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Slice Holomorphic Functions in the Unit Ball Having a Bounded L -Index in Direction

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Abstract: Let $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be a fixed direction. We consider slice holomorphic functions of several complex variables in the unit ball, i.e., we study functions that are analytic in the intersection of every slice $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ with the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| := \sqrt{|z_1|^2 + \dots + |z_n|^2} < 1\}$ for any $z^0 \in \mathbb{B}^n$. For this class of functions, there is introduced a concept of boundedness of L -index in the direction \mathbf{b} , where $L : \mathbb{B}^n \rightarrow \mathbb{R}_+$ is a positive continuous function such that $L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$, where $\beta > 1$ is some constant. For functions from this class, we describe a local behavior of modulus of directional derivatives on every ‘circle’ $\{z + t\mathbf{b} : |t| = r/L(z)\}$ with $r \in (0; \beta]$, $t \in \mathbb{C}$, $z \in \mathbb{C}^n$. It is estimated by the value of the function at the center of the circle. Other propositions concern a connection between the boundedness of L -index in the direction \mathbf{b} of the slice holomorphic function F and the boundedness of l_z -index of the slice function $g_z(t) = F(z + t\mathbf{b})$ with $l_z(t) = L(z + t\mathbf{b})$. In addition, we show that every slice holomorphic and joint continuous function in the unit ball has a bounded L -index in direction in any domain compactly embedded in the unit ball and for any continuous function $L : \mathbb{B}^n \rightarrow \mathbb{R}_+$.

Keywords: bounded index; bounded L -index in direction; slice function; analytic function; bounded L -index; unit ball; local behavior; maximum modulus

MSC: 32A10; 32A17; 32A37



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1. Introduction

The theory of entire functions of bounded index was initiated by the paper of B. Lepson [1]. An entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ is called a function of bounded index [1,2] if there exists $m_0 \in \mathbb{Z}_+$ such that, for all $z \in \mathbb{C}$ and, for all $p \in \mathbb{N}$, one has $\frac{|f^{(p)}(z)|}{p!} \leq \max_{0 \leq k \leq m_0} \left\{ \frac{|f^{(k)}(z)|}{k!} \right\}$. This theory has applications in the analytic theory of differential equations [3,4] and its systems [5] and the value distribution theory [6–8]. It is known that any entire function of bounded index [7] is a function of exponential type. Using a notion of bounded index for bivariate complex functions Nuray and Patterson [9] presented a series of sufficient conditions that ensure that exponential type is preserved. Another interesting application of this notion concerns summability methods. Nuray [10] presented necessary and sufficient conditions on four-dimensional matrix transformations that preserve entireness, bounded index, and absolute convergence of double sequences. He obtained general characterizations for four-dimensional RH-regular matrix transformations for the space of entire, bounded index, and absolutely summable double sequences.

Let $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be a fixed direction. Recently, a generalization of notion of bounded index [11–13] was proposed for so-called slice holomorphic functions in \mathbb{C}^n . There were two

classes considered of these functions: 1) $\tilde{\mathcal{H}}_{\mathbf{b}}$ is a class of functions which are holomorphic on every slice $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ for each $z^0 \in \mathbb{C}^n$; 2) $\mathcal{H}_{\mathbf{b}}$ is a subclass of functions from $\tilde{\mathcal{H}}_{\mathbf{b}}$ which are jointly continuous.

Those investigations were initiated by the following question of Favorov:

Problem 1 ([14]). *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be a given direction, $L : \mathbb{C}^n \rightarrow \mathbb{R}_+$ be a continuous function. Is it possible to replace the condition “ F is holomorphic in \mathbb{C}^n ” by the condition “ F is holomorphic on all slices $z^0 + t\mathbf{b}$ ” and to deduce all known properties of entire functions of bounded L -index in direction for this class of function class?*

There is a negative answer to Favorov’s question [14]. This relaxation of restrictions by the function F does not allow the proving of some theorems. It is known that any entire function has a bounded index in any bounded domain. An example of a slice holomorphic function was constructed having an unbounded index in a direction in some unbounded domain [14].

Note that joint continuity and slice holomorphy (in one direction \mathbf{b}) do not imply holomorphy in a whole n -dimensional complex space (see examples in [13]). For these classes, the theory of a bounded index in the direction was constructed in papers [11–13]. Particularly, growth estimates were obtained, and the described local behavior of holomorphic solutions of some partial differential equations [12]. These slice holomorphic functions in \mathbb{C}^n are some generalization of entire functions of several complex variables. Together with the class of entire functions, the analytic functions in the unit ball or in the polydisc are very important objects of investigations in the multidimensional complex analysis. Rudin [15] wrote that ‘The ball is the prototype of two important classes of regions that have been studied in depth, namely the strictly pseudoconvex domains and the bounded symmetric ones’. Thus, it leads to a general problem to construct a theory of bounded index for slice holomorphic functions in a bounded symmetric domain. In the paper, we consider this problem for the unit ball because it is an important model example of a bounded symmetric domain. Thus, we will study functions that are slice holomorphic in such a bounded domain as unit ball. Its symmetry simplifies many proofs and helps to select main ideas with a minimum of fuss and bother.

Moreover, functions analytic in the unit ball have a bounded index in any direction in a domain compactly embedded in the unit ball [16]. Despite this, the example from [14] can be easily generalized for the unit ball. In other words, there exist functions which are analytic on each slice $z + t\mathbf{b}$ ($z \in \mathbb{B}^n$, $t \in \mathbb{C}$) in the unit ball and which have unbounded index in a some domain compactly embedded in the unit ball.

Therefore, our goal is to construct theory of bounded index for functions which are slice holomorphic in the unit ball.

Let us introduce some notations and definitions.

Let $\mathbf{0} = (0, \dots, 0)$, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_+^* = [0, +\infty)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{0\}$ be a given direction, $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $L : \mathbb{B}^n \rightarrow \mathbb{R}_+$ be a continuous function such that, for all $z \in \mathbb{B}^n$

$$L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}, \quad \beta = \text{const} > 1. \quad (1)$$

For a given $z \in \mathbb{B}^n$, we denote $S_z = \{t \in \mathbb{C} : z + t\mathbf{b} \in \mathbb{B}^n\}$. Clearly, $\mathbb{D} = \mathbb{B}^1$.

The slice functions on S_z for fixed $z^0 \in \mathbb{B}^n$ we will denote as $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ and $l_{z^0}(t) = L(z^0 + t\mathbf{b})$ for $t \in S_z$.

Definition 1 ([16]). *An analytic function $F : \mathbb{B}^n \rightarrow \mathbb{C}$ is called a function of bounded L -index in a direction \mathbf{b} , if there exists $m_0 \in \mathbb{Z}_+$ such that, for every $m \in \mathbb{Z}_+$, and, for all $z \in \mathbb{B}^n$, one has*

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)} \leq \max_{0 \leq k \leq m_0} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}, \quad (2)$$

where $\partial_{\mathbf{b}}^0 F(z) = F(z)$, $\partial_{\mathbf{b}} F(z) = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j$, $\partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}} (\partial_{\mathbf{b}}^{k-1} F(z))$, $k \geq 2$.

The least such integer number m_0 , obeying (2), is called the L -index in the direction \mathbf{b} of the function F and is denoted by $N_{\mathbf{b}}(F, L)$.

