

Article

An Algebraic Inequality with Applications to Certain Chen Inequalities

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Abstract: We give a simple proof of the Chen inequality for the Chen invariant $\underbrace{\delta(2, \dots, 2)}_{k \text{ terms}}$ of submanifolds in Riemannian space forms.

Keywords: Riemannian space form; submanifold; Chen invariants; Chen inequalities

MSC: 53C40

1. Introduction

In [1,2], B.-Y. Chen introduced a string of Riemannian invariants, known as Chen invariants, which are different in nature from the classical Riemannian invariants. He established sharp relationships between these invariants and the squared mean curvature for submanifolds in Riemannian space forms, known as Chen inequalities (see [2]).

The proof uses an algebraic inequality, discovered by B.-Y. Chen in [1].

In the present paper, we obtain a different algebraic inequality that allows us to give simple proofs of certain Chen inequalities.

2. Preliminaries

The theory of Chen invariants and Chen inequalities was initiated by B.-Y. Chen [1,2].

Let (M, g) be an n -dimensional ($n \geq 2$) Riemannian manifold and ∇ its Levi-Civita connection. One denotes by R the Riemannian curvature tensor field on M . For any $p \in M$ and $\pi \subset T_p M$ a plane section, the *sectional curvature* $K(\pi)$ of π is defined by $K(\pi) = R(e_1, e_2, e_1, e_2)$, where we use the convention $R(e_1, e_2, e_1, e_2) = g(R(e_1, e_2)e_2, e_1)$, with $\{e_1, e_2\}$ an orthonormal basis of π . Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_p M$. The *scalar curvature* τ at p is given by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j (other authors consider $\tau(p) = \sum_{1 \leq i \neq j \leq n} K(e_i \wedge e_j)$).

The *Chen first invariant* δ_M is defined by

$$\delta_M(p) = \tau(p) - \inf\{K(\pi) | \pi \subset T_p M \text{ plane section}\}.$$

The Chen invariant $\delta(2, 2)$, given by

$$\delta(2, 2)(p) = \tau(p) - \inf\{K(\pi_1) + K(\pi_2) | \pi_1, \pi_2 \subset T_p M \text{ orthogonal plane sections}\},$$

was studied in [3].



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We shall consider the Chen invariant $\underbrace{\delta(2, \dots, 2)}_{k \text{ terms}}$, denoted by $\delta^k(2, \dots, 2)$, which is given by

$$\delta^k(2, \dots, 2)(p) = \tau(p) - \inf\{K(\pi_1) + \dots + K(\pi_k)\},$$

where π_1, \dots, π_k are mutually orthogonal plane sections at p .

Obviously, $\delta^1(2) = \delta_M$.

In the next section, we shall prove an algebraic inequality and study its equality case. As an application we shall give a simple proof of the Chen inequality for the invariant $\delta^k(2, \dots, 2)$.

3. An Algebraic Inequality

Proposition 1. Let $k, n \in \mathbb{N}^*$, $n \geq 2k$, and $a_1, a_2, \dots, a_n \in \mathbb{R}$. Then

$$\sum_{1 \leq i < j \leq n} a_i a_j - \sum_{i=1}^k a_{2i-1} a_{2i} \leq \frac{n-k-1}{2(n-k)} \left(\sum_{i=1}^n a_i \right)^2.$$

Moreover, the equality holds if and only if $a_{2i-1} + a_{2i} = a_j$, $1 \leq i \leq k$, $2k+1 \leq j \leq n$.

Proof. We shall prove the above Proposition by mathematical induction.

Let

$$P(n) : \sum_{1 \leq i < j \leq n} a_i a_j - \sum_{i=1}^k a_{2i-1} a_{2i} \leq \frac{n-k-1}{2(n-k)} \left(\sum_{i=1}^n a_i \right)^2,$$

with equality holding if and only if $a_{2i-1} + a_{2i} = a_j$, $1 \leq i \leq k$, $2k+1 \leq j \leq n$.

First we show that $P(2k)$ is true. Indeed

$$\sum_{1 \leq i < j \leq 2k} a_i a_j - \sum_{i=1}^k a_{2i-1} a_{2i} \leq \frac{k-1}{2k} \left(\sum_{i=1}^{2k} a_i \right)^2 \Leftrightarrow$$

$$(k-1)(a_1^2 + a_2^2 + \dots + a_{2k}^2) - 2 \sum_{1 \leq i < j \leq 2k} a_i a_j + 2k(a_1 a_2 + a_3 a_4 + \dots + a_{2k-1} a_{2k}) \geq 0 \Leftrightarrow$$

$$(a_1 + a_2 - a_3 - a_4)^2 + \dots + (a_1 + a_2 - a_{2k-1} - a_{2k})^2 + \dots + (a_{2k-3} + a_{2k-2} - a_{2k-1} - a_{2k})^2 \geq 0.$$

Clearly the equality holds if and only if $a_1 + a_2 = \dots = a_{2k-1} + a_{2k}$.

