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A Strong Convergence Theorem for Split Null Point Problem and Generalized Mixed Equilibrium Problem in Real Hilbert Spaces

Olawale Kazeem Oyewole [1,2] and Oluwatosin Temitope Mewomo [1,*]

1 School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban 4001, South Africa; 217079141@stu.ukzn.ac.za
2 DST-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), Johannesburg 2001, South Africa
* mewomoo@ukzn.ac.za

Abstract: In this paper, we study a schematic approximation of solutions of a split null point problem for a finite family of maximal monotone operators in real Hilbert spaces. We propose an iterative algorithm that does not depend on the operator norm which solves the split null point problem and also solves a generalized mixed equilibrium problem. We prove a strong convergence of the proposed algorithm to a common solution of the two problems. We display some numerical examples to illustrate our method. Our result improves some existing results in the literature.

Keywords: split feasibility problem; null point problem; generalized mixed equilibrium problem; monotone mapping; strong convergence; Hilbert space

MSC: 47H06; 47H09; 47H10; 46N10; 47J25

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H. The Equilibrium Problem (EP) in the sense of Blum and Oettli [1] is to find a point $x \in C$, such that

$$F(x,y) \geq 0,\ y \in C, \tag{1}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction. The EP unify many important problems, such as variational inequalities, fixed point problems, optimization problems, saddle point (minmax) problems, Nash equilibria problems and complimentarity problems [2–7]. It also finds applications in other fields of studies like physics, economics, engineering and so on [1,2,8–10]. The Generalized Mixed Equilibrium Problem (GMEP) (see e.g., [11]) is to find $x \in C$, such that

$$F(x,y) + \langle g(x), y - x\rangle + \phi(y) - \phi(x) \geq 0,\ \forall\, y\ \in C, \tag{2}$$

where $g : C \to H$ is a nonlinear mapping and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function. The solution set of (2) will be denoted $GMEP(F,g,\phi)$.

The GMEP includes as special cases, minimization problem, variational inequality problem, fixed point problem, nash equilibrium etc. GMEP (2) and these special cases have been studied in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces , see [11–21].

For a real Hilbert space H, the Variational Inclusion Problem (VIP) consists of finding a point $x^* \in H$ such that

$$0 \in Ax^*, \tag{3}$$

where $A : H \to 2^H$ is a multivalued operator. If A is a maximal monotone operator, then the VIP reduces to the Monotone Inclusion Problem (MIP). The MIP provides a general framework for the study of many important optimization problems, such as convex programming, variationa inequalities and so on.

For solving Problem (3), Martinet [22] introduced the Proximal Point Algorithm (PPA), which is given as follows: $x_0 \in H$ and

$$x_{n+1} = J^A_{r_n} x_n, \tag{4}$$

where $\{r_n\} \subset (0, +\infty)$ and $J^A_{r_n} = (I + r_n A)^{-1}$ is the resolvent of the maximal monotone operator A corresponding to the control sequence $\{r_n\}$. Several iterative algorithms have been proposed by authors in the literature for solving Problem (3) and related optimization problems, see [23–37].

Censor and Elfving [38] introduced the notion of Split Feasibility Problem (SFP). The SFP consists of finding a point

$$x^* \in C \quad \text{such that} \quad Lx^* \in Q, \tag{5}$$

where C and Q are nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and L is an $m \times n$ matrix. The SFP has been studied by researchers due to its applications in various field of science and technology, such as signal processing, intensity-modulated radiation therapy and medical image construction, for details, see [39,40]. In solving (5), Byrne [39] introduced the following iterative algorithm: let $x_0 \in \mathbb{R}^n$ be arbitrary,

$$x_{n+1} = P_C(x_n - \gamma L^*(I - P_Q)Lx_n), \tag{6}$$

where $\gamma \in (0, 2/||L||^2)$, L^* is the transpose of the matrix L, P_C and P_Q are nearest point mappings onto C and Q respectively. Lopez et al. [41] suggested the use of a stepsize γ_n in place of γ in Algorithm (6), where the stepsize does not depend on operator L. The stepsize γ_n is given as:

$$\gamma_n := \frac{\theta_n ||(I - P_Q)Lx_n||^2}{2||L^*(I - P_Q)Lx_n||^2}, \tag{7}$$

where $\theta_n \in (0, 4)$ and $L^*(I - P_Q)Lx_n \neq 0$. They proved a weak convergence theorem of the proposed algorithm. The authors in [41] noted that for L with higher dimensions, it may be hard to compute the operator norm and this may have effect on the iteration process. Instances of this effect can be observed in the CPU time. The algorithm with stepsizes improves the performance of the Byrne algorithm.

The Split Null Point Problem (SNPP) was introduced in 2012 by Byrne et al. [42]. These authors combined the concepts of VIP and SFP and defined SNPP as follows: Find $x^* \in H_1$ such that

$$0 \in A_1(x^*) \quad \text{and} \quad Lx^* \in H_2 \ \text{ such that } \quad 0 \in A_2(Lx^*), \tag{8}$$

where $A_i : H_i \to 2^{H_i}$, $i = 1, 2$ are maximal monotone operators, H_1 and H_2 are real Hilbert spaces. For solving (8), Byrne et al. [42] proposed the following iterative algorithm: For $r > 0$ and an arbitrary $x_0 \in H_1$,

$$x_{n+1} = J^{A_1}_r (x_n - \gamma L^*(I - J^{A_2}_r)Lx_n), \tag{9}$$

where $\gamma \in (0, 2/||L||^2)$. They prove a weak convergence of (9) to a solution of (8).

One of our aim in this work is to consider a generalization of Problem (3) in the following form: Find $x^* \in H$ such that

$$0 \in \bigcap_{i=1}^{N} A_i(x^*), \tag{10}$$

where A_i is a finite family of maximal monotone operators. There have been some iterative algorithms for approximating the solution of (10) in the literature, (see [37] and the references therein).

In this study, we consider the problem of finding the common solution of the GMEP (2) and the SNPP for a finite family of intersection of maximal monotone operator in the frame work of real Hilbert spaces. We consider the following generalization of the SNPP: Find $x^* \in C$ such that $x^* \in GMEP(F, g, \phi)$ and

$$x^* \in \bigcap_{i=1}^{N} A_i^{-1}(0) \quad \text{such that} \quad Lx^* \bigcap_{i=1}^{N} B_i^{-1}(0). \tag{11}$$

In our quest to obtain a common element in the solution set of problems (2) and (11), the following two research questions arise.

(1) Can we obtain an iterative algorithm which solves problem (11), without depending on the operator norm?
(2) Can we obtain a strong convergence theorem for the proposed algorithm to the solution of problem (11) ?

In this work, we give an affirmative answer to the questions above by introducing an iterative algorithm which solves (11). Further, we prove a strong convergence theorem of the proposed algorithm to the common solution of problem given by (11).

2. Preliminaries

In this section, we give some important definitions and Lemmas which are useful in establishing our main results.

From now, we denote by H a real Hilbert space, C a nonempty closed convex subset of H with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ respectively. We denote by $x_n \rightharpoonup x$ and $x_n \to x$ respectively the weak and strong convergence of a sequence $\{x_n\} \subset H$ to a point $x \in H$.

The nearest point mapping $P_C : H \to C$ is defined by $P_C x := \{x \in C : ||x - y|| = d_C(x), \forall y \in H\}$, where $d_C : H \to \mathbb{R}$ is the distance function of C. The mapping P_C is known to satisfy the inequality

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \ \forall\ x \in H \quad \text{and} \quad y \in C, \tag{12}$$

see e.g., [9,10] for details.

A point $x \in C$ is said to be a fixed point of a mapping $T : H \to H$, if $x = Tx$. We denote by $F(T)$ the set of fixed point of T. A mapping $f : C \to C$ is said to be a contraction, if there exists a constant $c \in (0, 1)$, such that

$$||f(x) - f(y)|| \leq c||x - y||, \ \forall\ \ x, y \in C. \tag{13}$$

If $c = 1$, then f is called nonexpansive.

