A Remark for the Hyers–Ulam Stabilities on n -Banach Spaces

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Abstract: In this article, we deal with stabilities of several functional equations in n -Banach spaces. For a surjective mapping f into a n -Banach space, we prove the generalized Hyers–Ulam stabilities of the cubic functional equation and the quartic functional equation for f in n -Banach spaces.

Keywords: n -Banach space; cubic mappings; quartic mappings; the generalized Hyers–Ulam stability

MSC: Primary 39B82; Secondary 46B20

1. Introduction

A question of the stability of functional equations concerning group homomorphisms was first raised by S. M. Ulam in 1940 [1]. In the next year, a partial affirmative answer to the question of Ulam was given by D. H. Hyers [2] for additive mappings on Banach spaces. Hyers' theorem was generalized by T. Aoki [3] for additive mapping. In 1978, Rassias [4] provided a generalization of the theorem for linear mappings by allowing the Cauchy differences to be unbounded. Subsequently, the result of Rassias' theorem was generalized by P. Găvruta [5], allowing the Cauchy difference controlled by a general unbounded function which is called *the generalized Hyers–Ulam stability*. On the other hand, Rassias and Šemrl found an example of a continuous real-valued function from $\mathbb{R}$ for which the Hyers–Ulam stability does not occur. See [6].

Let X and Y be real vector spaces and $f : X \rightarrow Y$ a mapping. For a cubic function $f(x) = cx^3$ ($c \in \mathbb{R}$, $X = Y = \mathbb{R}$), f clearly satisfies the following functional equation

$$f(x + 2y) + 3f(x) = 3f(x + y) + f(x - y) + 6f(y). \quad (1)$$

For this reason, it is natural that Equation (1) is called a cubic functional equation and every solution of Equation (1) is also called a cubic function. The general solution for Equation (1) was solved by J. M. Rassias [7] for a mapping from a real normed space to a Banach space. Jun et al. [8] proved that the cubic functional Equation (1) is equivalent to the following functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \quad (2)$$

In [9], Chu et al. extended the cubic functional equation to the following generalized form

$$\begin{aligned} & f\left(\sum_{j=1}^{n-1} x_j + 2x_n\right) + f\left(\sum_{j=1}^{n-1} x_j - 2x_n\right) + \sum_{j=1}^{n-1} f(2x_j) \\ & = 2f\left(\sum_{j=1}^{n-1} x_j\right) + 4\sum_{j=1}^{n-1} (f(x_j + x_n) + f(x_j - x_n)), \end{aligned}$$

where $n \geq 2$ is an integer, and they also proved the generalized Hyers–Ulam stability. The stability problem for cubic functional equations has been extensively investigated by many mathematicians (see [10–12].)

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In [13], J. M. Rassias introduced the functional equation as follows:

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y) \quad (3)$$

It is obvious that $f(x) = x^4$ is a solution of Equation (3), so we call Equation (3) a quartic functional equation. Chung and Sahoo [14] investigated the general solution of (3) and A. Najati [15] proved the generalized Hyers–Ulam stability for the quartic functional Equation (3) using the idea of Găvruta [5]. The stability results of quartic functional equations can be found in several other papers (see [16–18].) There are a number of papers and research monographs regarding various generalizations and applications of the generalized Hyers–Ulam stability of several functional equations. See [19–22]. Park investigated the generalized Hyers–Ulam stability for additive mappings, Jensen mappings and quadratic mappings in 2-Banach spaces in [23,24].

Misiak [25,26] introduced the notion of n -normed spaces which is one of the generalizations of normed spaces and 2-normed spaces. For more information of the phase spaces, we refer to the papers [27–30]. Recently, Chu et al. [31] studied the generalized Hyers–Ulam stabilities of the Cauchy functional equations, the Jensen functional equations and the quadratic functional equations on n -Banach spaces. In [32], Brzdęk and Ciepliński proved a fixed point theorem for operator acting on a class of functions with values in an n -Banach space. For study of the Hyers–Ulam stability, the extension to n -Banach spaces is valuable in terms of development of the field of functional equations.

Motivated by results in [31,32], we focus on the generalized Hyers–Ulam stabilities of several functional equations—in detail, the cubic functional equation expressed as Equation (2) and the quartic functional equation expressed as Equation (3) on n -Banach spaces. We prove the generalized Hyers–Ulam stabilities of the functional equations on the spaces.