Next, assume $P(n)$ and prove $P(n+1)$.

$$P(n+1) : \sum_{1 \leq i < j \leq n+1} a_i a_j - \sum_{i=1}^k a_{2i-1} a_{2i} \leq \frac{n-k}{2(n-k+1)} \left(\sum_{i=1}^{n+1} a_i \right)^2,$$

with equality holding if and only if $a_{2i-1} + a_{2i} = a_j$, $1 \leq i \leq k$, $2k+1 \leq j \leq n+1$.

By using $P(n)$, one has

$$\begin{aligned} & \sum_{1 \leq i < j \leq n+1} a_i a_j - \sum_{i=1}^k a_{2i-1} a_{2i} = \\ & = \sum_{1 \leq i < j \leq n} a_i a_j + a_{n+1} \left(\sum_{i=1}^n a_i \right) - \sum_{i=1}^k a_{2i-1} a_{2i} \leq \\ & \leq \frac{n-k-1}{2(n-k)} \left(\sum_{i=1}^n a_i \right)^2 + a_{n+1} \left(\sum_{i=1}^n a_i \right) \leq \end{aligned}$$

$$\leq \frac{n-k}{2(n-k+1)} \left(\sum_{i=1}^{n+1} a_i \right)^2.$$

The last inequality is equivalent to

$$\left[(n-k)a_{n+1} - \left(\sum_{i=1}^n a_i \right) \right]^2 \geq 0.$$

The equality holds if and only if

$$a_{2i-1} + a_{2i} = a_j, \quad 1 \leq i \leq k, \quad 2k+1 \leq j \leq n+1.$$

□

4. A Chen Inequality

As an application of Proposition 1, we give a simple proof of the Chen inequality for the Chen invariant $\delta^k(2, \dots, 2)$ of submanifolds in Riemannian space forms.

Let $\tilde{M}(c)$ be an m -dimensional Riemannian space form of constant sectional curvature c . The standard examples are the Euclidean space \mathbb{E}^m , the sphere S^m and the hyperbolic space H^m .

Let M be an n -dimensional submanifold of $\tilde{M}(c)$ and denote by h the second fundamental form of M in $\tilde{M}(c)$. Recall that the mean curvature vector $H(p)$ at $p \in M$ is given by

$$H(p) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i),$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_p M$.

The submanifold M is said to be *minimal* if $H(p) = 0, \forall p \in M$.

The Gauss equation is (see [4])

$$R(X, Y, Z, W) = c + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)),$$

for any vector fields X, Y, Z, W tangent to M .

Theorem 1. *Let $\tilde{M}(c)$ be an m -dimensional Riemannian space form of constant sectional curvature c and M an n -dimensional submanifold of $\tilde{M}(c)$. Then one has the following Chen inequality:*

$$\delta^k(2, \dots, 2) \leq \frac{n^2(n-k-1)}{2(n-k)} \|H\|^2 + \left[\frac{n(n-1)}{2} - k \right] c.$$

Moreover, the equality holds at a point $p \in M$ if and only if there exist suitable orthonormal bases $\{e_1, \dots, e_n\} \subset T_p M$ and $\{e_{n+1}, \dots, e_m\} \subset T_p^\perp M$ such that the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{2k-1} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & a_{2k} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & \mu \end{pmatrix}, \quad a_{2i-1} + a_{2i} = \mu, \quad 1 \leq i \leq k,$$

$$A_{e_r} = \begin{pmatrix} A_1^r & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & A_2^r & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & A_k^r & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \quad r = n + 2, \dots, m,$$

where A_j^r are symmetric 2×2 matrices with trace $A_j^r = 0, \forall j = 1, \dots, k$.

Proof. Let $p \in M, \pi_1, \dots, \pi_k \subset T_p M$ be mutually orthogonal plane sections and $\{e_1, e_2\} \subset \pi_1, \dots, \{e_{2k-1}, e_{2k}\} \subset \pi_k$ be orthonormal bases. We construct $\{e_1, \dots, e_{2k}, e_{2k+1}, \dots, e_n\} \subset T_p M$ and $\{e_{n+1}, \dots, e_m\} \subset T_p^\perp M$ orthonormal bases, respectively.

Denote by $h_{ij}^r = g(h(e_i, e_j), e_r), i, j = 1, \dots, n, r \in \{n + 1, \dots, m\}$, the components of the second fundamental form.