A mapping $T : H \to H$ is said to be firmly nonexpansive if, for all $x, y \in H$, the following holds

$$||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2, \tag{14}$$

where I is an identity mapping on H.

Lemma 1 ([43]). *Let $T : H \to H$ be a mapping. Then the following are equivalent:*

(i) *T is firmly nonexpansive,*
(ii) *$I - T$ is firmly nonexpansive,*
(iii) *$2T - I$ is nonexpansive,*
(iv) *$\langle x - y, Tx - Ty\rangle \geq ||Tx - Ty||^2$,*
(v) *$\langle (I - T)x - (I - T)y, Tx - Ty\rangle \geq 0$.*

A multivalued mapping $A : H \to 2^H$ is called monotone if for all $x, y \in H$, $u \in Ax$ and $v \in Ay$, we have

$$\langle x - y, u - v\rangle \geq 0. \tag{15}$$

A monotone mapping A is said to be maximal if its graph $G(A) := \{(x, u) \in H \times H : u \in Ax\}$ is not properly contained in the graph of any other monotone operator.

Let $A : H \to H$ be a single-valued mapping, then for a positive real number β, A is said to be β-inverse strongly monotone (β-ism), if

$$\langle x - y, Ax - Ay\rangle \geq \beta||Ax - Ay||^2, \quad \forall\ x, y \ \in H. \tag{16}$$

This class of monotone mapping have been widely studied in literature (see [44,45]) for more details. If A is a monotone operator, then we can define, for each $r > 0$, a nonexpansive single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A := (I + rA)^{-1}$ which is generally known as the resolvent of A, (see [46,47]). It is also known that $A^{-1}(0) = F(J_r^A)$, where $A^{-1}(0) = \{x \in H : 0 \in Ax\}$ and $F(J_r^A) = \{x \in H : J_r^A x = x\}$.

Lemma 2 ([6,48]). *Let H be a real Hilbert space. Then the following hold:*

(i) *$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle, \forall x, y \in H$,*
(ii) *$||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$, $x, y \in H$,*
(iii) *$||\lambda x + (1 - \lambda)y||^2 = \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)||x - y||^2, \forall x, y \in H$ and $\lambda \in [0, 1]$.*

The bifunction $F : C \times C \to \mathbb{R}$ will be assumed to admit the following restrictions:

(C1) $F(x, x) = 0$ for all $x \in C$;
(C2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(C3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(C4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 3 ([11]). *Let C be a nonempty closed convex subset of real Hilbert space H. Let F be a real valued bifunction on $C \times C$ admitting restrictions $C1 - C4$, $g : C \to H$ be a nonlinear mapping and let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower senicontinuous convex function. For any given $r > 0$ and $x \in H$, define a mapping $K_r^F : H \to C$ as*

$$K_r^F x = \{z \in C : F(z, y) + \langle g(z), y - z\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\ y \in \ C\}, \tag{17}$$

for all $x \in H$. Then the following conclusions hold:

(i) *for each $x \in H$, $K_r^F x \neq \emptyset$,*
(ii) *K_r^F is single valued,*
(iii) *K_r^F is firmly nonexpansive, i.e., for any $x, y \in H$*

$$||K_r^F x - K_r^F y||^2 \leq \langle K_r^F x - K_r^F y, x - y\rangle,$$

(iv) *$F(K_r^F(I - rg)) = GMEP(F, g, \phi)$,*
(v) *$GMEP(F, g, \phi)$ is closed and convex.*

Lemma 4 ([49,50]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-b_n)a_n + b_n c_n + d_n, \quad n \in \mathbb{N}, \tag{18}$$

where $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences of real numbers satisfying

(i) $\{b_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} b_n = \infty$;

(ii) $\limsup_{n\to\infty} c_n \leq 0$;

(iii) $d_n \geq 0$, $\sum_{n=0}^{\infty} d_n < \infty$.

Then, $\lim_{n=\infty} a_n = 0$.

3. Main Result

Throughout, we let $\Phi^{A_N}_{\lambda_{N,n}} = J^{A_N}_{\lambda_{N,n}} \circ J^{A_{N-1}}_{\lambda_{N-1,n}} \circ \cdots \circ J^{A_1}_{\lambda_{1,n}}$, where $\Phi^{A_0}_{\lambda_{0,n}} = I$. Define the stepsize γ_n by

$$\gamma_n = \begin{cases} \frac{\theta_n ||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2}{||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2}, & \text{if } L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n \neq 0, \\ \gamma, & \text{otherwise,} \end{cases} \tag{19}$$

where γ_n depends on $\theta_n \in [a,b] \subset (0,1)$ and γ is any nonnegative number.

Lemma 5. *Let H be a real Hilbert space and $A : H \to 2^H$ be a monotone mapping. Then for $0 < s \leq r$, we have*

$$||x - J^A_s x|| \leq 2||x - J^A_r x||.$$

Proof: Notice that $\frac{1}{s}(x - J^A_s) \in AJ^A_s x$ and $\frac{1}{r}(x - J^A_r x) \in AJ^A_r x$. Using the monotonicity of A, we have

$$\langle \frac{1}{s}(x - J^A_s x) - \frac{1}{r}(x - J^A_r x), J^A_s x - J^A_r x\rangle \geq 0.$$

That is

$$\langle x - J^A_s x - \frac{s}{r}(x - J^A_r x), J^A_s x - J^A_r x\rangle \geq 0,$$

which implies that

$$\langle x - J^A_s x, J^A_s x - J^A_r x\rangle \geq \frac{s}{r}\langle x - J^A_r x, J^A_s x - J^A_r x\rangle.$$

Using Lemma 2 (ii), we obtain

$$\frac{1}{2}(||x - J^A_r x||^2 - ||x - J^A_s x||^2 - ||J^A_s x - J^A_r x||^2) \geq \frac{s}{2r}(||x - J^A_r x||^2 + ||J^A_s x - J^A_r x||^2 - ||x - J^A_s x||^2),$$

that is

$$-\left(\frac{1}{2} + \frac{s}{2r}\right)||J^A_s x - J^A_r x||^2 \geq -\left(\frac{1}{2} - \frac{s}{2r}\right)||x - J^A_r x||^2 - \left(\frac{s}{2r} - \frac{1}{2}\right)||x - J^A_s x||^2$$

and

$$\left(\frac{r+s}{2r}\right)||J^A_s x - J^A_r x||^2 \leq \left(\frac{r-s}{2r}\right)||x - J^A_r x||^2 - \left(\frac{r-s}{2r}\right)||x - J^A_s x||^2.$$

Since $0 < s \leq r$, we obtain

$$||J_s^A x - J_r^A x||^2 \leq \left(\frac{r-s}{r+s}\right)||x - J_r^A x||^2,$$

which implies

$$||J_s^A x - J_r^A x|| \leq ||x - J_r^A x||. \tag{20}$$

Now, since $||x - J_s^A x|| \leq ||x - J_r^A x|| + ||J_r^A x - J_s^A x||$, by (20), we obtain

$$\begin{aligned} ||x - J_s^A x|| &\leq ||x - J_r^A x|| + ||x - J_r^A x|| \\ &= 2||x - J_r^A x||. \end{aligned}$$