If such m_0 does not exist, then we put $N_{\mathbf{b}}(F, L) = \infty$, and the function F is said to be of unbounded L -index in the direction \mathbf{b} in this case. Let $l : \mathbb{D} \rightarrow \mathbb{R}_+$ be a continuous function such that $l(z) > \frac{\beta}{1-|z|}$. For $n = 1$, $\mathbf{b} = 1$, $L(z) \equiv l(z)$ ($z \in \mathbb{D}$) the inequality (2) defines an analytic function in the unit disc of bounded l -index with the l -index $N(F, l) \equiv N_1(F, l)$ (see [17]).

Let $N_{\mathbf{b}}(F, L, z^0)$ stands for the L -index in the direction \mathbf{b} of the function F at the point z^0 , i.e., it is the least integer m_0 , for which the inequality (2) is satisfied at this point $z = z^0$. By analogy, the notation $N(f, l, z^0)$ is defined if $n = 1$, i.e., in the case of analytic functions in the unit disc.

There are many papers on entire and slice holomorphic functions of bounded L -index in the direction. Methods of investigation of properties of these functions often use the restriction of the function to the slices $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$. For fixed $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ and $z^0 \in \mathbb{C}^n$, using considerations from the one-dimensional case, mathematicians obtain the estimates which are uniform in $z^0 \in \mathbb{C}^n$. This is a short description of the main idea.

Please note that the positivity and the continuity of the function L are weak restrictions to deduce constructive results. Thus, we assume additional restrictions to the function L .

Let us denote

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z \in \mathbb{B}^n} \sup_{t_1, t_2 \in S_z} \left\{ \frac{L(z + t_1 \mathbf{b})}{L(z + t_2 \mathbf{b})} : |t_1 - t_2| \leq \frac{\eta}{\min\{L(z + t_1 \mathbf{b}), L(z + t_2 \mathbf{b})\}} \right\}.$$

By $Q_{\mathbf{b}}(\mathbb{B}^n)$, we denote a class of positive continuous functions $L : \mathbb{B}^n \rightarrow \mathbb{R}_+$, satisfying the condition

$$\forall \eta \in [0; \beta] : \lambda_{\mathbf{b}}(\eta) < +\infty. \quad (3)$$

Moreover, it is sufficient to require validity of (3) for one value $\eta \in (0; \beta]$.

In addition, we denote by $\langle a, c \rangle = \sum_{j=1}^n a_j \bar{c}_j$ the scalar product in \mathbb{B}^n , where $a, c \in \mathbb{B}^n$.

Let $\tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions which are holomorphic on every slices $\{z^0 + t\mathbf{b} : t \in S_{z^0}\}$ for each $z^0 \in \mathbb{B}^n$ and let $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions from $\tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ which are joint continuous. The notation $\partial_{\mathbf{b}} F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e., for every $p \in \mathbb{N}$ $\partial_{\mathbf{b}}^p F(z) = g_z^{(p)}(0)$, where $g_z(t) = F(z + t\mathbf{b})$ is an analytic function of complex variable $t \in \mathbb{C}$ for given $z \in \mathbb{B}^n$. In this research, we will often call this derivative as a directional derivative because, if F is an analytic function in \mathbb{B}^n , then the derivatives of the function $g_z(t)$ matches with directional derivatives of the function F .

Please note that, if $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$, then for every $p \in \mathbb{N}$ $\partial_{\mathbf{b}}^p F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$. It can be proved by using Cauchy's formula.

Together, the hypothesis on joint continuity and the hypothesis on holomorphy in one direction do not imply holomorphy in whole n -dimensional complex unit ball. We give some examples to demonstrate it. For $n = 2$, let $f : \mathbb{D} \rightarrow \mathbb{C}$ be an analytic function, $g : \mathbb{D} \rightarrow \mathbb{C}$ be a continuous function. Then, $f(z_1)g(z_2)$, $f(z_1) \pm g(z_2)$ are functions that are holomorphic in the direction $(1, 0)$ and are joint continuous in \mathbb{B}^2 . Moreover, the function $f(z_1 \cdot g(z_2))$ has the same properties if $|g(z)| = 1$. If, in addition, we have performed an affine transformation

$$\begin{cases} z_1 = b_2 z'_1 + b_1 z'_2, \\ z_2 = b_2 z'_1 - b_1 z'_2 \end{cases}$$

then the new functions are also holomorphic in the direction (b_1, b_2) and are joint continuous in \mathbb{B}^2 , where $|b_1 b_2| = 1/2$.

Definition 2. A function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is said to be of bounded L -index in the direction \mathbf{b} , if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and each $z \in \mathbb{B}^n$ the inequality (2) is true.

All notations, introduced above for analytic functions of bounded L -index in the direction, remain for functions from $\tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$.

2. Sufficient Sets

Now, we prove several assertions that establish a connection between functions of bounded L -index in direction and functions of a bounded l -index of one variable. The similar results were obtained for analytic functions in the unit ball [18] and for slice holomorphic functions in \mathbb{C}^n [13]. The next proofs use ideas from the mentioned papers.

Proposition 1. If a function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L -index in the direction \mathbf{b} , then, for every $z \in \mathbb{B}^n$, the analytic function $g_z(t)$ is of bounded l_z -index and $N(g_z, l_z) \leq N_{\mathbf{b}}(F, L)$.

Proof. Let $z \in \mathbb{B}^n$, $g(t) \equiv g_z(t)$, $l(t) \equiv l_z(t)$. As for all $p \in \mathbb{N}$

$$g^{(p)}(t) = \partial_{\mathbf{b}}^p F(z + t\mathbf{b}), \quad (4)$$

then, by the definition of the boundedness of the L -index in the direction \mathbf{b} for all $t \in S_z$ and $p \in \mathbb{Z}_+$, we obtain

$$\begin{aligned} \frac{|g^{(p)}(t)|}{p!l^p(t)} &= \frac{|\partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})|}{p!L^p(z^0 + t\mathbf{b})} \leq \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z^0 + t\mathbf{b})|}{k!L^k(z^0 + t\mathbf{b})} : 0 \leq k \leq N_{\mathbf{b}}(F, L) \right\} = \\ &= \max \left\{ \frac{|g^{(k)}(t)|}{k!l^k(t)} : 0 \leq k \leq N_{\mathbf{b}}(F, L) \right\}. \end{aligned}$$

Hence, we obtain that $g(t)$ is of bounded l -index and $N(g, l) \leq N_{\mathbf{b}}(F, L)$. Proposition 1 is proved. \square

The equality (4) implies that the following proposition holds.

Proposition 2. If a function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L -index in the direction \mathbf{b} , then

$$N_{\mathbf{b}}(F, L) = \max \{N(g_z, l_z) : z \in \mathbb{B}^n\}.$$

Theorem 1. A function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L -index in the direction \mathbf{b} if and only if there exists a number $M > 0$ such that for all $z \in \mathbb{B}^n$ the function $g_z(t)$ is of bounded l_z -index with $N(g_z, l_z) \leq M < +\infty$, as a function of variable $t \in \mathbb{C}$. Thus, $N_{\mathbf{b}}(F, L) = \max \{N(g_z, l_z) : z \in \mathbb{B}^n\}$.

Proof. The necessity follows from Proposition 1.