The contents of paper: In Section 2, we recall definitions and lemma in n -Banach spaces to investigate the generalized Hyers–Ulam stabilities on the spaces. In Section 3, we investigate the generalized Hyers–Ulam stability problem in n -Banach spaces. The problems for the generalized Hyers–Ulam stability related on the cubic functional equation in n -Banach spaces are studied in Section 3.1. We also deal with applications of the stabilities for the functional equations on the spaces. In Section 3.2, we focus on the the quartic functional equation and prove the generalized stability on the n -Banach spaces.

2. Preliminaries

In this section, we recall definitions and lemma in n -Banach spaces as a preliminary step toward the main theorems.

Definition 1 ([25,26]). Let X be a real linear space with $\dim X \geq n$ and $\|\cdot, \dots, \cdot\| : X^n \rightarrow \mathbb{R}$ be a function. Then $(X, \|\cdot, \dots, \cdot\|)$ is called a linear n -normed space if

$$(nN_1) \quad \|x_1, \dots, x_n\| = 0 \Leftrightarrow x_1, \dots, x_n \text{ are linearly dependent};$$

$$(nN_2) \quad \|x_1, \dots, x_n\| = \|x_{j_1}, \dots, x_{j_n}\| \text{ for every permutation } (j_1, \dots, j_n) \text{ of } (1, \dots, n);$$

$$(nN_3) \quad \|\alpha x_1, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|;$$

$$(nN_4) \quad \|x + y, x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$$

for all $\alpha \in \mathbb{R}$ and all $x, y, x_1, \dots, x_n \in X$. The function $\|\cdot, \dots, \cdot\|$ is called an n -norm on X .

Definition 2 ([31]). Let $\{x_\ell\}$ be a sequence in a linear n -normed space X . The sequence $\{x_\ell\}$ is said to be n -convergent in X if there exists an element $x \in X$ such that

$$\lim_{\ell \rightarrow \infty} \|x_\ell - x, y_2, \dots, y_n\| = 0$$

for all $y_2, \dots, y_n \in X$. In this case, we say that a sequence $\{x_\ell\}$ converges to the limit x , simply denoted by $\lim_{\ell \rightarrow \infty} x_\ell = x$ with a slight abuse of notation.

Definition 3 ([31]). A sequence $\{x_\ell\}$ in a linear n -normed space X is called an n -Cauchy sequence if for any $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $s, t \geq N$, $\|x_s - x_t, y_2, \dots, y_n\| < \varepsilon$ for all $y_2, \dots, y_n \in X$. For convenience, we will write $\lim_{s,t \rightarrow \infty} \|x_s - x_t, y_2, \dots, y_n\| = 0$ for an n -Cauchy sequence $\{x_\ell\}$. An n -Banach space is defined to be a linear n -normed space in which every n -Cauchy sequence is n -convergent.

The following lemma is a useful toolbox for a linear n -normed space.

Lemma 1 ([31]). Let $(X, \|\cdot, \dots, \cdot\|)$ be a linear n -normed space and $x \in X$. Then

- (1) If $\|x, x_2, \dots, x_n\| = 0$ for all $x_2, \dots, x_n \in X$, then $x = 0$.
- (2) $\|x, x_2, \dots, x_n\| - \|y, x_2, \dots, x_n\| \leq \|x - y, x_2, \dots, x_n\|$ for all $x, y, x_2, \dots, x_n \in X$.
- (3) if a sequence $\{x_m\}$ is convergent in X , then

$$\lim_{m \rightarrow \infty} \|x_m, y_2, \dots, y_n\| = \left\| \lim_{m \rightarrow \infty} x_m, y_2, \dots, y_n \right\|$$

for all $y_2, \dots, y_n \in X$.

From now on, let X be a real linear space and let $(Z, \|\cdot, \dots, \cdot\|)$ be an n -Banach space unless otherwise stated.

3. Main Results

In this section, we present the generalized Hyers–Ulam stabilities for the several functional equations in n -Banach spaces. We solve the problems for the stabilities and consider applications of the results in n -Banach spaces.

3.1. Stability of the Cubic Functional Equation

We start this subsection by investigating the generalized Hyers–Ulam stability for the cubic functional Equation (2) in n -Banach spaces. For convenience, we use the notation $D_f(x, y)$ as follows:

$$D_f(x, y) := f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)$$

for all $x, y \in X$. If $D_f(x, y) = 0$, then the function f is a solution of the cubic functional equation. Thus, $D_f(x, y)$ is an approximate remainder of the functional Equation (2) and acts as a perturbation of the equation. We use this approximate remainder to solve the generalized Hyers–Ulam stability for the cubic functional equation in n -Banach spaces.