Lemma 6. *Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $L : H_1 \to H_2$ be a bounded linear operator. Assume F is a real valued bifunction on $C \times C$ which admits condition C1-C4. Let $\phi : H_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous convex function, g be a β-inverse strongly monotone mapping and $f : H_1 \to R$ be a differentiable function, such that ∇f is a contraction with coefficient $c \in (0,1)$. For $i = 1, 2 \cdots, N$, let $A_i : H_1 \to 2^{H_1}$ and $B_i : H_2 \to 2^{H_2}$ be finite families of monotone mappings. Assume $\Omega = GMEP(F, g, \phi) \cap \Gamma \neq \emptyset$, where $\Gamma = \{x^* \in H_1 : 0 \in \bigcap_{i=1}^N A_i(x^*)$ and $Lx^* \in H_2 : 0 \in \bigcap_{i=1}^N B_i(Lx^*)\}$. For an arbitrary $x_0 \in H_1$, let $\{x_n\} \subset H_1$ be a sequence defined iteratively by*

$$\begin{cases} F(u_n, y) + \langle g(u_n), y - u_n\rangle + \phi(y) - \phi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \; y \in H_1, \\ z_n = u_n - \gamma_n L^*(I - \Phi_{\lambda_{i,n}}^{B_i})Lu_n, \\ x_{n+1} = \alpha_n \nabla f(z_n) + (1 - \alpha_n)\Phi_{\lambda_{i,n}}^{A_i} z_n, \end{cases} \tag{21}$$

where $\{r_n\}$ is a nonnegative sequence of real numbers, $\{\alpha_n\}$ and $\{\lambda_{i,n}\}$ are sequences in $(0,1)$, γ_n is a nonnegative sequence defined by (19), satisfying the following restrictions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < \lambda_i \leq \lambda_{i,n}$;
(iii) $0 < a \leq r_n \leq b < 2\beta$.

Then $\{x_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded.

Proof. Observe that u_n can be rewritten as $u_n = K_{r_n}^F(x_n - r_n g(x_n))$ for each n. Fix $p \in \Omega$. Since $p = K_{r_n}^F(p - r_n p)$, g is β-inverse strongly monotone and $r_n \in (0, 2\beta)$, for any $n \in \mathbb{R}$, we have from (21) and Lemma 2 (ii) that

$$\begin{aligned} ||u_n - p||^2 &= ||K_{r_n}^F(x_n - r_n g(x_n)) - K_{r_n}^F(p - r_n g(p))||^2 \\ &\leq ||x_n - r_n g(x_n) - (p - r_n g(p))||^2 \\ &= ||(x_n - p) - r_n(g(x_n) - g(p)||^2 \\ &= ||x_n - p||^2 - 2r_n\langle x_n - p, g(x_n) - g(p)\rangle + r_n^2||g(x_n) - g(p)||^2 \qquad (22) \\ &\leq ||x_n - p||^2 - 2\beta r_n||g(x_n) - g(p)||^2 + r_n^2||g(x_n) - g(p)||^2 \\ &= ||x_n - p||^2 - r_n(2\beta - r_n)||g(x_n) - g(p)||^2 \\ &\leq ||x_n - p||^2. \end{aligned}$$

Also by Lemma 2, we have

$$\begin{aligned}
||z_n - p||^2 &= ||u_n - \gamma_n L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n - p||^2 \\
&= ||u_n - p - \gamma_n L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&= ||u_n - p||^2 - 2\gamma_n \langle u_n - p, L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n\rangle + \gamma_n^2 ||L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&= ||u_n - p||^2 - 2\gamma_n \langle Lu_n - Lp, (I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n\rangle + \gamma_n^2 ||L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 \quad (23) \\
&\leq ||u_n - p||^2 - 2\gamma_n ||(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 + \gamma_n^2 ||L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&\leq ||u_n - p||^2 - \gamma_n ||(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 + \gamma_n^2 ||L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 \\
&= ||u_n - p||^2 - \gamma_n [||(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 - \gamma_n ||L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2].
\end{aligned}$$

Using the definition of γ_n, we obtain

$$||z_n - p||^2 \leq ||u_n - p||^2, \quad (24)$$

hence, $||z_n - p|| \leq ||u_n - p|| \leq ||x_n - p||$.

Further, we obtain that

$$\begin{aligned}
||x_{n+1} - p|| &= ||\alpha_n \nabla f(z_n) + (1 - \alpha_n)\Phi^{A_i}_{\lambda_{i,n}} z_n - p|| \\
&= ||\alpha_n (\nabla f(z_n) - p) + (1 - \alpha_n)(\Phi^{A_i}_{\lambda_{i,n}} z_n - p)|| \\
&\leq \alpha_n ||\nabla f(z_n) - p|| + (1 - \alpha_n)||\Phi^{A_i}_{\lambda_{i,n}} z_n - p|| \\
&\leq \alpha_n ||\nabla f(z_n) - \nabla f(p)|| + \alpha_n ||\nabla f(p) - p|| + (1 - \alpha_n)||z_n - p|| \quad (25) \\
&\leq \alpha_n c||z_n - p|| + \alpha_n ||\nabla f(p) - p|| + ||(1 - \alpha_n)||z_n - p|| \\
&= (1 - \alpha_n(1 - c))||z_n - p|| + \alpha_n ||\nabla f(p) - p|| \\
&\leq (1 - \alpha_n(1 - c))||x_n - p|| + \frac{\alpha_n(1 - c)}{1 - c}||\nabla f(p) - p||.
\end{aligned}$$

Let $K = \max\{||x_0 - p||, \frac{||\nabla f(p) - p||}{1 - c}\}$. We show that $||x_n - p|| \leq K$ for all $n \geq 0$. Indeed, we see that $||x_0 - p|| \leq K$. Now suppose $||x_j - p|| \leq K$ for some $j \in \mathbb{N}$. Then, we have that

$$\begin{aligned}
||x_{j+1} - p|| &\leq (1 - \alpha_j(1 - c))||x_j - p|| + \frac{\alpha_j(1 - c)||\nabla f(p) - p||}{1 - c} \\
&\leq (1 - \alpha_j(1 - c))K + \alpha_j(1 - c)K \quad (26) \\
&\leq K.
\end{aligned}$$

By induction, we obtain that $||x_n - p|| \leq K$ for all n. Therefore $\{x_n\}$ is bounded, consequently $\{z_n\}$ and $\{u_n\}$ are bounded. □

Theorem 1. *Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and $L : H_1 \to H_2$ be a bounded linear operator. Assume F is a real valued bifunction on $C \times C$ which admits condition C1-C4. Let $\phi : H_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous function, g be a β-inverse strongly monotone mapping and $f : H_1 \to \mathbb{R}$ be a differentiable function, such that ∇f is a contraction with coefficient $c \in (0, 1)$. For $i = 1, 2 \cdots, N$, let $A_i : H_1 \to 2^{H_1}$ and $B_i : H_2 \to 2^{H_2}$ be finite families of monotone mappings. Assume $\Omega = GMEP(F, g, \phi) \cap \Gamma \neq \emptyset$, where $\Gamma = \{p \in H_1 : 0 \in \bigcap_{i=1}^N A_i(p) \text{ and } Lp \in H_2 : 0 \in \bigcap_{i=1}^N B_i(Lp)\}$. For an arbitrary $x_0 \in H_1$, let $\{x_n\} \subset H_1$ be a sequence defined iteratively by (21) satisfying the conditions of Lemma 6. Then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega \nabla f(p)$.*