By the Gauss equation, we have

$$\begin{aligned} \tau &= \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j) = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_i, e_j) = \\ &= \frac{n(n-1)}{2}c + \sum_{r=n+1}^m \sum_{1 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2]. \end{aligned}$$

The Gauss equation also implies

$$\begin{aligned} K(\pi_i) &= K(e_{2i-1} \wedge e_{2i}) = R(e_{2i-1}, e_{2i}, e_{2i-1}, e_{2i}) = \\ &= c + \sum_{r=n+1}^m [h_{2i-1, 2i-1}^r h_{2i, 2i}^r - (h_{2i-1, 2i}^r)^2], \quad \forall i = 1, \dots, k. \end{aligned}$$

Then we get

$$\begin{aligned} \tau - \sum_{i=1}^k K(\pi_i) &= \\ &= \left[\frac{n(n-1)}{2} - k \right]c + \sum_{r=n+1}^m \left[\sum_{1 \leq i < j \leq n} h_{ii}^r h_{jj}^r - \sum_{i=1}^k h_{2i-1, 2i-1}^r h_{2i, 2i}^r \right] - \\ &\quad - \sum_{r=n+1}^m \sum_{\substack{1 \leq i < j \leq n, \\ (i, j) \neq (1, 2), \dots, (2k-1, 2k)}} (h_{ij}^r)^2. \end{aligned}$$

By using the algebraic inequality from the previous section, we obtain

$$\begin{aligned} \tau - \sum_{i=1}^k K(\pi_i) &\leq \frac{n-k-1}{2(n-k)} \sum_{r=n+1}^m \left(\sum_{i=1}^n h_{ii}^r \right)^2 + \left[\frac{n(n-1)}{2} - k \right]c = \\ &= \frac{n^2(n-k-1)}{2(n-k)} \|H\|^2 + \left[\frac{n(n-1)}{2} - k \right]c, \end{aligned}$$

which implies the desired inequality.

If the equality case holds at a point $p \in M$, then we have equalities in all the inequalities in the proof, i.e.,

$$\begin{cases} h_{2i-1, 2i-1}^r + h_{2i, 2i}^r = h_{jj}^r, \quad 1 \leq i \leq k, \quad 2k+1 \leq j \leq n, \\ h_{ij}^r = 0, \quad \forall 1 \leq i < j \leq n, \quad (i, j) \neq (1, 2), \dots, (2k-1, 2k), \end{cases}$$

for any $r \in \{n + 1, \dots, m\}$.

We choose e_{n+1} parallel to $H(p)$. Then the shape operators take the above forms. \square

Corollary 1. *Let $\tilde{M}(c)$ be an m -dimensional Riemannian space form of constant sectional curvature c and M an n -dimensional submanifold of $\tilde{M}(c)$. If there exists a point $p \in M$ such that $\delta^k(2, \dots, 2)(p) > \left[\frac{n(n-1)}{2} - k\right]c$, then M is not minimal.*

For $k = 1$, one derives Chen’s first inequality (see [1]).

Corollary 2. *Let $\tilde{M}(c)$ be an m -dimensional Riemannian space form of constant sectional curvature c and M an n -dimensional submanifold of $\tilde{M}(c)$. Then one has*

$$\inf K \geq \tau - \frac{n-2}{2} \left[\frac{n^2}{n-1} \|H\|^2 + (n+1)c \right].$$

Equality holds if and only if, with respect to suitable frame fields $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$, the shape operators take the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & \mu - a & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix},$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \dots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad r = n + 2, \dots, m.$$

Example 1. *The generalized Clifford torus.*

$$\text{Let } T = S^k\left(\frac{1}{\sqrt{2}}\right) \times S^k\left(\frac{1}{\sqrt{2}}\right) \subset S^{2k+1} \subset \mathbb{E}^{2k+2}.$$

It is known that T is a minimal hypersurface of S^{2k+1} , but a non-minimal submanifold of \mathbb{E}^{2k+2} .

$$\text{Obviously } \delta^k(2, \dots, 2) = \tau = 2k(k - 1).$$

Then $T \subset S^{2k+1}$ does not satisfy the equality case of Theorem 1.

If we consider $T \subset \mathbb{E}^{2k+2}$, then it satisfies the equality case of Theorem 1.

Remark 1. *By using the inequality from Proposition 1, we can obtain Chen inequalities for the invariant $\delta^k(2, \dots, 2)$ on submanifolds in other ambient spaces, for instance, complex space forms, Sasakian space forms, Hessian manifolds of constant Hessian curvature, etc.*

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