Now, in the following theorem, we present a solution of stability for the cubic functional equation in the spaces.

Theorem 1. Let $\varphi : X^{n+1} \rightarrow \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 0, x_2, \dots, x_n)}{8^i} < \infty, \quad \lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y, x_2, \dots, x_n)}{8^n} = 0$$

for all $x, y, x_2, \dots, x_n \in X$. Suppose that a function $f : X \rightarrow Z$ be a surjective mapping satisfying

$$\|D_f(x, y), z_2, \dots, z_n\| \leq \varphi(x, y, x_2, \dots, x_n) \quad (4)$$

for all $x, y, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. Then there is a unique cubic mapping $C : X \rightarrow Z$ such that

$$\|f(x) - C(x), z_2, \dots, z_n\| \leq \frac{1}{16} \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 0, x_2, \dots, x_n)}{8^i} \quad (5)$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$.

We call the function f the pseudo-cubic function for the error function φ , and the solution function C is the cubic function induced from the pseudo-cubic function f .

Proof. Let $z_i = f(x_i)$ ($i = 2, 3, \dots, n$). First, take $y = 0$ in (4) to have

$$\left\| \frac{f(2x)}{8} - f(x), z_2, \dots, z_n \right\| \leq \frac{1}{16} \varphi(x, 0, x_2, \dots, x_n) \quad (6)$$

for all $x, x_2, \dots, x_n \in X$. Replacing x by $2x$ in (6) and dividing by 8, we obtain

$$\left\| \frac{f(2^2x)}{8^2} - f(x), z_2, \dots, z_n \right\| \leq \frac{1}{16} [\varphi(x, 0, x_2, \dots, x_n) + \frac{\varphi(2x, 0, x_2, \dots, x_n)}{8}] \quad (7)$$

for all $x, x_2, \dots, x_n \in X$. Using the induction on n , we get that

$$\left\| \frac{f(2^n x)}{8^n} - f(x), z_2, \dots, z_n \right\| \leq \frac{1}{16} \sum_{i=0}^{n-1} \frac{\varphi(2^i x, 0, x_2, \dots, x_n)}{8^i} \quad (8)$$

for all $x, x_2, \dots, x_n \in X$. For $0 \leq m < n$, divide inequality (8) by 8^m and also replace x by $2^m x$ to find that

$$\begin{aligned} \left\| \frac{f(2^{n+m} x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, z_2, \dots, z_n \right\| &= \frac{1}{8^m} \left\| \frac{f(2^{n+m} x)}{8^n} - f(2^m x), z_2, \dots, z_n \right\| \\ &\leq \frac{1}{16 \cdot 8^m} \sum_{i=0}^{n-1} \frac{\varphi(2^{i+m} x, 0, x_2, \dots, x_n)}{8^i} \\ &\leq \frac{1}{16} \sum_{i=m}^{n-1} \frac{\varphi(2^i x, 0, x_2, \dots, x_n)}{8^i} \end{aligned}$$

for all $x, x_2, \dots, x_n \in X$. We then obtain

$$\lim_{m, n \rightarrow \infty} \left\| \frac{f(2^{n+m} x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, z_2, \dots, z_n \right\| = 0$$

for all $x_2, \dots, x_n \in X$. Since f is surjective, by Lemma 1, the sequence $\{\frac{1}{8^n} f(2^n x)\}$ is an n -Cauchy sequence in Z . Therefore, we may define a mapping $C : X \rightarrow Z$ by

$$C(x) := \lim_{n \rightarrow \infty} \frac{1}{8^n} f(2^n x)$$

for all $x \in X$. By letting $n \rightarrow \infty$ in (8), we arrive at the formula (5). To show that the mapping $C : X \rightarrow Z$ satisfies Equation (2), replace x, y with $2^n x, 2^n y$, respectively, in (4) and divide by 8^n ; then it follows that

$$\begin{aligned} 8^{-n} \|f(2^n(2x+y)) + f(2^n(2x-y)) - 2f(2^n(x+y)) - 2f(2^n(x-y)) - 12f(2^n x), z_2, \dots, z_n\| \\ \leq 8^{-n} \varphi(2^n x, 2^n y, x_2, \dots, x_n) \end{aligned} \quad (9)$$

for all $x, x_2, \dots, x_n \in X$. Taking the limit as $n \rightarrow \infty$ in (9), we immediately obtain that the mapping C satisfies (2).