Proof. We observe from (21), that

$$
\begin{aligned}
||x_{n+1}-p||^2 &= \langle \alpha_n \nabla f(z_n) + (1-\alpha_n)(\Phi^{A_i}_{\lambda_{i,n}} z_n - p), x_{n+1}-p\rangle \\
&= \alpha_n \langle \nabla f(z_n), x_{n+1}-p\rangle + (1-\alpha_n)\langle \Phi^{A_i}_{\lambda_{i,n}} z_n - p, x_{n+1}-p\rangle \\
&= \alpha_n \langle \nabla f(z_n) - \nabla f(p), x_{n+1}-p\rangle + \alpha_n\langle \nabla f(p) - p, x_{n+1}-p\rangle + (1-\alpha_n)\langle \Phi^{A_i}_{\lambda_{i,n}} z_n - p, x_{n+1}-p\rangle \\
&\leq \alpha_n ||\nabla f(z_n) - \nabla f(p)|| \cdot ||x_{n+1}-p|| + (1-\alpha_n)||\Phi^{A_i}_{\lambda_{i,n}} z_n - p|| \cdot ||x_{n+1}-p|| \\
&\quad + \alpha_n \langle \nabla f(p) - p, x_{n+1}-p\rangle \\
&\leq \frac{\alpha_n}{2}\Big(||\nabla f(z_n) - \nabla f(p)||^2 + ||x_{n+1}-p||^2\Big) + \left(\frac{1-\alpha_n}{2}\right)\Big(||\Phi^{A_i}_{\lambda_{i,n}} z_n - p||^2 + ||x_{n+1}-p||^2\Big) \\
&\quad + \alpha_n \langle \nabla f(p) - p, x_{n+1}-p\rangle \\
&\leq \frac{\alpha_n c^2}{2}||z_n-p||^2 + \frac{\alpha_n}{2}||x_{n+1}-p||^2 + \frac{(1-\alpha_n)}{2}||z_n-p||^2 + \frac{(1-\alpha_n)}{2}||x_{n+1}-p||^2 \\
&\quad + \alpha_n \langle \nabla f(p) - p, x_{n+1}-p\rangle \\
&\leq \frac{[1-\alpha_n(1-c^2)]}{2}||z_n-p||^2 + \frac{1}{2}||x_{n+1}-p||^2 + \alpha_n\langle \nabla f(p) - p, x_{n+1}-p\rangle \\
&\leq \frac{[1-\alpha_n(1-c^2)]}{2}||u_n-p||^2 + \frac{1}{2}||x_{n+1}-p||^2 + \alpha_n\langle \nabla f(p) - p, x_{n+1}-p\rangle,
\end{aligned} \tag{27}
$$

that is

$$
\begin{aligned}
||x_{n+1}-p||^2 &\leq [1-\alpha_n(1-c^2)]||u_n-p||^2 + \alpha_n(1-c^2)\left(\frac{2}{(1-c^2)}\langle \nabla f(p) - p, x_{n+1}-p\rangle\right) \\
&\leq [1-\alpha_n(1-c^2)]||x_n-p||^2 + \alpha_n(1-c^2)\left(\frac{2}{(1-c^2)}\langle \nabla f(p) - p, x_{n+1}-p\rangle\right).
\end{aligned} \tag{28}
$$

From now the rest of the proof shall be divide into two cases.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - p||\}$ is not monotonically increasing. Then by Lemma 6, we have that $\{||x_n - p||\}$ is convergent. From (21), we have by Lemma 2 that

$$
\begin{aligned}
||x_{n+1}-p||^2 &= ||\alpha_n \nabla f(z_n) + (1-\alpha_n)\Phi^{A_i}_{\lambda_{i,n}} z_n - p||^2 \\
&= ||\alpha_n \nabla (f(z_n) - p) + (1-\alpha_n)(\Phi^{A_i}_{\lambda_{i,n}} z_n - p)||^2 \\
&= \alpha_n||\nabla f(z_n) - p||^2 + (1-\alpha_n)||\Phi^{A_i}_{\lambda_{i,n}} z_n - p||^2 - \alpha_n(1-\alpha_n)||\nabla f(z_n) - \Phi^{A_i}_{\lambda_{i,n}} z_n||^2 \\
&\leq \alpha_n||\nabla f(z_n) - p||^2 + (1-\alpha_n)||z_n-p||^2.
\end{aligned} \tag{29}
$$

Thus,

$$
||z_n-p||^2 \geq ||x_{n+1}-p||^2 - \alpha_n(||\nabla f(z_n) - p||^2 - ||z_n-p||^2). \tag{30}
$$

From (23), we have that

$$
\begin{aligned}
\gamma_n\big[||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 - \gamma_n||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2\big] \\
&\leq ||u_n-p||^2 - ||z_n-p||^2 \\
&\leq ||u_n-p||^2 - ||x_{n+1}-p||^2 + \alpha_n(||\nabla f(z_n) - p||^2 - ||z_n-p||^2) \\
&\leq ||x_n-p||^2 - ||x_{n+1}-p||^2 + \alpha_n(||\nabla f(z_n) - p||^2 - ||z_n-p||^2),
\end{aligned}
$$

by using restriction (i) in Lemma 6, we have

$$
\lim_{n\to\infty} \gamma_n\big[||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2 - \gamma_n||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2\big] = 0. \tag{31}
$$

Using (19), we have that

$$\left[||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2-\gamma_n||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2\right]=\theta_n(1-\theta_n)\frac{||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^4}{||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2}, \tag{32}$$

thus by (31), we obtain

$$\theta_n(1-\theta_n)\frac{||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^4}{||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2}\to 0,\ \text{ as }\ n\to\infty.$$

Therefore, since $\theta_n\in(0,1)$, we obtain

$$\lim_{n\to\infty}\frac{||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||^2}{||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||}=0. \tag{33}$$

Notice that $||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||\leq||L^*||\cdot||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||$, which implies

$$||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||\leq\frac{||L^*||\cdot||(I-\Phi_{\lambda_{i,n}})Lu_n||^2}{||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||},$$

by (33), we obtain

$$\lim_{n\to\infty}||(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||=0, \tag{34}$$

consequently,

$$\lim_{n\to\infty}||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||=0. \tag{35}$$

From (21), we see that

$$\begin{aligned}||z_n-u_n|| &= ||u_n-\gamma_nL^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n-u_n||\\ &\leq \gamma_n||L^*(I-\Phi^{B_i}_{\lambda_{i,n}})Lu_n||.\end{aligned}$$

By (35), we get that

$$\lim_{n\to\infty}||z_n-u_n||=0. \tag{36}$$

Furthermore, we have from (21),

$$\begin{aligned}||x_{n+1}-z_n|| &= ||\alpha_n\nabla f(z_n)+(1-\alpha_n)\Phi^{A_i}_{\lambda_{i,n}}z_n-z_n||\\ &= ||\alpha_n(\nabla f(z_n)-z_n)+(1-\alpha_n)(\Phi^{A_i}_{\lambda_{i,n}}z_n-z_n)||\\ &\leq \alpha_n||\nabla f(z_n)-z_n||+(1-\alpha_n)||\Phi^{A_i}_{\lambda_{i,n}}z_n-z_n||\\ &\leq \alpha_n||\nabla f(z_n)-\nabla f(p)||+\alpha_n||\nabla f(p)-z_n||+(1-\alpha_n)||z_n-\Phi^{A_i}_{\lambda_{i,n}}z_n||\\ &\leq \alpha_nc||z_n-p||+\alpha_n||\nabla f(p)-z_n||+(1-\alpha_n)||z_n-\Phi^{A_i}_{\lambda_{i,n}}z_n||.\end{aligned} \tag{37}$$

Observe from (21), that

$$||\Phi^{A_i}_{\lambda_{i,n}}z_n-p||\geq||x_{n+1}-p||-\alpha_n||\nabla f(z_n)-\Phi^{A_i}_{\lambda_{i,n}}z_n||,$$

using the nonexpansivity of $\Phi^{A_i}_{\lambda_{i,n}}$, we obtain that

$$\begin{aligned} 0 \leq\ & ||z_n - p|| - ||\Phi^{A_i}_{\lambda_{i,n}} z_n - \Phi^{A_i}_{\lambda_{i,n}} p|| \\ =\ & ||z_n - p|| - ||\Phi^{A_i}_{\lambda_{i,n}} z_n - p|| \\ \leq\ & ||x_n - p|| - ||x_{n+1} - p|| + \alpha_n ||\nabla f(z_n) - \Phi^{A_i}_{\lambda_{i,n}} z_n||. \end{aligned}$$

