Omega_1$ is bounded.
**Case 2**. $\|(u,v)\|_{X\times Y} \leq \big|u(0)\big| + b\|N_2 u\|_\infty$.

From (10) and ($H_1$), we obtain

$$
\begin{aligned}
&\|(u,v)\|_{X\times Y} \leq \big|u(0)\big| + b\|N_2 u\|_\infty \\
&\leq A + p\|v(t)\|_Y + b\|N_2 u\|_\infty \\
&= A + p\|v(t)\|_Y + b\big\|g\big(u(t), D^\alpha_{0+}u(t), (D^\beta_{0+}(D^\alpha_{0+}u))(t)\big)\big\|_\infty \\
&\leq A + p\|v(t)\|_Y + b\big[q_1\|u(t)\|_\infty + q_2\|D^\lambda_{0+}u(t)\|_\infty + q_3\|(D^\mu_{0+}(D^\lambda_{0+}u))(t)\|_\infty\big] \\
&\leq A + p\|v(t)\|_Y + b(q_1+q_2+q_3)\|u(t)\|_Y \\
&\leq A + \big(p + b\Sigma^3_{i=1}q_i\big)\cdot\|(u,v)\|_{X\times Y}.
\end{aligned}
$$

By ($H_4$), we can obtain

$$\|(u,v)\|_{X\times Y} \leq \frac{1}{1-q-b\Sigma^3_{i=1}q_i} := M.$$

Therefore, $\Omega_1$ is bounded.
**Case 3**. $\|(u,v)\|_\infty \leq \big|v(0)\big| + a\|N_1 v\|_\infty$.

According to (11) and ($H_4$), by a similar proof of Case 2, we can derive

$$\|(u,v)\|_{X\times Y} \leq \frac{A}{1-q-a\Sigma^3_{i=1}p_i} := M.$$

Therefore, $\Omega_1$ is bounded.
**Case 4**. $\|(u,v)\|_Y \leq \big|v(0)\big| + b\|N_2 u\|_\infty$.

By a similar proof of Case 1, we obtain

$$\|(u,v)\|_{X\times Y} \leq \frac{A}{1-q-b\Sigma^3_{i=1}q_i} := M.$$

Therefore, $\Omega_1$ is bounded.
According the above arguments, we prove that $\Omega_1$ is bounded. □

**Lemma 7.** $\Omega_2 = \big\{(u,v) \in \mathrm{Ker}L : N(u,v) \in \mathrm{Im}L\big\}$ *is bounded.*

**Proof.** Let $(u,v) \in \mathrm{Ker}L$; thus, we have $u = c, v = d, c, d \in \mathbb{R}$. In view of $N(u,v) = (N_1 v, N_2 u) \in \mathrm{Im}L = \mathrm{Ker}Q$, we have

$$Q_1(N_1 v) = Q_2(N_2 u) = 0.$$

By ($H_2$), there exit constants $t_0,\ t_1 \in [0,1]$ such that

$$|u(t_0)| = |c| \leq A, \quad |v(t_1)| = |d| \leq A.$$

Hence, $\Omega_2$ is bounded. □

**Lemma 8.** $\Omega_3 = \big\{(u,v) \in \mathrm{Ker}L : \lambda(u,v) + (1-\lambda)QN(u,v) = (0,0), \lambda \in [0,1]\big\}$ *is bounded.*

**Proof.** Let $(u,v) \in \mathrm{Ker}L$; therefore, we have $u = c, v = d, c, d \in \mathbb{R}$ and

$$\lambda c + (1-\lambda)Q_1 N_1(d) = 0,\ \lambda d + (1-\lambda)Q_2 N_2(c) = 0. \tag{12}$$

If $\lambda = 0$, similar to the proof of Lemma 7, we can obtain $|c| \leq A$ and $|d| \leq A$. If $\lambda = 1$, we have $c = d = 0$. If $\lambda \in (0,1)$, then we can also obtain $|c| \leq A$ and $|d| \leq A$. Otherwise, if $|c| > A$ or $|d| > A$, in view of the first part of ($H_2$), we obtain

$$\lambda c^2 + c(1-\lambda)Q_1N_1(d) = 0,\ \lambda d^2 + d(1-\lambda)Q_2N_2(c) = 0,$$

which contradicts (12). Thus, $\Omega_3$ is bounded.