Now, let $D : X \rightarrow Z$ be another cubic mapping satisfying (5). Then we have

$$\begin{aligned} \|C(x) - D(x), z_2, \dots, z_n\| &= 8^{-n} \|C(2^n x) - D(2^n x), z_2, \dots, z_n\| \\ &\leq 8^{-n} (\|C(2^n x) - f(2^n x), z_2, \dots, z_n\| + \|f(2^n x) - D(2^n x), z_2, \dots, z_n\|) \\ &\leq \frac{1}{8} \sum_{i=0}^{\infty} \frac{\varphi(2^{i+n} x, 0, x_2, \dots, x_n)}{8^{n+i}} \end{aligned}$$

which tends to zero as $k \rightarrow \infty$ for all $x, z_2, \dots, z_n \in X$. By Lemma 1, we conclude that $C(x) = D(x)$ for all $x \in X$. This completes the proof of the theorem. $\square$

As an application of Theorem 1, we obtain a stability of Equation (2) in the following corollary.

Corollary 1. Assume that $(X, \|\cdot\|)$ is a real normed space and that $(Z, \|\cdot, \dots, \cdot\|)$ is a linear n -normed space. Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ and $p, q < 3$ and let $f : X \rightarrow Z$ be a surjective mapping satisfying

$$\|D_f(x, y), z_2, \dots, z_n\| \leq \theta(\|x\|^p + \|y\|^q)\|x_2\|^r \cdots \|x_n\|^r$$

for all $x, y, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. Then there is a unique cubic mapping $C : X \rightarrow Z$ such that

$$\|f(x) - C(x), z_2, \dots, z_n\| \leq \frac{\theta}{16 - 2^{p+1}} \|x\|^p \|x_2\|^r \cdots \|x_n\|^r$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$.

Proof. The assertion follows from Theorem 1 by setting

$$\varphi(x, y, x_2, \dots, x_n) = \theta(\|x\|^p + \|y\|^q)\|x_2\|^r \cdots \|x_n\|^r$$

for all $x, y, x_2, \dots, x_n \in X$. $\square$

In the next theorem we also deal with a solution of the cubic functional equation in n -Banach spaces under different conditions. Next, we investigate the change of conditions for the pseudo-cubic function f and the error function φ , and also obtain a stability of Equation (2) in the following theorem (compare with Theorem 1).

Theorem 2. Let $\varphi : X^{n+1} \rightarrow \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty} 8^i \varphi\left(\frac{x}{2^{i+1}}, 0, x_2, \dots, x_n\right) < \infty, \quad \lim_{n \rightarrow \infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, x_2, \dots, x_n\right) = 0$$

for all $x, y, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. Suppose that a function $f : X \rightarrow Z$ be a surjective mapping satisfying (4). Then there is a unique cubic mapping $C : X \rightarrow Z$ such that

$$\|f(x) - C(x), z_2, \dots, z_n\| \leq \frac{1}{2} \sum_{i=0}^{\infty} 8^i \varphi\left(\frac{x}{2^{i+1}}, 0, x_2, \dots, x_n\right) \quad (10)$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$.

Proof. Let $z_i = f(x_i)$ for each $i = 2, \dots, n$. Take $x = \frac{x}{2}$ in (6) and multiply by eight to have

$$\|f(x) - 8f\left(\frac{x}{2}\right), z_2, \dots, z_n\| \leq \frac{1}{2} \varphi\left(\frac{x}{2}, 0, x_2, \dots, x_n\right) \quad (11)$$

for all $x, x_2, \dots, x_n \in X$. By replacing x with $\frac{x}{2}$ in (11) and multiplying by eight, we get

$$\|f(x) - 8^2 f\left(\frac{x}{2^2}\right), z_2, \dots, z_n\| \leq \frac{1}{2} \varphi\left(\frac{x}{2}, 0, x_2, \dots, x_n\right) + 4\varphi\left(\frac{x}{2^2}, 0, x_2, \dots, x_n\right)$$

for all $x, x_2, \dots, x_n \in X$. Then we can find a unique cubic mapping $C : X \rightarrow Z$ defined by

$$C(x) := \lim_{n \rightarrow \infty} 8^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$, as in the proof of Theorem 4. This completes the proof. $\square$

Using the above theorem, we immediately get the following corollary.