Using restriction (i) in Lemma 6, the boundedness of $\{z_n\}$ and the convergence of $\{||x_n - p||\}$, we have that $||z_n - p|| - ||\Phi^{A_i}_{\lambda_{i,n}} z_n - p|| \to 0$ as $n \to \infty$. Thus by the strong nonexpansivity of $\Phi^{A_i}_{\lambda_{i,n}}$, we get that

$$\lim_{n\to\infty} ||z_n - \Phi^{A_i}_{\lambda_{i,n}} z_n|| = 0.$$

Using this and restriction (i) of Lemma 6 in (38), we get

$$\lim_{n\to\infty} ||x_{n+1} - z_n|| = 0. \tag{38}$$

Observe from (28), that

$$-||u_n - p||^2 \leq -||x_{n+1} - p||^2 - \alpha_n(1 - c^2)||u_n - p||^2 + 2\alpha_n \langle \nabla f(p) - p, x_{n+1} - p\rangle, \tag{39}$$

since $||u_n - x_n||^2 \leq ||x_n - p||^2 - ||u_n - p||^2$, using (39), we have that

$$||u_n - x_n||^2 \leq ||x_n - p||^2 - ||x_{n+1} - p||^2 - \alpha_n(1 - c^2)||u_n - p||^2 + 2\alpha_n \langle \nabla f(p) - p, x_{n+1} - p\rangle,$$

thus, by restriction (i) in Lemma 6, we obtain

$$\lim_{n\to\infty} ||u_n - x_n|| = ||K^F_{r_n} x_n - x_n|| = 0. \tag{40}$$

Combining (36) and (40), we obtain

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \tag{41}$$

Moreover, since

$$||x_{n+1} - x_n|| \quad \leq \quad ||x_{n+1} - z_n|| + ||z_n - x_n||,$$

we have that

$$||x_{n+1} - x_n|| \to 0 \text{ as } n \to \infty. \tag{42}$$

Furthermore,

$$\begin{aligned} ||x_n - \Phi^{A_i}_{\lambda_{i,n}} x_n|| \quad &\leq \quad ||x_n - x_{n+1}|| + ||x_{n+1} - \Phi^{A_i}_{\lambda_{i,n}} z_n|| + ||\Phi^{A_i}_{\lambda_{i,n}} - \Phi^{A_i}_{\lambda_{i,n}} x_n|| \\ &\leq \quad ||x_n - x_{n+1}|| + ||x_{n+1} - \Phi^{A_i}_{\lambda_{i,n}} z_n|| + ||z_n - x_n||, \end{aligned} \tag{43}$$

but

$$\begin{aligned} ||x_{n+1} - \Phi^{A_i}_{\lambda_{i,n}} z_n|| \quad &= \quad ||\alpha_n \nabla f(z_n) + (1 - \alpha_n)\Phi^{A_i}_{\lambda_{i,n}} z_n - \Phi^{A_i}_{\lambda_{i,n}} z_n|| \\ &= \quad \alpha_n ||\nabla f(z_n) - \Phi^{A_i}_{\lambda_{i,n}} z_n|| \to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence, by substituting this, (41) and (42) into (43), we obtain

$$\lim_{n\to\infty} ||(I - \Phi^{A_i}_{\lambda_{i,n}})x_n|| = 0.$$

Since $0 < \lambda_i \leq \lambda_{i,n}$, we have by Lemma 5, that

$$\lim_{n\to\infty} ||(I - \Phi^{A_i}_{\lambda_i})x_n|| = 0. \quad (44)$$

Now, since $\{x_n\}$ is bounded in H_1, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^* \in H_1$. First, we show that $x^* \in \cap_{i=1}^N A_i^{-1}(0)$. Consider for each $j \in \mathbb{N}$,

$$\begin{aligned} ||(I - \Phi^{A_i}_{\lambda_i})x^*||^2 &\leq \langle (I - \Phi^{A_i}_{\lambda_i})x^*, x^* - x_{n_j}\rangle + \langle (I - \Phi^{A_i}_{\lambda_i})x^*, x_{n_j} - \Phi^{A_i}_{\lambda_i}x_{n_j}\rangle \\ &\quad + \langle (I - \Phi^{A_i}_{\lambda_i})x^*, \Phi^{A_i}_{\lambda_i}x_{n_j} - \Phi^{A_i}_{\lambda_i}x^*\rangle. \end{aligned} \quad (45)$$

Since $\{x_{n_j}\} \subset \{x_n\}$, as a consequence of (44), we have

$$\lim_{j\to\infty} ||x_{n_j} - \Phi^{A_i}_{\lambda_i}x_{n_j}|| = 0. \quad (46)$$

Therefore, using $x_{n_j} \rightharpoonup x^*$ and (46) in (45), we have

$$\lim_{n\to\infty} ||(I - \Phi^{A_i}_{\lambda_i})x^*|| = 0. \quad (47)$$

Thus, $x^* = \Phi^{A_i}_{\lambda_i}x^*$ and hence $x^* \in \cap_{i=1}^N A_i^{-1}(0)$.

Secondly, we show that $Lx^* \in \cap_{i=1}^N B_i^{-1}(0)$. Consider again for each $j \in \mathbb{N}$,

$$\begin{aligned} ||(I - \Phi^{B_i}_{\lambda_i})Lx^*||^2 &\leq \langle (I - \Phi^{B_i}_{\lambda_i})Lx^*, Lx^* - Lz_{n_j}\rangle + \langle (I - \Phi^{B_i}_{\lambda_i})Lx^*, Lz_{n_j} - \Phi^{B_i}_{\lambda_i}Lz_{n_j}\rangle \\ &\quad + \langle (I - \Phi^{B_i}_{\lambda_i})Lx^*, \Phi^{B_i}_{\lambda_i}Lz_{n_j} - \Phi^{B_i}_{\lambda_i}Lx^*\rangle, \end{aligned} \quad (48)$$

observe that,

$$\begin{aligned} ||(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n|| &\leq ||(I - \Phi^{B_i}_{\lambda_{i,n}})Lz_n - (I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n|| + ||(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n|| \\ &\leq ||Lz_n - Lu_n|| + ||\Phi^{B_i}_{\lambda_{i,n}}Lz_n - \Phi^{B_i}_{\lambda_{i,n}}Lu_n|| + ||(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n|| \\ &\leq 2||L||||z_n - u_n|| + ||(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n||, \end{aligned}$$

which by (34) and (36), implies

$$\lim_{n\to\infty} ||(I - \Phi^{B_i}_{\lambda_{i,n}})Lz_n|| = 0.$$

Again, since $0 < \lambda_i \leq \lambda_{i,n}$, we have by Lemma 5, that

$$\lim_{n\to\infty} ||(I - \Phi^{B_i}_{\lambda_i})Lz_n|| = 0. \quad (49)$$

So for any subsequence $\{z_{n_j}\} \subset \{z_n\}$, we also have that

$$\lim_{n\to\infty} ||(I - \Phi^{B_i}_{\lambda_i})Lz_{n_j}|| = 0. \quad (50)$$

Thus, by the linearity and continuity of L, $Lx_{n_j} \rightharpoonup Lx^*$ as $j \to \infty$ and $||z_n - x_n|| \to 0$ as $n \to \infty$ implies $Lz_{n_j} \rightharpoonup Lx^*$ as $j \to \infty$. Hence from (49), we have

$$\lim_{n\to\infty} ||(I - \Phi^{B_i}_{\lambda_i})Lx^*|| = 0. \quad (51)$$

Therefore, $Lx^* = \Phi^{B_i}_{\lambda_i} Lx^*$, that is $Lx^* \in \bigcap_{i=1}^N B_i^{-1}(0)$. Further, we show that $x^* \in GMEP(F, g, \phi)$. From (40), we have $u_{n_j} \rightharpoonup x^*$. Since $u_n = K^F_{r_n}(x_n - r_n g(x_n))$, for any $y \in C$, we have

$$F(u_n, y) + \langle g(u_n), y - u_n\rangle + \phi(y) - \phi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0. \tag{52}$$

It follows from condition $(C2)$ of the bifunction F, that

$$\langle g(u_n), y - u_n\rangle\phi(y) - \phi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n).$$