Sufficiency. Since $N(g_z, l_z) \leq M$, there exists $\max \{N(g_z, l_z) : z \in \mathbb{B}^n\}$. We denote $N_{\mathbf{b}}(F, L) = \max \{N(g_z, l_z) : z \in \mathbb{B}^n\} < +\infty$. Suppose that $N_{\mathbf{b}}(F, L)$ is not the L -index in the direction \mathbf{b} of the function $F(z)$. It means that there exists $n^* > N_{\mathbf{b}}(F, L)$ and $z^* \in \mathbb{B}^n$ such that

$$\frac{|\partial_{\mathbf{b}}^{n^*} F(z^*)|}{n^*!L^{n^*}(z^*)} > \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z^*)|}{k!L^k(z^*)} : 0 \leq k \leq N_{\mathbf{b}}(F, L) \right\}. \quad (5)$$

Since, for $g_z(t) = F(z + t\mathbf{b})$, we have $g_z^{(p)}(t) = \partial_{\mathbf{b}}^p F(z + t\mathbf{b})$, the inequality (5) can be rewritten as $\frac{|g_{z^*}^{(n^*)}(0)|}{n^*!l_{z^*}^{n^*}(0)} > \max \left\{ \frac{|g_{z^*}^{(k)}(0)|}{k!l_{z^*}^k(0)} : 0 \leq k \leq N_{\mathbf{b}}(F, L) \right\}$, but it is impossible (it contradicts that all l_z -indices $N(g_{z_0}, l_z)$ are not greater than $N_{\mathbf{b}}(F, L)$). Thus, $N_{\mathbf{b}}(F, L)$ is the L -index in the direction \mathbf{b} of the function $F(z)$. Theorem 1 is proved. \square

However, maximum can be calculated on a set A with a property $\bigcup_{z^0 \in A} \{z^0 + t\mathbf{b} : t \in S_{z^0}\} = \mathbb{B}^n$. Thus, the following assertion is valid.

Lemma 1. *If a function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ has bounded L -index in the direction, \mathbf{b} and j_0 are chosen with $b_{j_0} \neq 0$ then $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{C}^n, z_{j_0}^0 = 0\}$ and if $\sum_{j=1}^n b_j \neq 0$ then $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{C}^n, \sum_{j=1}^n z_j^0 = 0\}$.*

Proof. We prove that, for every $z \in \mathbb{B}^n$, there exist $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ with $z = z^0 + t\mathbf{b}$ and $z_{j_0}^0 = 0$. Put $t = z_{j_0}/b_{j_0}$, $z_j^0 = z_j - t\mathbf{b}_j$, $j \in \{1, 2, \dots, n\}$. Clearly, $z_{j_0}^0 = 0$ for this choice.

However, the point z^0 may not be contained in \mathbb{B}^n . However, there exists $t \in \mathbb{C}$ that $z^0 + t\mathbf{b} \in \mathbb{B}^n$. Let $z^0 \notin \mathbb{B}^n$ and $|z| = R_1 < 1$. Therefore, $|z^0 + t\mathbf{b}| = |z - \frac{z_{j_0}}{b_{j_0}}\mathbf{b} + t\mathbf{b}| = |z + (t - \frac{z_{j_0}}{b_{j_0}})\mathbf{b}| \leq |z| + |t - \frac{z_{j_0}}{b_{j_0}}| \cdot |\mathbf{b}| \leq R_1 + |t - \frac{z_{j_0}}{b_{j_0}}| \cdot |\mathbf{b}| < 1$. Thus, $|t - \frac{z_{j_0}}{b_{j_0}}| < \frac{1-R_1}{|\mathbf{b}|}$.

In the second part, we prove that, for every $z \in \mathbb{B}^n$, there exist $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ such that $z = z^0 + t\mathbf{b}$ and $\sum_{j=1}^n z_j^0 = 0$. Put $t = \frac{\sum_{j=1}^n z_j}{\sum_{j=1}^n b_j}$ and $z_j^0 = z_j - t\mathbf{b}_j$, $1 \leq j \leq n$. Thus, the following equality is valid: $\sum_{j=1}^n z_j^0 = \sum_{j=1}^n (z_j - t\mathbf{b}_j) = \sum_{j=1}^n z_j - \sum_{j=1}^n b_j t = 0$.

Lemma 1 is proved. \square

Note that, for a given $z \in \mathbb{B}^n$, we can pick uniquely $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ such that $\sum_{j=1}^n z_j^0 = 0$ and $z = z^0 + t\mathbf{b}$.

Remark 1. *If, for some $z^0 \in \mathbb{C}^n$ $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\} \cap \mathbb{B}^n = \emptyset$, then we put $N(g_{z^0}, l_{z^0}) = 0$.*

Theorem 2. *Let $A_0 \subset \mathbb{C}^n$ be such that $\bigcup_{z \in A_0} \{z + t\mathbf{b} : t \in S_z\} = \mathbb{B}^n$. A function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L -index in the direction \mathbf{b} if and only if there exists $M > 0$ such that for all $z^0 \in A_0$ the function $g_{z^0}(t)$ is of bounded l_{z^0} -index with $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of variable $t \in S_{z^0}$ and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.*

Proof. By Theorem 1, the analytic function F is of bounded the L -index in the direction \mathbf{b} if and only if there exists a number $M > 0$ such that, for every $z^0 \in \mathbb{B}^n$, the function $g_{z^0}(t)$ is of bounded l_{z^0} -index $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of variable $t \in S_{z^0}$. However, in view of property of the set A_0 for every $z^0 + t\mathbf{b}$, there exist $\tilde{z}^0 \in A_0$ and $\tilde{t} \in S_{\tilde{z}^0}$ such that $z^0 + t\mathbf{b} = \tilde{z}^0 + \tilde{t}\mathbf{b}$. In other words, for all $p \in \mathbb{Z}_+$, $(g_{z^0}(t))^{(p)} = (g_{\tilde{z}^0}(\tilde{t}))^{(p)}$. However, \tilde{t} depends on t . Thus, the condition that $g_{z^0}(t)$ is of bounded l_{z^0} -index for all $z^0 \in \mathbb{B}^n$ is equivalent to the condition $g_{\tilde{z}^0}(t)$ is of bounded $l_{\tilde{z}^0}$ -index for all $\tilde{z}^0 \in A_0$. \square

Remark 2. *An intersection of arbitrary hyperplane $H = \{z \in \mathbb{C}^n : \langle z, c \rangle = 1\}$ and the set $\mathbb{B}_{\mathbf{b}}^n = \{z + \frac{1-\langle z, c \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} : z \in \mathbb{B}^n\}$, where $\langle \mathbf{b}, c \rangle \neq 0$, satisfies conditions of Theorem 2.*

We prove that, for every $w \in \mathbb{B}^n$, there exist $z \in H \cap \mathbb{B}_{\mathbf{b}}^n$ and $t \in \mathbb{C}$ such that $w = z + t\mathbf{b}$.

Choosing $z = w + \frac{1-\langle w, c \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} \in H \cap \mathbb{B}_{\mathbf{b}}^n$, $t = \frac{\langle w, c \rangle - 1}{\langle \mathbf{b}, c \rangle}$, we obtain

$$z + t\mathbf{b} = w + \frac{1 - \langle w, c \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} + \frac{\langle w, c \rangle - 1}{\langle \mathbf{b}, c \rangle} \mathbf{b} = w.$$

Theorem 3 requires replacement of the space $\tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ by the space $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$. In other words, we use joint continuity in its proof.

Theorem 3. *Let $\bar{A} = \mathbb{B}^n$, i.e., A be an everywhere dense set in \mathbb{B}^n and let a function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$. The function F is of bounded L -index in the direction \mathbf{b} if and only if there exists $M > 0$ such*

that, for all $z^0 \in A$, a function $g_{z^0}(t)$ is of bounded l_{z^0} -index $N(g_{z^0}, l_{z^0}) \leq M < +\infty$ and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$.