Corollary 2. Assume that $(X, \|\cdot\|)$ is a real normed space and that $(Z, \|\cdot, \dots, \cdot\|)$ is a linear n -normed space. Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$, and $p, q > 3$. Let $f : X \rightarrow Z$ be a surjective mapping satisfying

$$\|D_f(x, y), z_2, \dots, z_n\| \leq \theta(\|x\|^p + \|y\|^q)\|x_2\|^r \cdots \|x_n\|^r$$

for all $x, y, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. Then there is a unique cubic mapping $C : X \rightarrow Z$ such that

$$\|f(x) - C(x), z_2, \dots, z_n\| \leq \frac{\theta}{2^{p+1} - 16} \|x\|^p \|x_2\|^r \cdots \|x_n\|^r$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$.

Proof. The proof follows from Theorem 2 with

$$\varphi(x, y, x_2, \dots, x_n) = \theta(\|x\|^p + \|y\|^q)\|x_2\|^r \cdots \|x_n\|^r$$

for all $x, y, x_2, \dots, x_n \in X$. $\square$

3.2. Stability of the Quartic Functional Equation

In this subsection, we discuss the generalized Hyers–Ulam stability of the quartic functional Equation (3) in n -Banach spaces. For $x, y \in X$, we define $E_f(x, y)$ given by

$$E_f(x, y) := f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) + 6f(y).$$

The difference $E_f(x, y)$ means an approximate remainder of the functional Equation (3).

Now we provide important consequences for the stability of the quartic functional equation.

Theorem 3. Let $\varphi : X^{n+1} \rightarrow \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty} 2^{-4k} \varphi(2^i x, 0, x_2, \dots, x_n) < \infty \text{ and } \lim_{n \rightarrow \infty} 2^{-4n} \varphi(2^n x, 2^n y, x_2, \dots, x_n) = 0,$$

for all $x, y, x_2, \dots, x_n \in X$. Suppose that a function $f : X \rightarrow Z$ be a surjective mapping satisfying

$$\|E_f(x, y), z_2, \dots, z_n\| \leq \delta + \varphi(x, y, x_2, \dots, x_n) \quad (12)$$

for all $x, y, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$, where $\delta \geq 0$. Then there is a unique quartic mapping $T : X \rightarrow Z$ such that

$$\|f(x) - T(x), z_2, \dots, z_n\| \leq \frac{1}{30} \delta + \frac{1}{32} \sum_{i=0}^{\infty} 2^{-4k} \varphi(2^i x, 0, x_2, \dots, x_n) + \frac{1}{5} \|f(0), z_2, \dots, z_n\| \quad (13)$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$.

From now on we call the function f the pseudo-quartic function for φ , and the solution function T is the quartic function induced from the pseudo-quartic function f .

Proof. Let $z_i = f(x_i)$ for each $i = 2, \dots, n$. By letting $y = 0$ in (12), we get

$$\|2^{-4} f(2x) - f(x) + \frac{3}{16} f(0), z_2, \dots, z_n\| \leq \frac{1}{32} \delta + \frac{1}{32} \varphi(x, 0, x_2, \dots, x_n), \quad (14)$$

for all $x, x_2, \dots, x_n \in X$. If we replace x by $2^{n-1}x$ in (14) and divide both sides of (14) by 2^{4n-4} , we have that

$$\|2^{-4n}f(2^n x) - 2^{4-4n}f(2^{n-1}x) + 3 \cdot 2^{-4n}f(0), z_2, \dots, z_n\| \leq \frac{\delta}{2^{4n+1}} + \frac{1}{2^{4n+1}}\varphi(2^{n-1}x, 0, x_2, \dots, x_n)$$

for all $x, x_2, \dots, x_n \in X$ and integers $n \geq 1$. Therefore, for all integers $0 \leq m < n$, we obtain

$$\begin{aligned} & \left\| \sum_{k=m+1}^n [2^{-4k}f(2^k x) - 2^{4-4k}f(2^{k-1}x) + 3 \cdot 2^{-4k}f(0)], z_2, \dots, z_n \right\| \\ & \leq \sum_{k=m+1}^n \left\| [2^{-4k}f(2^k x) - 2^{4-4k}f(2^{k-1}x) + 3 \cdot 2^{-4k}f(0)], z_2, \dots, z_n \right\| \\ & \leq \delta \sum_{k=m+1}^n 2^{-4k-1} + \sum_{k=m+1}^n 2^{-4k-1}\varphi(2^{k-1}x, 0, x_2, \dots, x_n) \end{aligned}$$