Replacing n by n_j, we have

$$\langle g(u_{n_j}), y - u_{n_j}\rangle + \frac{1}{r_{n_j}}\langle y - u_{n_j} - x_{n_j}\rangle \geq F(y, u_{n_j}) + \phi(u_{n_j}) - \phi(y). \tag{53}$$

Let $y_t = ty + (1-t)x^*$ for all $t \in (0, 1]$ and $y \in C$. Then we have $y_t \in C$. So from (53), we have

$$\begin{aligned} \langle g(y_t), y_t - u_{n_j}\rangle \;\; &\geq \;\; \langle y_t - u_{n_j}, g(y_t)\rangle - \langle y_t - u_{n_j}, g(x_{n_j})\rangle - \left\langle y_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle + F(y_t, u_{n_j}) + \phi(u_{n_j}) - \phi(y_t) \\ &= \;\; \langle y_t - u_{n_j}, g(y_t) - g(u_{n_j})\rangle + \langle y_t - u_{n_j}, g(u_{n_j}) - g(x_{n_j})\rangle - \left\langle y_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle \\ &\quad + F(y_t, u_{n_j}) + \phi(u_{n_j}) - \phi(y_t). \end{aligned} \tag{54}$$

Since $\lim_{n\to\infty} ||u_n - x_n|| = 0$, we obtain $||g(u_{n_j}) - g(x_{n_j})|| \to 0$ as $n \to \infty$. Moreover, since g is monotone, we have $\langle y_t - u_{n_j}, g(y_t) - g(u_{n_j})\rangle \geq 0$. Therefore by $(C4)$ of the bifunction F and the weak lower semicontinuity of ϕ, taking the limit of (54), we obtain

$$\langle y_t - x^*, g(y_t)\rangle \geq F(y_t, x^*) + \phi(x^*) - \phi(y_t). \tag{55}$$

Using $(C1)$ of bifunction F and (55), we obtain

$$\begin{aligned} 0 &= F(y_t, y_t) + \phi(y_t) - \phi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, x^*) + t\phi(y) + (1-t)\phi(x^*) - \phi(y_t) \\ &= t[F(y_t, y) + \phi(y) - \phi(y_t)] + (1-t)[F(y_t, x^*) + \phi(x^*) - \phi(y_t)] \\ &\leq t[F(y_t, y) + \phi(y) - \phi(y_t)] + (1-t)\langle y_t - x^*, g(y_t)\rangle \\ &\leq t[F(y_t, y) + \phi(y) - \phi(y_t)] + (1-t)t\langle y - x^*, g(y_t)\rangle, \end{aligned}$$

this implies that

$$F(y_t, y) + (1-t)\langle y - x^*, g(y_t)\rangle + \phi(y) - \phi(y_t) \geq 0. \tag{56}$$

By letting $t \to 0$, we have

$$F(x^*, y) + \langle g(x^*), y - x^*\rangle + \phi(y) - \phi(x^*)\rangle \geq 0, \; y \in C, \tag{57}$$

which implies $x^* \in GMEP(F, g, \phi)$.

Finally we show that $x_n \to p = P_\Omega \nabla f(p)$. Let $\{x_{n_j}\}$ be subsequence of $\{x_n\}$, such that $x_{n_j} \rightharpoonup x^*$ and

$$\limsup_{n\to\infty} \frac{2}{1-c^2}\langle \nabla f(p) - p, x_{n+1} - p\rangle = \lim_{j\to\infty} \frac{2}{1-c^2}\langle \nabla f(p) - p, x_{n_j+1} - p\rangle, \tag{58}$$

since $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$ and $x_{n_j} \rightharpoonup x^*$, it follows that $x_{n_j+1} \rightharpoonup x^*$. Consequently, we obtain by (12), that

$$\limsup_{n\to\infty} \frac{2}{1-c^2}\langle \nabla f(p) - p, x_{n+1} - p\rangle = \frac{2}{1-c^2}\langle \nabla f(p) - p, x^* - p\rangle \leq 0. \tag{59}$$

By using Lemma 4 in (28), we conlude that $||x_n - p|| \to 0$ as $n \to \infty$. Thus, $x_n \to p$ as $n \to \infty$, ditto for both $\{u_n\}$ and $\{z_n\}$.

Case 2: Let $\Gamma_n = ||x_n - p||$ be monotonically nondecreasing. Define $\tau : \mathbb{N} \to \mathbb{N}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is nondecreasing, $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall\ n \geq n_0.$$

By using similar argument as in Case 1, we make the following conclusions

$$\lim_{n\to\infty} ||(I - \Phi_{\lambda_i}^{B_i})Lx_{\tau(n)}|| = 0,$$

$$\lim_{n\to\infty} ||L^*(I - \Phi_{\lambda_i}^{B_i})Lx_{\tau(n)}|| = 0,$$

$$\lim_{n\to\infty} ||u_{\tau(n)} - x_{\tau(n)}|| = 0,$$

$$\lim_{n\to\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0$$

and

$$\limsup_{n\to\infty} \frac{2}{1-c^2}\langle \nabla f(p) - p, x_{\tau(n)+1} - p\rangle \leq 0. \tag{60}$$

Using the boundedness of $\{x_{\tau(n)}\}$, we can obtain a subsequence of $\{x_{\tau(n)}\}$ which converges weakly to $x^* \in \bigcap_{i=1}^N A_i^{-1}(0)$, $Lx^* \in \bigcap_{i=1}^N B_i^{-1}(0)$ and $x^* \in GMEP(F, \phi, g)$. Therefore, it follows from (28), that

$$\begin{aligned} ||x_{\tau(n)+1} - p||^2 \quad &\leq \quad [1 - \alpha_{\tau(n)}(1-c^2)]||x_{\tau(n)} - p||^2 \\ &\quad + \alpha_{\tau(n)}(1-c^2)\left(\frac{2}{1-c^2}\nabla f(p) - p, x_{\tau(n)+1} - p\rangle\right). \end{aligned} \tag{61}$$

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we obtain $||x_{\tau(n)} - x_{\tau(n)+1}|| \leq 0$. Thus, from (61), we obtain

$$\alpha_{\tau(n)}(1-c^2)||x_{\tau(n)} - p||^2 \leq \alpha_{\tau(n)}(1-c^2)\left(\frac{2}{1-c^2}\nabla f(p) - p, x_{\tau(n)+1} - p\rangle\right). \tag{62}$$

We note that $\alpha_{\tau(n)}(1-c^2) > 0$, then from (62), we get

$$\lim_{n\to\infty} ||x_{\tau(n)} - p||^2 \leq 0.$$

This implies

$$\lim_{n\to\infty} ||x_{\tau(n)} - p||^2 = 0,$$

hence

$$\lim ||x_{\tau(n)} - p|| = 0.$$

Using this and $\lim_{n\to\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0$, we obtain

$$||x_{\tau(n)+1} - p|| \leq ||x_{\tau(n)+1} - x_{\tau(n)}|| + ||x_{\tau(n)} - p|| \to 0, \text{ as } n \to \infty.$$

Further, for $n \geq n_0$, we clearly observe that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$, (i.e., $\tau(n) < n$). Since $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(u) + 1 \leq j \leq n$. Consequently, for all $n \geq n_0$

$$0 \leq \Gamma_n \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}. \tag{63}$$

Using (63), we conclude that $\lim_{n\to\infty} ||x_n - p|| = 0$, that is $x_n \to p$. □

The following are some consequences of our main theorem.
Let $u = \nabla f(z_n)$ in (21), we have the following corollary:

Corollary 1. *Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and $L : H_1 \to H_2$ be a bounded linear operator. Assume F is a real valued bifunction on $C \times C$ which admits condition C1-C4. Let $\phi : H_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous function, g be a β-inverse strongly monotone mapping. For $i = 1, 2 \cdots, N$, let $A_i : H_1 \to 2^{H_1}$ and $B_i : H_2 \to 2^{H_2}$ be finite families of monotone mappings. Assume $\Omega = GMEP(F, g, \phi) \cap \Gamma \neq \emptyset$, where $\Gamma = \{p \in H_1 : 0 \in \bigcap_{i=1}^N A_i(p)$ and $Lp \in H_2 : 0 \in \bigcap_{i=1}^N B_i(Lp)\}$. For an arbitrary $u, x_0 \in H_1$, let $\{x_n\} \subset H_1$ be a sequence defined iteratively by*

$$\begin{cases} F(u_n, y) + \langle g(u_n), y - u_n\rangle + \phi(y) - \phi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ y \in H_1, \\ z_n = u_n - \gamma_n L^*(I - \Phi^{B_i}_{\lambda_{i,n}})Lu_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)\Phi^{A_i}_{\lambda_{i,n}} z_n, \end{cases} \tag{64}$$

where $\{r_n\}$ is a nonnegative sequence of real numbers, $\{\alpha_n\}$and $\{\lambda_{i,n}\}$ are sequences in $(0, 1)$, γ_n is a nonnegative sequence defined by (19), satisfying the following restrictions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $0 < \lambda_i \leq \lambda_{i,n}$;
(iii) $0 < a \leq r_n \leq b < 2\beta$.

Then x_n converges strongly to $p \in \Omega$, where $p = P_\Omega \nabla f(p)$.

For $i = 1, 2$, we obtain the following result for approximation a common solution of a split null point for a sum of monotone operators and generalized mixed equilibrium problem.

Corollary 2. *Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and $L : H_1 \to H_2$ be a bounded linear operator. Assume F is a real valued bifunction on $C \times C$ which admits condition C1-C4. Let $\phi : H_1 \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous function, g be a β-inverse strongly monotone mapping. For $i = 1, 2$, let $A_i : H_1 \to 2^{H_1}$ and $B_i : H_2 \to 2^{H_2}$ be finite families of monotone mappings. Assume $\Omega = GMEP(F, g, \phi) \cap \Gamma \neq \emptyset$, where $\Gamma = \{p \in H_1 : 0 \in \bigcap_{i=1}^2 A_i(p)$ and $Lp \in H_2 : 0 \in \bigcap_{i=1}^2 B_i(Lp)\}$. For an arbitrary $u, x_0 \in H_1$, let $\{x_n\} \subset H_1$ be a sequence defined iteratively by*

$$\begin{cases} F(u_n, y) + \langle g(u_n), y - u_n\rangle + \phi(y) - \phi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ y \in H_1, \\ z_n = u_n - \gamma_n L^*(I - (J^{B_2}_{\lambda_{2,n}} \circ J^{B_1}_{\lambda_{1,n}}))Lu_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(J^{A_2}_{\lambda_{2,n}} \circ J^{A_1}_{\lambda_{1,n}})z_n, \end{cases} \tag{65}$$

where $\{r_n\}$ is a nonnegative sequence of real numbers, $\{\alpha_n\}$and $\{\lambda_{i,n}\}$ are sequences in $(0,1)$, γ_n is a nonnegative sequence defined by (19), satisfying the following restrictions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0;$
(ii) $0 < \lambda_i \leq \lambda_{i,n};$
(iii) $0 < a \leq r_n \leq b < 2\beta.$

Then x_n converges strongly to $p \in \Omega$, where $p = P_\Omega \nabla f(p)$.

4. Numerical Example

In this section, we provide some numerical examples. The algorithm was coded in MATLAB 2019a on a Dell i7 Dual core 8.00 GB(7.78 GB usable) RAM laptop.

Example 1. *Let $E_1 = E_2 = C = Q = \ell_2(\mathbb{R})$ be the linear spaces of 2-summable sequences $\{x_j\}_{j=1}^{\infty}$ of scalars in $\mathbb{R}$, that is*

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2 \cdots, x_j \cdots), \ \ x_j \in \mathbb{R} \ \ and \ \ \sum_{j=1}^{\infty} |x_i|^2 < \infty \right\},$$

with the inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{j=1}^{\infty} x_j y_j$ and the norm $|| \cdot || : \ell_2 \to \mathbb{R}$ by $||x|| := \sqrt{\sum_{i=1}^{\infty} |x_j|^2}$, where $x = \{x_j\}_{j=1}^{\infty}$, $y = \{y_j\}_{j=1}^{\infty}$. Let $L : \ell_2 \to \ell_2$ be given by $Lx = \big(x_1, x_2, \cdots, x_j, \cdots,\big)$ for all $x = \{x_i\}_{i=1}^{\infty} \in \ell_2$, then $L^* y = \big(y_1, y_2, \cdots, y_j, \cdots,\big)$ for each $y = \{y_i\}_{\infty} \in \ell_2$.

Let $f(x) = \dfrac{1}{2} x(s)^2, \ \forall x \in \ell_2$, it is easy to that f is differentiable with $\nabla f = x$. For each $i = 1, 2 \cdots N$, define $A_i(x) : \ell_2 \to \ell_2$ and $B_i(x) : \ell_2 \to \ell_2$ by $A_i(x) = ix$ and $B_i(x) = \frac{2}{3} ix$ respectively for all $x \in \ell_2$.

For each $u, v \in \ell_2$, define the bifunction $F : C \times C \to \mathbb{R}$ by $F(u, v) = uv + 15v - 15u - u^2$, the function $g : C \to H_1$ by $g(u) = u, \ \forall u \in H_1$ and $\phi : H_1 \to \mathbb{R} \cup \{+\infty\}$ by $\phi(u) = 0$, for each $u \in H_1$. For each $x \in C$, we have the following steps to get $\{u_n\}$: Find u such that

$$\begin{aligned}
0 &\leq F(u, v) + \langle g(u), v - u \rangle + \phi(v) - \phi(u) + \frac{1}{r} \langle v - u, u - x \rangle \\
&= uv + 15v - 15u - u^2 + v - u + \frac{1}{r} \langle v - u, u - x \rangle \\
&= uv + 16v - 16u - u^2 + \frac{1}{r} \langle v - u, u - x \rangle \\
&= (u + 16)(v - u) + \frac{1}{r} \langle v - u, u - x \rangle \\
&= (v - u) \left(u + 16 + \frac{1}{r} \langle v - u, u - x \rangle \right)
\end{aligned}$$

for all $v \in C$. Hence, by Lemma 3 (2), it follows that $u = \dfrac{x - 16r}{r + 1}$. Therefore, $u_n = \dfrac{x_n - 16r_n}{r_n + 1}$.

For $i = 1, 2$, choose the sequences $\alpha_n = \dfrac{1}{n+1}$, $r_n = \dfrac{1}{2n^2 - 1}$, $\lambda_{i,n} = \dfrac{1}{in + 2}$ and $\gamma = 0.25$. We obtain the graph of errors against the number of iterations for different values of x_0. The following cases are presented in Figure 1 below:

Case 1 $x_0 = (0.435, 0.896, 1.004, \cdots 0, \cdots)$,
Case 2 $x_0 = (-0.987, 0.615, -2.804, \cdots 0, \cdots)$,

Case 3 $x_0 = (3.45, 6.000, 1.53, \cdots 0, \cdots)$.