Proof. The necessity follows from Theorem 1.

Sufficiency. Since $\overline{A} = \mathbb{B}^n$, then, for every $z^0 \in \mathbb{B}^n$, there exists a sequence $z^{(m)}$, which $z^{(m)} \rightarrow z^0$ as $m \rightarrow +\infty$ and $z^{(m)} \in A$ for all $m \in \mathbb{N}$. However, $F(z + t\mathbf{b})$ is of bounded l_z -index for all $z \in \overline{A}$ as a function of variable t . This is why, in view of the definition of bounded l_z -index, there exists $M > 0$ that for all $z \in A$, $t \in \mathbb{C}$, $p \in \mathbb{Z}_+$

$$\frac{|g_z^{(p)}(t)|}{p!l_z^p(t)} \leq \max\left\{\frac{|g_z^{(k)}(t)|}{k!l_z^k(t)} : 0 \leq k \leq M\right\}.$$

Substituting instead of z a sequence $z^{(m)} \in A$, $z^{(m)} \rightarrow z^0$, we obtain that, for every $m \in \mathbb{N}$,

$$\frac{|\partial_{\mathbf{b}}^p F(z^{(m)} + t\mathbf{b})|}{p!L^p(z^{(m)} + t\mathbf{b})} \leq \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^{(m)} + t\mathbf{b})|}{k!L^k(z^{(m)} + t\mathbf{b})} : 0 \leq k \leq M\right\}.$$

However, F and $\partial_{\mathbf{b}}^p F$ are continuous in \mathbb{B}^n for all $p \in \mathbb{N}$ and L is a positive continuous function. Thus, in the obtained expression, the limiting transition is possible as $m \rightarrow +\infty$ ($z^{(m)} \rightarrow z^0$). Evaluating the limit as $m \rightarrow +\infty$, we obtain that, for all $z^0 \in \mathbb{B}^n$, $t \in \mathbb{C}$, $m \in \mathbb{Z}_+$

$$\frac{|\partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})|}{p!L^p(z^0 + t\mathbf{b})} \leq \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0 + t\mathbf{b})|}{k!L^k(z^0 + t\mathbf{b})} : 0 \leq k \leq M\right\}.$$

This inequality implies that $F(z + t\mathbf{b})$ is of bounded $L(z + t\mathbf{b})$ -index as a function of variable t for every given $z \in \mathbb{B}^n$. Applying Theorem 1, we obtain the desired conclusion. Theorem 3 is proved. \square

Remark 2 and Theorem 3 yield the following corollary.

Corollary 1. Let A_0 be such that its closure is $\overline{A_0} = \{z \in \mathbb{C}^n : \langle z, c \rangle = 1\} \cap \mathbb{B}_{\mathbf{b}}^n$, where $\langle c, \mathbf{b} \rangle \neq 0$, $\mathbb{B}_{\mathbf{b}}^n = \{z + \frac{1 - \langle z, c \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} : z \in \mathbb{B}^n\}$. A function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L -index in the direction \mathbf{b} if and only if there exists number $M > 0$ such that, for all $z^0 \in A_0$, the function $g_{z^0}(t)$ is of bounded l_{z^0} -index with $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of variable $t \in S_{z^0}$. In addition, $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.

Proof. In view of Remark 2 in Theorem 2, we can take $B_0 = \{z \in \mathbb{C}^n : \langle z, c \rangle = 1\} \cap \{z + \frac{1 - \langle z, c \rangle}{\langle \mathbf{b}, c \rangle} \mathbf{b} : z \in \mathbb{B}^n\}$, where $\langle c, \mathbf{b} \rangle \neq 0$. Let A_0 be a dense set in B_0 , $\overline{A_0} = B_0$. Repeating considerations of Theorem 3, we obtain the desired conclusion.

Indeed, the necessity follows from Theorem 1 (in this theorem, the same condition is satisfied for all $z^0 \in \mathbb{C}^n$, and we need this condition for all $z^0 \in A_0$).

To prove the sufficiency, we use the density of the set A_0 . Obviously, for every $z^0 \in B_0$, there exists a sequence $z^{(m)} \rightarrow z^0$ and $z^{(m)} \in A_0$. However, $g_z(t)$ is of bounded l_z -index for all $z \in A_0$. Taking the conditions of Corollary 1 into account, for some $M > 0$ and for all $z \in A_0$, $t \in \mathbb{C}$, $p \in \mathbb{Z}_+$, the following inequality holds: $\frac{|g_z^{(p)}(t)|}{p!l_z^p(t)} \leq \max\left\{\frac{|g_z^{(k)}(t)|}{k!l_z^k(t)} : 0 \leq k \leq M\right\}$.

Substituting an arbitrary sequence $z^{(m)} \in A$, $z^{(m)} \rightarrow z^0$ instead of $z \in A^0$, we have

$$\frac{|g_{z^{(m)}}^{(p)}(t)|}{p!l_{z^{(m)}}^p(t)} \leq \max\left\{\frac{|g_{z^{(m)}}^{(k)}(t)|}{k!l_{z^{(m)}}^k(t)} : 0 \leq k \leq M\right\}, \text{ that is,}$$

$$\frac{|\partial_{\mathbf{b}}^p F(z^{(m)} + t\mathbf{b})|}{L^p(z^{(m)} + t\mathbf{b})} \leq \max_{0 \leq k \leq M} \frac{|\partial_{\mathbf{b}}^k F(z^{(m)} + t\mathbf{b})|}{k!L^k(z^{(m)} + t\mathbf{b})}.$$

However, F is an analytic function in \mathbb{B}^n , L is a positive continuous. Thus, we calculate a limit as $m \rightarrow +\infty$ ($z^m \rightarrow z$). For all $z^0 \in B_0$, $t \in S_{z^0}$, $m \in \mathbb{Z}_+$, we have

$$\frac{|\partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})|}{L^p(z^0 + t\mathbf{b})} \leq \max_{0 \leq k \leq M} \frac{|\partial_{\mathbf{b}}^k F(z^0 + t\mathbf{b})|}{k! L^k(z^0 + t\mathbf{b})}.$$

Therefore, $F(z^0 + t\mathbf{b})$ is of a bounded $L(z^0 + t\mathbf{b})$ -index as a function of t at each $z^0 \in \mathbb{B}^n$. By Theorem 3 and Remark 2, F is of bounded L -index in the direction \mathbf{b} . \square

Proposition 3. Let (r_p) be a positive sequence such that $r_p \rightarrow 1$ as $p \rightarrow \infty$, $D_p = \{z \in \mathbb{C}^n : |z| = r_p\}$, A_p be a dense set in D_p (i.e., $\overline{A_p} = D_p$) and $A = \bigcup_{p=1}^{\infty} A_p$. A function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L -index in the direction \mathbf{b} if and only if there exists number $M > 0$ such that, for all $z^0 \in A$, the function $g_{z^0}(t)$ is of bounded l_{z^0} -index $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of the variable $t \in S_{z^0}$. In addition, $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$.

Proof. Theorem 1 implies the necessity of this theorem.