and so

$$\begin{aligned} & \|2^{-4n}f(2^n x) - 2^{-4m}f(2^m x), z_2, \dots, z_n\| \\ & \leq 3 \cdot \|f(0), z_2, \dots, z_n\| \sum_{k=m+1}^n 2^{-4k} + \delta \sum_{k=m+1}^n 2^{-4k-1} + \sum_{k=m+1}^n 2^{-4k-1}\varphi(2^{k-1}x, 0, x_2, \dots, x_n) \end{aligned} \quad (15)$$

for all $x, x_2, \dots, x_n \in X$. In a similar way as in the proof of Theorem 1, we can show that the sequence $\{2^{-4n}f(2^n x)\}$ is an n -Cauchy sequence in Z for all $x \in X$. Define a mapping $T : X \rightarrow Z$ by

$$T(x) := \lim_{n \rightarrow \infty} 2^{-4n}f(2^n x)$$

for all $x \in X$. By (15), we have the inequality (13). It follows from (12) that

$$\begin{aligned} & \|T(2x + y) + T(2x - y) - 4T(x + y) - 4T(x - y) - 24T(x) + 6T(y), z_2, \dots, z_n\| \\ & = \lim_{n \rightarrow \infty} 2^{-4n} \|f(2^n(2x + y)) + f(2^n(2x - y)) - 4f(2^n(x + y)) - 4f(2^n(x - y)) \\ & \quad - 24f(2^n x) + 6f(2^n y), z_2, \dots, z_n\| \\ & \leq \lim_{n \rightarrow \infty} 2^{-4n} \varphi(2^n x, 2^n y, x_2, \dots, x_n) = 0 \end{aligned}$$

for all $x, y, x_2, \dots, x_n \in X$. This implies that $T : X \rightarrow Z$ is a quartic mapping. Let $Q : X \rightarrow Z$ be another quartic mapping satisfying (13). Therefore we have

$$\begin{aligned} \|T(x) - Q(x), z_2, \dots, z_n\| &= \lim_{n \rightarrow \infty} 2^{-4n} \|f(2^n x) - Q(2^n x), z_2, \dots, z_n\| \\ &\leq \lim_{n \rightarrow \infty} 2^{-4n} \left(\frac{1}{30}\delta + \frac{1}{32}\tilde{\varphi}(2^n x) + \frac{1}{5}\|f(0), z_2, \dots, z_n\| \right) \\ &= \frac{1}{32} \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} 2^{-4k} \varphi(2^k x, 0, x_2, \dots, x_n) = 0 \end{aligned}$$

for all $x, y, x_2, \dots, x_n \in X$. It follows from Lemma 1 that $T(x) = Q(x)$ for all $x \in X$. This proves the uniqueness of T . $\square$

Now we show a simple application of Theorem 3 to obtain a stability of Equation (3).

Corollary 3. Assume that $(X, \|\cdot\|)$ is a real normed space and that $(Z, \|\cdot, \dots, \cdot\|)$ is a linear n -normed space. Let $\varepsilon, \delta, \theta, \in [0, \infty)$, $p, q, r \in (0, \infty)$ and $p, q < 4$. Let $f : X \rightarrow Z$ be a surjective mapping satisfying

$$\|E_f(x, y), z_2, \dots, z_n\| \leq \delta + (\varepsilon\|x\|^p + \theta\|y\|^q)\|x_2\|^r \cdots \|x_n\|^r$$

for all $x, y, z_2, \dots, z_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. Then there is a unique quartic mapping $T : X \rightarrow Z$ such that

$$\|f(x) - T(x), z_2, \dots, z_n\| \leq \frac{\delta}{24} + \frac{\varepsilon}{32 - 2^{p+1}} \|x\|^p \|x_2\|^r \cdots \|x_n\|^r$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$.

Proof. It is a consequence of Theorem 3 with

$$\varphi(x, y, x_2, \dots, x_n) = (\varepsilon \|x\|^p + \theta \|y\|^q) \|x_2\|^r \cdots \|x_n\|^r$$

for all $x, y, x_2, \dots, x_n \in X$. $\square$

Next we consider the changes of conditions of the pseudo-quartic function and the error function, and we find a solution of the quartic functional equation in n -Banach spaces. We prove the existence of a solution of Equation (3) in n -Banach spaces.