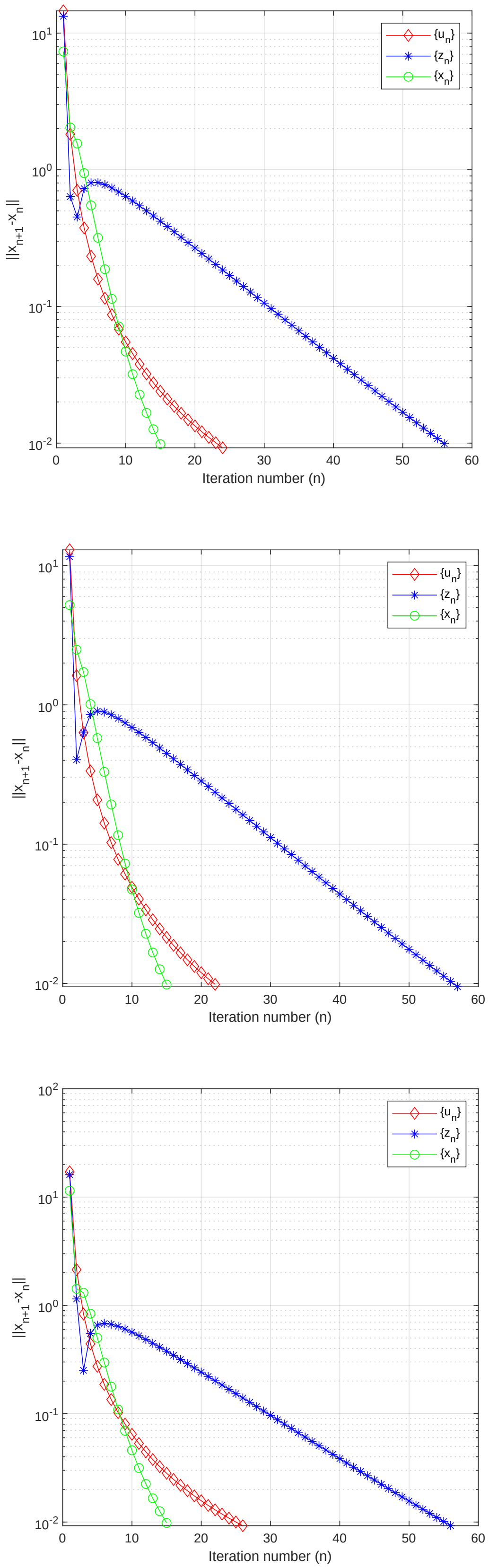


Figure 1. Case 1 (**top**); Case 2 (**middle**); Case 3 (**bottom**).

Example 2. *Let $H_1 = H_2 = \mathbb{R}^2$ be endowed with an inner product $\langle x, y\rangle = x \cdot y = x_1y_1 + x_2y_2$, where $x = (x_1, x_2), y = (y_1, y_2)$ and the euclidean norm. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L(x) = (x_1 + x_2, 2x_1 + 2x_2)$, $x = (x_1, x_2)$ and $f(x) = \frac{1}{4}x^2$. For each $i = 1, 2 \cdots N$, define $A_i(x) : \mathbb{R}^2 \to \mathbb{R}^2$ and $B_i(x) : \mathbb{R}^2 \to \mathbb{R}^2$ by $A_i(x) = ix$ and $B_i(x) = \frac{2}{3}ix$ respectively, where $x = (x_1, x_2)$. Let $y = (y_1, y_2), z = (z_1, z_2) \in \mathbb{R}^2$. Define $F(z, y) = -3z^2 + 2zy + y^2$, $g(z) = z$ and $\phi(z) = z$. By simple calculation, we obtain that*

$$u_n = \frac{x_n}{8r_n + 1}.$$

Choose the sequences $\alpha_n = \frac{1}{\sqrt{2n^2+3}}$, $r_n = \frac{n-1}{2n^2-1}$, $\lambda_{i,n} = \frac{1}{in+2}$ and $\gamma = 0.25$. For $i = 1, 2$, (21) becomes

$$\begin{cases} F(u_n, y) + \langle g(u_n), y - u_n\rangle + \phi(y) - \phi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ y \in H_1, \\ z_n = u_n - \gamma_n L^*(I - J^{B_1}_{\lambda_n} \circ J^{B_2}_{\lambda_n})Lu_n, \\ x_{n+1} = \frac{1}{\sqrt{2n^2+3}}\nabla f(z_n) + \left(1 - \frac{1}{\sqrt{2n^2+3}}\right) J^{A_1}_{\lambda_n} \circ J^{A_2}_{\lambda_n} z_n, \end{cases} \tag{66}$$

We make different choices of our initial value as follow:

Case 1, $x = (0.5, 1)$, Case 2, $x = (-0.05, 0.5)$, and Case 3, $x = (-1.5, 1.0)$.

We use $\|x_{n+1} - x_n\|^2 < 2 \times 10^{-3}$ as our stopping criterion and plot the graphs of errors against the number of iterations. See Figure 2.

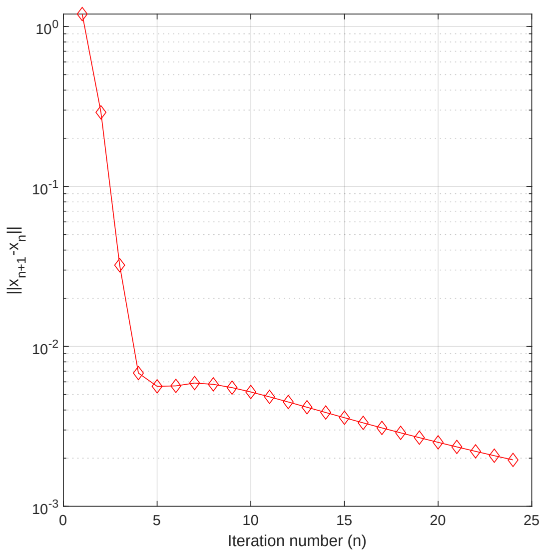
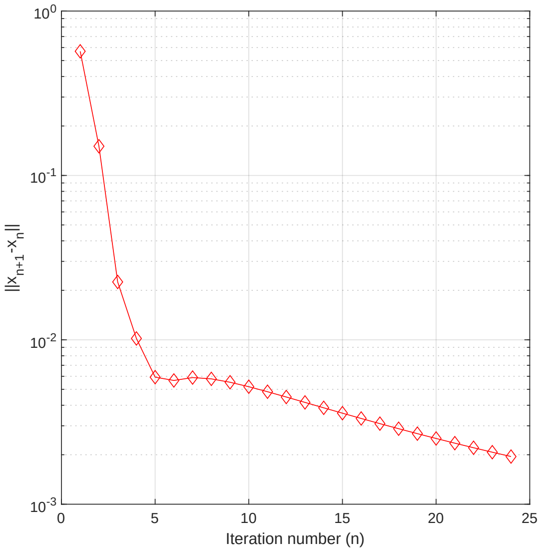


Figure 2. *Cont.*



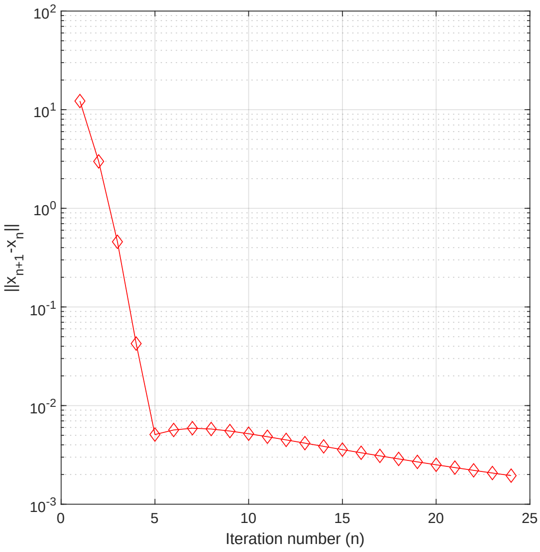


Figure 2. Case 1 (**top**); Case 2 (**middle**); Case 3 (**bottom**).

5. Conclusions

This paper considered the approximation of common solutions of a split null point problem for a finite family of maximal monotone operators and generalized mixed equilibrium problem in real Hilbert spaces. We proposed an iterative algorithm which does not depend on the prior knowledge of the operator norm as being used by many authors in the literature [39,42]. We proved a strong convergence of the proposed algorithm to a common solution of the two problems. We displayed some numerical examples to illustrate our method. Our result improves some existing results in the literature.

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