Sufficiency. It is easy to prove $\{z + t\mathbf{b} : t \in S_z, z \in A\} = \mathbb{B}^n$. Furthermore, we repeat arguments with the proof of sufficiency in Theorem 3 and obtain the desired conclusion. \square

3. Local Behavior of Directional Derivative

The following proposition is important in the theory of functions of bounded index. It initializes series of propositions that are necessary to prove the logarithmic criterion of the index boundedness. It was first obtained by G. H. Fricke [19] for entire functions of bounded index. Later, the proposition was generalized for entire functions of bounded l -index [20], analytic functions of bounded l -index [21], and entire functions of bounded L -index in direction [22], functions analytic in a polydisc [23] or in a ball [24] with bounded L -index in joint variables, for slice holomorphic functions in \mathbb{C}^n [13] and for holomorphic bivariate entire functions [25].

Theorem 4. Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$. A function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L -index in the direction \mathbf{b} if and only if for each $\eta \in (0; \beta]$, there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that, for every $z \in \mathbb{B}^n$, there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, and

$$\max \left\{ \left| \partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b}) \right| : |t| \leq \frac{\eta}{L(z)} \right\} \leq P_1 \left| \partial_{\mathbf{b}}^{k_0} F(z) \right|. \quad (6)$$

Proof. Our proof is based on the proof of the appropriate theorem for analytic functions in the unit ball having bounded L -index in direction [16] and for slice holomorphic functions in \mathbb{C}^n [13].

Necessity. Let $N_{\mathbf{b}}(F; L) \equiv N < +\infty$. Let $[a]$, $a \in \mathbb{R}$ stands for the integer part of the number a in this proof. We denote

$$q(\eta) = [2\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}] + 1.$$

For $z \in \mathbb{B}^n$ and $p \in \{0, 1, \dots, q(\eta)\}$ we put

$$R_p^{\mathbf{b}}(z, \eta) = \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z + t\mathbf{b})|}{k! L^k(z + t\mathbf{b})} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\},$$

$$\tilde{R}_p^{\mathbf{b}}(z, \eta) = \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z + t\mathbf{b})|}{k! L^k(z)} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\}.$$

However, $|t| \leq \frac{p\eta}{q(\eta)L(z)} \leq \frac{\eta}{L(z)}$, then $\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right) \leq \lambda_{\mathbf{b}}(\eta)$. It is clear that $R_p^{\mathbf{b}}(z, \eta)$, $\tilde{R}_p^{\mathbf{b}}(z, \eta)$ are well-defined. Moreover,

$$\begin{aligned} R_p^{\mathbf{b}}(z, \eta) &= \\ &= \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} \left(\frac{L(z)}{L(z+t\mathbf{b})} \right)^k : 0 \leq k \leq N, |t| \leq \frac{p\eta}{q(\eta)L(z)} \right\} \leq \\ &\leq \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} \left(\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right) \right)^k : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\} \leq \\ &\leq \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} (\lambda_{\mathbf{b}}(\eta))^k : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\} \leq \\ &\leq (\lambda_{\mathbf{b}}(\eta))^N \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\} = \\ &= \tilde{R}_p^{\mathbf{b}}(z, \eta) (\lambda_{\mathbf{b}}(\eta))^N, \end{aligned} \quad (7)$$

$$\begin{aligned} \tilde{R}_p^{\mathbf{b}}(z, \eta) &= \\ &= \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} \left(\frac{L(z+t\mathbf{b})}{L(z)} \right)^k : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\} \leq \\ &\leq \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} \left(\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right) \right)^k : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\} \leq \\ &\leq \max \left\{ (\lambda_{\mathbf{b}}(\eta))^k \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\} \leq \\ &\leq (\lambda_{\mathbf{b}}(\eta))^N \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} : |t| \leq \frac{p\eta}{q(\eta)L(z)}, 0 \leq k \leq N \right\} = \\ &= R_p^{\mathbf{b}}(z, \eta) (\lambda_{\mathbf{b}}(\eta))^N. \end{aligned} \quad (8)$$

Let $k_p^z \in \mathbb{Z}$, $0 \leq k_p^z \leq N$, and $t_p^z \in S_z$, $|t_p^z| \leq \frac{p\eta}{q(\eta)L(z)}$ be such that

$$\tilde{R}_p^{\mathbf{b}}(z, \eta) = \frac{|\partial_{\mathbf{b}}^{k_p^z} F(z+t_p^z \mathbf{b})|}{k_p^z! L^{k_p^z}(z)}. \quad (9)$$

However, for every given $z \in \mathbb{B}^n$, the function $g_z(t) = F(z+t\mathbf{b})$ and its derivatives are analytic as functions of variable t . Then, by the maximum modulus principle, the equality (9) holds for t_p^z such that $|t_p^z| = \frac{p\eta}{q(\eta)L(z)}$. We set $\tilde{t}_p^z = \frac{p-1}{p} t_p^z$. Then,

$$|\tilde{t}_p^z| = \frac{(p-1)\eta}{q(\eta)L(z)}, \quad (10)$$

$$|\tilde{t}_p^z - t_p^z| = \frac{|t_p^z|}{p} = \frac{\eta}{q(\eta)L(z)}. \quad (11)$$

It follows from (10) and the definition of $\tilde{R}_{p-1}^{\mathbf{b}}(z, \eta)$ that

$$\tilde{R}_{p-1}^{\mathbf{b}}(z, \eta) \geq \frac{|\partial_{\mathbf{b}}^{k_p^z} F(z+\tilde{t}_p^z \mathbf{b})|}{k_p^z! L^{k_p^z}(z)}.$$

Therefore,

$$\begin{aligned} 0 \leq \tilde{R}_p^{\mathbf{b}}(z, \eta) - \tilde{R}_{p-1}^{\mathbf{b}}(z, \eta) &\leq \frac{\left| \partial_{\mathbf{b}}^{k_p^z} F(z+t_p^z \mathbf{b}) \right| - \left| \partial_{\mathbf{b}}^{k_p^z} F(z+\tilde{t}_p^z \mathbf{b}) \right|}{k_p^z! L^{k_p^z}(z)} = \\ &= \frac{1}{k_p^z! L^{k_p^z}(z)} \int_0^1 \frac{d}{ds} \left| \partial_{\mathbf{b}}^{k_p^z} F(z+(\tilde{t}_p^z + s(t_p^z - \tilde{t}_p^z)) \mathbf{b}) \right| ds. \end{aligned} \quad (12)$$

For every analytic complex-valued function of real variable $\varphi(s)$, $s \in \mathbb{R}$, the inequality $\left| \frac{d}{ds} \varphi(s) \right| \leq \left| \frac{d}{ds} \varphi(s) \right|$ holds, where $\varphi(s) \neq 0$. Applying this inequality to (12) and using the mean value theorem, we obtain

$$\begin{aligned} & \tilde{R}_p^{\mathbf{b}}(z, t_0, \eta) - \tilde{R}_{p-1}^{\mathbf{b}}(z, t_0, \eta) \leq \\ & \leq \frac{|t_p^z - \tilde{t}_p^z|}{k_p^z! L^{k_p^z}(z)} \int_0^1 \left| \partial_{\mathbf{b}}^{k_p^z+1} F(z + (\tilde{t}_p^z + s(t_p^z - \tilde{t}_p^z)) \mathbf{b}) \right| ds = \\ & = \frac{|t_p^z - \tilde{t}_p^z|}{k_p^z! L^{k_p^z}(z)} \left| \partial_{\mathbf{b}}^{k_p^z+1} F(z + (\tilde{t}_p^z + s^*(t_p^z - \tilde{t}_p^z)) \mathbf{b}) \right| = \\ & = L(z)(k_p^z + 1) |t_p^z - \tilde{t}_p^z| \frac{|\partial_{\mathbf{b}}^{k_p^z+1} F(z + (\tilde{t}_p^z + s^*(t_p^z - \tilde{t}_p^z)) \mathbf{b})|}{(k_p^z + 1)! L^{k_p^z+1}(z)}, \end{aligned}$$

where $s^* \in [0, 1]$. The point $\tilde{t}_p^z + s^*(t_p^z - \tilde{t}_p^z)$ belongs to the set

$$\left\{ t \in \mathbb{C} : |t| \leq \frac{p\eta}{q(\eta)L(z)} \leq \frac{\eta}{L(z)} \right\}.$$