Theorem 4. Let $\varphi : X^{n+1} \rightarrow \mathbb{R}^+$ be a function such that

$$\sum_{i=0}^{\infty} 2^{4i} \varphi\left(\frac{x}{2^{i+1}}, 0, x_2, \dots, x_n\right) < \infty, \quad \lim_{n \rightarrow \infty} 2^{4n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, x_2, \dots, x_n\right) = 0, \quad (16)$$

for all $x, y, x_2, \dots, x_n \in X$. Suppose that a function $f : X \rightarrow Z$ be a surjective mapping satisfying

$$\|E_f(x, y), z_2, \dots, z_n\| \leq \varphi(x, y, x_2, \dots, x_n) \quad (17)$$

for all $x, y, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. Then there is a unique quartic mapping $T : X \rightarrow Z$ such that

$$\|f(x) - T(x), z_2, \dots, z_n\| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{4i} \varphi\left(\frac{x}{2^{i+1}}, 0, x_2, \dots, x_n\right) \quad (18)$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$.

Proof. It follows from (16) that $\varphi(0, 0) = 0$. Thus, we have $f(0) = 0$ from (17). By letting $y = 0$ in (17), we get

$$\|2^{-4}f(2x) - f(x), z_2, \dots, z_n\| \leq \frac{1}{32} \varphi(x, 0, x_2, \dots, x_n) \quad (19)$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. Through replacing x by $\frac{x}{2}$ in (19) and multiplying by 2^4 , we obtain

$$\|f(x) - 2^4 f\left(\frac{x}{2}\right), z_2, \dots, z_n\| \leq \frac{1}{2} \varphi\left(\frac{x}{2}, 0, x_2, \dots, x_n\right) \quad (20)$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. By (20), we have

$$\|f(x) - 2^8 f\left(\frac{x}{2^2}\right), z_2, \dots, z_n\| \leq \frac{1}{2} \varphi\left(\frac{x}{2}, 0, x_2, \dots, x_n\right) + 2^3 \varphi\left(\frac{x}{2^2}, 0, x_2, \dots, x_n\right)$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. As in the proof of Theorem 3, we can find a unique quartic mapping $T : X \rightarrow Z$ defined by

$$T(x) := \lim_{n \rightarrow \infty} 2^{4n} f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. This completes the proof. $\square$

As an application of Theorem 1, we obtain a stability of Equation (2) in the following corollary.

Corollary 4. Assume that $(X, \|\cdot\|)$ is a real normed space and that $(Z, \|\cdot, \dots, \cdot\|)$ is a linear n -normed space. Let $\epsilon, \theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ and $p, q > 4$. Let $f : X \rightarrow Z$ be a surjective mapping satisfying

$$\|E_f(x, y), z_2, \dots, z_n\| \leq (\epsilon \|x\|^p + \theta \|y\|^q) \|x_2\|^r \cdots \|x_n\|^r$$

for all $x, y, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$. Then there is a unique quartic mapping $T : X \rightarrow Z$ such that

$$\|f(x) - T(x), z_2, \dots, z_n\| \leq \frac{\epsilon}{2^{p+1} - 32} \|x\|^p \|x_2\|^r \cdots \|x_n\|^r$$

for all $x, x_2, \dots, x_n \in X$, where $z_i = f(x_i)$ for each $i = 2, \dots, n$.

Proof. It is a direct consequence of Theorem 4 with

$$\varphi(x, y, x_2, \dots, x_n) = (\epsilon \|x\|^p + \theta \|y\|^q) \|x_2\|^r \cdots \|x_n\|^r$$

for all $x, y, x_2, \dots, x_n \in X$. $\square$

4. Conclusions

In this paper, we considered the cubic functional equation and quartic functional equation in n -Banach spaces. We dealt with stabilities of the functional equations in n -Banach spaces. For a surjective mapping f into an n -Banach space, called a pseudo-cubic function or a pseudo-quartic function, we solved the stability problem for the cubic functional equations and the quartic functional equations for f , as we demonstrated the existence of the solutions of the functional equations. As applications, we got the solutions of the generalized Hyers–Ulam stabilities under the changes of conditions of the pseudo-functions and the error functions. Our results about the equations in n -Banach spaces are a new approach and are key extensions for the study of functional equations, where the novelty of our results lies.

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