If the second part of ($H_2$) holds, then we can prove the set

$$\Omega_3' = \left\{(u,v) \in \mathrm{Ker}L : -\lambda(u,v) + (1-\lambda)QN(u,v) = (0,0), \lambda \in [0,1]\right\}$$

is bounded. $\square$

**Theorem 2.** *Suppose ($H_1$)–($H_3$) hold, then the problem* (1) *and* (2) *has at least one solution in $Y$.*

**Proof.** Let $\Omega$ be a bounded open subset of $Y$, such that $\cup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$. It follows from Lemma 5, that $N$ is $L$-compact on $\Omega$. By Lemmas 6–8, we obtain:

(1) $L(u,v) \neq \lambda N(u,v)$, for every $(u,v,\lambda) \in [(\mathrm{dom}L \setminus \mathrm{Ker}L) \cap \partial\Omega] \times (0,1)$.
(2) $N(u,v) \notin \mathrm{Im}L$ for every $u \in \mathrm{Ker}L \cap \partial\Omega$.
(3) Let $H\big((u,v),\lambda\big) = \pm\lambda I(u,v) + (1-\lambda)JQN(u,v)$. Here, we let $I$ and the isomorphism $J : \mathrm{Im}\, Q \to \ker L$, which are both identical operators. Via the homotopy property of degree, we obtain that

$$\begin{aligned} \deg(JQN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(H(\cdot,0), \Omega \cap \ker L, 0) \\ &= \deg(H(\cdot,1), \Omega \cap \ker L, 0) \\ &= \deg(I, \Omega \cap \ker L, 0) = 1 \neq 0. \end{aligned}$$

Applying Theorem 1, we conclude that $L(u,v) = N(u,v)$ has at least one solution in $\mathrm{dom}\, L \cap \overline{\Omega}$. $\square$

## 4. Example

Let us consider the following coupled system of fractional jerk equations at resonance

$$\begin{cases} D_{0+}^{0.1}x_1(t) = x_2(t), \\ D_{0+}^{0.3}x_2(t) = x_3(t), \\ D_{0+}^{0.5}x_3(t) = f(y_1,y_2,y_3), \\ D_{0+}^{0.7}y_1(t) = y_2(t), \\ D_{0+}^{0.4}y_2(t) = y_3(t), \\ D_{0+}^{0.2}y_3(t) = g(x_1,x_2,x_3), \\ x_1(0) = \sum_{i=1}^{3} a_i x_1(\xi_i), x_2(0) = x_3(0) = 0, \\ y_1(0) = \sum_{i=1}^{3} b_i y_1(\eta_i), y_2(0) = y_3(0) = 0, \end{cases} \tag{13}$$

where

$$f(y_1,y_2,y_3) = \frac{|y_1|}{10} + \frac{\sin y_2}{20} + \frac{\cos y_3}{20},$$
$$g(x_1,x_2,x_3) = \frac{\sin^2 x_1}{16} + \frac{\sin x_2}{32} + \frac{|x_3|}{32},$$

and $a_1 = a_2 = \frac{1}{4}$, $a_3 = \frac{1}{2}$, $b_1 = b_2 = b_3 = \frac{1}{3}$, $\xi_1 = 0.3$, $\xi_2 = 0.5$, $\xi_3 = 0.7$, $\eta_1 = 0.2$, $\eta_2 = 0.4$, and $\eta_1 = 0.6$. Corresponding to BVP (1) and (2), we have that $\alpha = 0.1$, $\beta = 0.5$, $\gamma = 0.3$, $\lambda = 0.7$, $\mu = 0.4$, and $\nu = 0.2$.

It is easy to see that

$$p_1 = \frac{1}{10}, = p_2 = \frac{1}{20}, p_3 = \frac{1}{20}, a = \frac{1}{\Gamma(\gamma+1)} = \frac{1}{\Gamma(1.3)} \approx 1.11$$
$$q_1 = \frac{1}{16}, q_2 = \frac{1}{32}, q_3 = \frac{1}{16}, b = \frac{1}{\Gamma(\nu+1)} = \frac{1}{\Gamma(1.2)} \approx 1.09.$$

By direct calculation, we can obtain

$$p = \frac{p_1+p_2+p_3}{\Gamma\big(\alpha+\beta+\gamma+1\big)} = \frac{0.2}{\Gamma(1.9)} \approx 0.21, p + a\sum_{i=1}^{3} p_i \approx 0.65, \; q + b\sum_{i=1}^{3} q_i \approx 0.28,$$
$$q = \frac{q_1+q_2+q_3}{\Gamma\big(\lambda+\mu+\nu+1\big)} = \frac{0.125}{\Gamma(2.3)} \approx 0.11, p + b\sum_{i=1}^{3} q_i \approx 0.38, \; q + a\sum_{i=1}^{3} p_i \approx 0.55.$$

We take $A = 18$, then we can obtain that ($H_1$)–($H_3$) hold. Hence, all the conditions of Theorem 2 are satisfied, and consequently BVP (13) has at least one solution.

## 5. Conclusions

In this paper, we investigated a resonant boundary value problem of the system of jerk differential equations with the fractional order. The interesting point is that two fractional jerk equations are coupled. By coincidence degree theory due to Mawhin, the existence result is proved. Our result obtained in this paper is new and complements the existing literature on the topic. As far as our work is concerned, we mainly concentrated on the existence of solutions. To the best of our knowledge, some results were not considered for fractional coupled jerk equations with resonant conditions, such as unique solutions, positive solutions, and numerical solutions. In future research, we will study the corresponding problem, and we hope to be able to make some progress. The corresponding physical interpretation of fractional coupled jerk equations, in order to compare the presented results on integer-order jerk system, is proposed as future work.

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