Using the definition of the boundedness of L -index in direction, the definition of $q(\eta)$, inequalities (7) and (11), for $k_p^z \leq N$, we have

$$\begin{aligned} & \tilde{R}_p^{\mathbf{b}}(z, \eta) - \tilde{R}_{p-1}^{\mathbf{b}}(z, \eta) \leq \frac{|\partial_{\mathbf{b}}^{k_p^z+1} F(z + (\tilde{t}_p^z + s^*(t_p^z - \tilde{t}_p^z)) \mathbf{b})|}{(k_p^z + 1)! L^{k_p^z+1}(z + (\tilde{t}_p^z + s^*(t_p^z - \tilde{t}_p^z)) \mathbf{b})} \times \\ & \times \left(\frac{L(z + (\tilde{t}_p^z + s^*(t_p^z - \tilde{t}_p^z)) \mathbf{b})}{L(z)} \right)^{k_p^z+1} L(z)(k_p^z + 1) |t_p^z - \tilde{t}_p^z| \leq \eta \frac{N+1}{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^{N+1} \times \\ & \times \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z + (\tilde{t}_p^z + s^*(t_p^z - \tilde{t}_p^z)) \mathbf{b})|}{k! L^k(z + (\tilde{t}_p^z + s^*(t_p^z - \tilde{t}_p^z)) \mathbf{b})} : 0 \leq k \leq N \right\} \leq \eta \frac{N+1}{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^{N+1} R_p^{\mathbf{b}}(z, \eta) \leq \\ & \leq \frac{\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}}{[2\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}] + 1} \tilde{R}_p^{\mathbf{b}}(z, \eta) \leq \frac{1}{2} \tilde{R}_p^{\mathbf{b}}(z, \eta) \end{aligned}$$

It follows that $\tilde{R}_p^{\mathbf{b}}(z, \eta) \leq 2\tilde{R}_{p-1}^{\mathbf{b}}(z, \eta)$. Using inequalities (7) and (8), we obtain for $R_p^{\mathbf{b}}(z, \eta)$

$$R_p^{\mathbf{b}}(z, \eta) \leq 2(\lambda_{\mathbf{b}}(\eta))^N \tilde{R}_{p-1}^{\mathbf{b}}(z, \eta) \leq 2(\lambda_{\mathbf{b}}(\eta))^{2N} R_{p-1}^{\mathbf{b}}(z, \eta).$$

Hence,

$$\begin{aligned} & \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z + t\mathbf{b})|}{k! L^k(z + t\mathbf{b})} : |t| \leq \frac{\eta}{L(z)}, 0 \leq k \leq N \right\} = R_{q(\eta)}^{\mathbf{b}}(z, \eta) \leq \\ & \leq 2(\lambda_{\mathbf{b}}(\eta))^{2N} R_{q(\eta)-1}^{\mathbf{b}}(z, \eta) \leq (2(\lambda_{\mathbf{b}}(\eta))^{2N})^2 R_{q(\eta)-2}^{\mathbf{b}}(z, \eta) \leq \\ & \leq \dots \leq (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} R_0^{\mathbf{b}}(z, \eta) = \\ & = (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} : 0 \leq k \leq N \right\}. \end{aligned} \quad (13)$$

Let $k_z \in \mathbb{Z}$, $0 \leq k_z \leq N$, and $\tilde{t}_z \in \mathbb{C}$, $|\tilde{t}_z| = \frac{\eta}{L(z)}$ be such that

$$\frac{|\partial_{\mathbf{b}}^{k_z} F(z)|}{k_z! L^{k_z}(z)} = \max_{0 \leq k \leq N} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)},$$

and

$$|\partial_{\mathbf{b}}^{k_z} F(z + \tilde{t}_z \mathbf{b})| = \max \{ |\partial_{\mathbf{b}}^{k_z} F(z + t\mathbf{b})| : |t| \leq \eta/L(z) \}.$$

Inequality (13) implies

$$\begin{aligned} \frac{|\partial_{\mathbf{b}}^{k_z} F(z + \tilde{t}_z \mathbf{b})|}{k_z! L^{k_z}(z + \tilde{t}_z \mathbf{b})} &\leq \max \left\{ \frac{|\partial_{\mathbf{b}}^{k_z} F(z + t \mathbf{b})|}{k_z! L^{k_z}(z + t \mathbf{b})} : |t| = \frac{\eta}{L(z)} \right\} \leq \\ &\leq \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z + t \mathbf{b})|}{k! L^k(z + t \mathbf{b})} : |t| = \frac{\eta}{L(z)}, 0 \leq k \leq N \right\} \leq \\ &\leq (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} \frac{|\partial_{\mathbf{b}}^{k_z} F(z)|}{k_z! L^{k_z}(z)}. \end{aligned}$$

Hence,

$$\begin{aligned} &\max \left\{ |\partial_{\mathbf{b}}^{k_z} F(z + t \mathbf{b})| : |t| \leq \eta / L(z) \right\} \leq \\ &\leq (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} \frac{L^{k_z}(z + \tilde{t}_z \mathbf{b})}{L^{k_z}(z)} |\partial_{\mathbf{b}}^{k_z} F(z)| \leq \\ &\leq (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^N |\partial_{\mathbf{b}}^{k_z} F(z)| \leq \\ &\leq (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^N |\partial_{\mathbf{b}}^{k_z} F(z)|. \end{aligned}$$

Thus, we obtain (6) with $n_0 = N_{\mathbf{b}}(F, L)$ and

$$P_1(\eta) = (2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^N > 1.$$

Sufficiency. Suppose that, for each $\eta \in (0; \beta]$, there exists $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that, for every $z \in \mathbb{B}^n$, there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, for which inequality (6) holds. We choose $\eta > 1$ and $j_0 \in \mathbb{N}$ such that $P_1 \leq \eta^{j_0}$. For given $z \in \mathbb{B}^n$, $k_0 = k_0(z)$ and $j \geq j_0$ by Cauchy's formula for $F(z + t \mathbf{b})$ as a function of one variable t

$$\partial_{\mathbf{b}}^{k_0+j} F(z) = \frac{j!}{2\pi i} \int_{|t|=\eta/L(z)} \frac{\partial_{\mathbf{b}}^{k_0} F(z + t \mathbf{b})}{t^{j+1}} dt.$$

Therefore, in view of (6), we have

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{j!} \leq \frac{L^j(z)}{\eta^j} \max \left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t \mathbf{b})| : |t| = \frac{\eta}{L(z)} \right\} \leq P_1 \frac{L^j(z)}{\eta^j} |\partial_{\mathbf{b}}^{k_0} F(z)|,$$

Hence, for all $j \geq j_0$, $z \in \mathbb{B}^n$,

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)! L^{k_0+j}(z)} \leq \frac{j! k_0!}{(j+k_0)!} \frac{P_1}{\eta^j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \leq \eta^{j_0-j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)}.$$

Since $k_0 \leq n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ are independent of z and t_0 , this inequality means that a function F has a bounded L -index in the direction \mathbf{b} and $N_{\mathbf{b}}(F, L) \leq n_0 + j_0$. The proof of Theorem 4 is complete. \square

4. Application of Theorem on Local Behavior of Functions Having Bounded L -Index in Direction

Below, we consider an application of Theorem 4. This theorem implies the next proposition that describes the boundedness of L -index in direction for an equivalent function to L .

Proposition 4. Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $\frac{1}{\beta} < \theta_1 \leq \theta_2 < +\infty$, $\theta_1 L(z) \leq L^*(z) \leq \theta_2 L(z)$. A function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of bounded L^* -index in the direction \mathbf{b} if and only if F is of bounded L -index in the direction \mathbf{b} .

Proof. Obviously, if $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$ and $\theta_1 L(z) \leq L^*(z) \leq \theta_2 L(z)$, then $L^* \in Q_{\mathbf{b}}(\mathbb{B}^n)$ with $\beta^* \in [\theta_1 \beta; \theta_2 \beta]$ and $\beta^* > 1$ instead of $\beta > 1$. Let $N_{\mathbf{b}}(F, L^*) < +\infty$. Therefore, by Theorem 4 for each η^* , $0 < \eta^* < \beta \theta_2$, there exist $n_0(\eta^*) \in \mathbb{Z}_+$ and $P_1(\eta^*) \geq 1$ such that, for every $z \in \mathbb{B}^n$, $t_0 \in S_z$ and some k_0 , $0 \leq k_0 \leq n_0$, inequality (6) is valid with L^* and η^* instead of L and η . Taking $\eta^* = \theta_2 \eta$, we obtain

$$\begin{aligned} P_1 |\partial_{\mathbf{b}}^{k_0} F(z)| &\geq \max \left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \eta^* / L^*(z) \right\} \geq \\ &\geq \max \left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \eta / L(z) \right\}. \end{aligned}$$

Therefore, by Theorem 4, the function $F(z)$ is of bounded L -index in the direction \mathbf{b} . The converse assertion is obtained by replacing L on L^* . \square

Proposition 5. Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $m \in \mathbb{C} \setminus \{0\}$. A function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ is of a bounded L -index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if $F(z)$ is of a bounded L -index in the direction $m\mathbf{b}$.

Proof. Let a function $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be of a bounded L -index in the direction \mathbf{b} . By Theorem 4, $(\forall \eta > 0) (\exists n_0(\eta) \in \mathbb{Z}_+) (\exists P_1(\eta) \geq 1) (\forall z \in \mathbb{B}^n) (\exists k_0 = k_0(z) \in \mathbb{Z}_+, 0 \leq k_0 \leq n_0)$, and the following inequality is valid

$$\max \left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \eta / L(z) \right\} \leq P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|. \quad (14)$$

Since $\frac{\partial^k F}{\partial (m\mathbf{b})^k} = (m)^k \frac{\partial^k F}{\partial \mathbf{b}^k}$, inequality (14) is equivalent to the inequality

$$\max \left\{ |m|^{k_0} |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \eta / L(z) \right\} \leq P_1 |m|^{k_0} |\partial_{\mathbf{b}}^{k_0} F(z)|$$

as well as to the inequality

$$\max \left\{ |\partial_{m\mathbf{b}}^{k_0} F(z + \frac{t}{m} m\mathbf{b})| : |t/m| \leq \eta / (|m|L(z)) \right\} \leq P_1 |\partial_{m\mathbf{b}}^{k_0} F(z)|.$$

Denoting $t^* = \frac{t}{m}$, $\eta^* = \frac{\eta}{|m|}$, we obtain

$$\max \left\{ |\partial_{m\mathbf{b}}^{k_0} F(z + t^* m\mathbf{b})| : |t^*| \leq \eta^* / L(z) \right\} \leq P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|.$$

By Theorem 4, the function $F(z)$ is of bounded L -index in the direction \mathbf{b} . The converse assertion can be proved similarly. \square

Please note that Proposition 4 can be slightly refined. The following proposition is easily deduced from (2).

Proposition 6. Let $L_1(z)$, $L_2(z)$ be positive continuous functions, $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a function of bounded L_1 -index in the direction \mathbf{b} , for all $z \in \mathbb{B}^n$ the inequality $L_1(z) \leq L_2(z)$ holds. Then, $N_{\mathbf{b}}(L_2, F) \leq N_{\mathbf{b}}(L_1, F)$.

Using Fricke's idea [26], we deduce a modification of Theorem 4. Our proof is similar to proof in [27]. This theorem gives weaker sufficient conditions of boundedness of L -index in the direction in comparison of Theorem 4. Unlike the last assertion, it turns out that, under appropriate conditions imposed on the slice holomorphic function, it is sufficient to demand the validity of the corresponding inequality only for a circle with a certain given value of the radius but not for all values of the radius from the segment $[0; \beta]$. In other words, the universal quantifier in the sufficient conditions of Theorem 4 are replaced by the existential quantifier in Theorem 5.

Theorem 5. Let $L \in Q_{\mathbf{b}}(\mathbb{B}^n)$, $F \in \tilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$. If there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that, for any $z \in \mathbb{B}^n$, there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, and

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \eta/L(z)\} \leq P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|,$$

then the function F has bounded L -index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.

Proof. Besides the mentioned paper of Fricke [26], our proof is similar to proofs in [13] (slice holomorphic functions in \mathbb{C}^n).

Assume that there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that, for any $z \in \mathbb{B}^n$, there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, and

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \frac{\eta}{L(z)}\} \leq P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|. \quad (15)$$

If $\eta \in (1, \beta]$, then we choose $j_0 \in \mathbb{N}$ such that $P_1 \leq \eta^{j_0}$. In addition, for $\eta \in (0; 1]$, we choose $j_0 \in \mathbb{N}$ such that $\frac{j_0!k_0!}{(j_0+k_0)!} P_1 < 1$. The j_0 is well-defined because

$$\frac{j_0!k_0!}{(j_0+k_0)!} P_1 = \frac{k_0!}{(j_0+1)(j_0+2) \cdots (j_0+k_0)} P_1 \rightarrow 0, \quad j_0 \rightarrow \infty.$$

Applying integral Cauchy's formula to the function $g_z(t) = F(z + t\mathbf{b})$ as an analytic function of one complex variable t for $j \geq j_0$, we obtain that, for every $z \in \mathbb{B}^n$, there exists $k_0 = k_0(z)$, $0 \leq k_0 \leq n_0$, and

$$\partial_{\mathbf{b}}^{k_0+j} F(z) = \frac{j!}{2\pi i} \int_{|t|=\frac{\eta}{L(z)}} \frac{\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})}{t^{j+1}} dt.$$

Taking into account (15), we deduce

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{j!} \leq \frac{L^j(z)}{\eta^j} \max\left\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| = \frac{\eta}{L(z)}\right\} \leq P_1 \frac{L^j(z)}{\eta^j} |\partial_{\mathbf{b}}^{k_0} F(z)|. \quad (16)$$

In view of choice j_0 with $\eta \in (1, \beta]$, for all $j \geq j_0$, one has

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)!L^{k_0+j}(z)} \leq \frac{j!k_0!}{(j+k_0)!} \frac{P_1}{\eta^j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!L^{k_0}(z + t_0\mathbf{b})} \leq \eta^{j_0-j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!L^{k_0}(z)}.$$

Since $k_0 \leq n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ do not depend on z , and $z \in \mathbb{B}^n$ is arbitrary, the last inequality is equivalent to the assertion that F has bounded L -index in the direction \mathbf{b} and $N_{\mathbf{b}}(F, L) \leq n_0 + j_0$.

If $\eta \in (0, 1)$, then, from (16), it follows that for all $j \geq j_0$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)!L^{k_0+j}(z)} \leq \frac{j!k_0!P_1}{(j+k_0)! \eta^j k_0!L^{k_0}(z)} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0!L^{k_0}(z)}$$

or in view of choice j_0

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)!} \frac{\eta^{k_0+j}}{L^{k_0+j}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!} \frac{\eta^{k_0}}{L^{k_0}(z)}.$$

Thus, the function F is of bounded \tilde{L} -index in the direction \mathbf{b} , where $\tilde{L}(z) = \frac{L(z)}{\eta}$. Then, by Proposition 4, the function F has bounded L -index in the direction \mathbf{b} , if $\eta\beta > 1$. When $\eta \leq \frac{1}{\beta}$, we choose arbitrary $\gamma > \frac{1}{\eta\beta}$. By Proposition 4, the function F is of bounded

L_1 -index in the direction \mathbf{b} , where $L_1(z) = \eta\gamma\tilde{L}(z)$. Then, by Proposition 5, the function F has bounded L_1 -index in the direction $\gamma\mathbf{b}$. Since $\partial_{\gamma\mathbf{b}}^k F = \gamma^k \partial_{\mathbf{b}}^k F$ and $L_1^k(z) = \gamma^k L^k(z)$, in inequality (2), with the definition of L -index boundedness in direction, the corresponding multiplier γ is reduced. Hence, the function F is of bounded L -index in the direction \mathbf{b} . The theorem is proved. \square

5. L -Index in Direction in a Domain Compactly Embedded in the Unit Ball

In this section, we consider an application of Theorem 4.

Let D be an arbitrary bounded domain in \mathbb{B}^n such that $\text{dist}(D, \mathbb{B}^n) > 0$. If inequality (2) holds for all $z \in D$ instead \mathbb{B}^n , then the function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ is called a function of bounded L -index in the direction \mathbf{b} in the domain D . The least such integer m_0 is called the L -index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ in domain D and is denoted by $N_{\mathbf{b}}(F, L, D) = m_0$. The notation \overline{D} stands for a closure of the domain D .

Lemma 2. Let D be a bounded domain in \mathbb{B}^n such that $d = \text{dist}(D, \mathbb{B}^n) = \inf_{z \in D} (1 - |z|) > 0$, $\beta > 1$, $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ be an arbitrary direction. If $L: \mathbb{B}^n \rightarrow \mathbb{R}_+$ is a continuous function such that $L(z) \geq \frac{\beta|\mathbf{b}|}{d}$ and a function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ are such that $(\forall z^0 \in \overline{D}): F(z^0 + t\mathbf{b}) \neq 0$, then $N_{\mathbf{b}}(F, L, D) < \infty$.

Proof. This proof is similar to proof in [13] for slice entire functions in \mathbb{C}^n .

For every fixed $z^0 \in \overline{D}$, we expand the analytic function $F(z^0 + t\mathbf{b})$ in a power series by powers of t in the disc $\{t \in \mathbb{C} : |t| \leq \frac{1}{L(z^0)}\}$

$$F(z^0 + t\mathbf{b}) = \sum_{m=0}^{\infty} \frac{\partial_{\mathbf{b}}^m F(z^0)}{m!} t^m. \quad (17)$$

The quantity $\frac{|\partial_{\mathbf{b}}^m F(z^0)|}{m!}$ is the modulus of a coefficient of the power series (17) at the point $t \in \mathbb{C}$ such that $|t| = \frac{1}{L(z^0)}$. Since $F(z)$ is an analytic function, for every $z_0 \in \overline{D}$

$$\frac{|\partial_{\mathbf{b}}^m F(z^0)|}{m! L^m(z^0)} \rightarrow 0 \quad (m \rightarrow \infty),$$

i.e., there exists $m_0 = m(z^0, \mathbf{b})$ such that inequality (2) holds at the point $z = z^0$ for all $m \in \mathbb{Z}_+$.

We prove that $\sup\{m_0: z^0 \in \overline{D}\} < +\infty$. On the contrary, we assume that the set of all values m_0 is unbounded in \mathbb{Z}_+ , i.e., $\sup\{m_0: z^0 \in \overline{D}\} = +\infty$. Hence, for every $m \in \mathbb{Z}_+$, there exists $z^{(m)} \in \overline{D}$ and $p_m > m$

$$\frac{|\partial_{\mathbf{b}}^{p_m} F(z^{(m)})|}{p_m! L^{p_m}(z^{(m)})} > \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z^{(m)})|}{k! L^k(z^{(m)})} : 0 \leq k \leq m \right\}. \quad (18)$$

Since $\{z^{(m)}\} \subset \overline{D}$, there exists a subsequence $z'^{(m)} \rightarrow z' \in \overline{G}$ as $m \rightarrow +\infty$. By Cauchy's integral formula

$$\frac{\partial_{\mathbf{b}}^p F(z)}{p!} = \frac{1}{2\pi i} \int_{|t|=r} \frac{F(z + t\mathbf{b})}{t^{p+1}} dt$$

for any $p \in \mathbb{N}$, $z \in D$. Rewrite (18) as follows:

$$\begin{aligned} & \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z^{(m)})|}{k! L^k(z^{(m)})} : 0 \leq k \leq m \right\} < \\ & < \frac{1}{L^{p_m}(z^{(m)})} \int_{|t|=r/L(z^{(m)})} \frac{|F(z^{(m)} + t\mathbf{b})|}{|t|^{p_m+1}} |dt| \leq \frac{1}{r^{p_m}} \max\{|F(z)| : z \in D_r\}, \end{aligned} \quad (19)$$

where $D_r = \bigcup_{z^* \in \overline{D}} \{z \in \mathbb{C}^n : |z - z^*| \leq \frac{|\mathbf{b}|r}{L(z^*)}\}$. We can choose $r \in (1, \beta)$, because $g_{z(m)}(t) = F(z^{(m)} + t\mathbf{b})$ is an analytic function in $S_{z(m)}$. Evaluating the limit for every directional derivative of fixed order in (19) as $m \rightarrow \infty$, we obtain

$$(\forall k \in \mathbb{Z}_+): \quad \frac{|\partial_{\mathbf{b}}^k F(z')|}{k! L^k(z')} \leq \overline{\lim}_{m \rightarrow \infty} \frac{1}{r^k p_m} \max\{|F(z)| : z \in D_r\} \leq 0.$$

Thus, all derivatives in the direction \mathbf{b} of the function F at the point z' equal 0 and $F(z') = 0$. In view of (17), $F(z' + t\mathbf{b}) \equiv 0$. It is a contradiction. \square

6. Discussion

The proposed approach can be used in an analytic theory of partial differential equations. For cases of entire functions and analytic functions, it is known that similar results allow for deducing sufficient conditions by coefficients of partial differential equations and its systems providing index boundedness of every analytic solution. In addition, it gives growth estimates, local behavior, and some value distribution for these functions. By analogy, we hope that the obtained results allow for obtaining similar applications for slice holomorphic functions in the unit ball in the future investigations.

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