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Article

A Class of Extended Fractional Derivative Operators and Associated Generating Relations Involving Hypergeometric Functions

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Abstract: Recently, an extended operator of fractional derivative related to a generalized Beta function was used in order to obtain some generating relations involving the extended hypergeometric functions [1]. The main object of this paper is to present a further generalization of the extended fractional derivative operator and apply the generalized extended fractional derivative operator to derive linear and bilinear generating relations for the generalized extended Gauss, Appell and Lauricella hypergeometric functions in one, two and more variables. Some other properties and relationships involving the Mellin transforms and the generalized extended fractional derivative operator are also given.

Keywords: gamma and beta functions; Eulerian integrals; generating functions; hypergeometric functions; Appell–Lauricella hypergeometric functions; fractional derivative operators; Mellin transforms

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1. Introduction, Definitions and Preliminaries

For the sake of clarity and easy readability, we find it to be natural and convenient to divide this introductory section into three parts (or subsections). In Part 1.1, we introduce the extended Beta, Gamma and hypergeometric functions. Part 1.2 deals with the familiar Riemann–Liouville fractional derivative operator and its generalizations, which are motivated essentially by the definition in Part 1.1 for the extended Beta function. In the third subsection (Part 1.3), we then introduce the extended Appell hypergeometric functions in two variables, which were recently investigated in conjunction with the family of the extended Riemann–Liouville fractional derivative operators defined in Part 1.2.

1.1. The Extended Beta, Gamma and Hypergeometric Functions

Extensions of a number of well-known special functions were investigated recently by several authors (see, for example [2–7]). In particular, Chaudhry *et al.* [3] gave the following interesting extension of the classical Beta function $B(\alpha, \beta)$:

$$B(\alpha, \beta; p) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \exp\left(-\frac{p}{t(1-t)}\right) dt \tag{1.1}$$

$$\big(\min\{\Re(\alpha), \Re(\beta)\} > 0;\ \Re(p) \geqq 0\big)$$

so that, clearly,

$$B(\alpha, \beta) = B(\alpha, \beta; 0)$$

Here, and in what follows, such arguments as (for example) $-\frac{p}{t(1-t)}$ in the Definition (1.1) are motivated by the connection of the extended Beta function $B(\alpha, \beta; p)$ with the Macdonald (or modified Bessel) function $K_\nu(z)$ (see, for details, [8,9]).

Making use of the extended Beta function $B(\alpha, \beta; p)$ defined by (1.1), Chaudhry *et al.* [8] introduced the extended hypergeometric function as follows:

$$F_p(a, b; c; z) = \frac{1}{B(b, c-b)} \sum_{n=0}^{\infty} (a)_n\, B(b+n, c-b; p)\, \frac{z^n}{n!} \tag{1.2}$$

$$\big(|z| < 1;\ \Re(c) > \Re(b) > 0;\ \Re(p) \geqq 0\big)$$

where $(\lambda)_n$ denotes the Pochhammer symbol or the *shifted factorial*, which is defined (for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_0$) by

$$(\lambda)_0 := 1 \qquad \text{and} \qquad (\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1) \qquad (n \in \mathbb{N};\ \lambda \in \mathbb{C}) \tag{1.3}$$

it being understood *conventionally* that $(0)_0 := 1$.

Among several interesting and potentially useful properties of the extended hypergeometric function $F_p(a, b; c; z)$ defined by (1.2), the following integral representation of the Pfaff–Kummer type was also given by Chaudhry *et al.* [8, p. 592, Equation (3.2)]:

$$F_p(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left(-\frac{p}{t(1-t)}\right) dt \tag{1.4}$$

$$\big(\Re(p) > 0;\ p = 0 \quad \text{and} \quad |\arg(1-z)| < \pi;\ \Re(c) > \Re(b) > 0\big)$$

Obviously, for the Gauss hypergeometric function ${}_2F_1$, we have

$$F_0(a, b; c; z) = {}_2F_1(a, b; c; z)$$

The following *further* generalizations of the extended Gamma function $\Gamma_p(z)$ (see, for details [9]), the extended Beta function $B_p(\alpha, \beta)$ and the extended Gauss hypergeometric function F_p were considered more recently by Özergin *et al.* [7]:

$$\Gamma_p^{(\rho,\sigma)}(z) := \int_0^\infty t^{z-1}\ {}_1F_1\left(\rho; \sigma; -t - \frac{p}{t}\right) dt \tag{1.5}$$

$$\big(\min\{\Re(z), \Re(\rho), \Re(\sigma)\} > 0;\ \Re(p) \geqq 0\big)$$

$$B^{(\rho,\sigma)}(\alpha, \beta; p) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\ {}_1F_1\left(\rho; \sigma; -\frac{p}{t(1-t)}\right) dt \tag{1.6}$$

$$\big(\min\{\Re(\alpha), \Re(\beta), \Re(\rho), \Re(\sigma)\} > 0;\ \Re(p) \geqq 0\big)$$

and

$$F_p^{(\rho,\sigma)}(a, b; c; z) := \frac{1}{B(b, c-b)} \sum_{n=0}^\infty (a)_n\ B^{(\rho,\sigma)}(b+n, c-b; p)\ \frac{z^n}{n!} \tag{1.7}$$

$$\big(|z| < 1;\ \min\{\Re(\rho), \Re(\sigma)\} > 0;\ \Re(c) > \Re(b) > 0;\ \Re(p) \geqq 0\big)$$

respectively, ${}_1F_1$ being the (Kummer's) *confluent* hypergeometric function. The following integral representation of the Pfaff–Kummer type was also given by Özergin *et al.* [7]:

$$F_p^{(\rho,\sigma)}(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\ (1-zt)^{-a}\ {}_1F_1\left(\rho; \sigma; -\frac{p}{t(1-t)}\right) dt \tag{1.8}$$

$$\big(\Re(p) > 0;\ p = 0 \quad \text{and} \quad |\arg(1-z)| < \pi;\ \Re(c) > \Re(b) > 0;\ \min\{\Re(\rho), \Re(\sigma)\} > 0\big)$$

Clearly, since

$${}_1F_1(\rho; \rho; z) = {}_0F_0(\text{——}\,;\text{——}\,; z) = \exp(z) \tag{1.9}$$

the special cases of Equations (1.6–1.8) when $\rho = \sigma$ would immediately yield Equations (1.1), (1.2) and (1.4), respectively.

Throughout our present investigation, it is *tacitly* assumed that σ and various other lower (or denominator) parameters are not zero or negative integers (that is, no zeros appear in the denominators).

1.2. The Riemann–Liouville Fractional Derivative Operator and Its Generalizations

For the *Riemann–Liouville fractional derivative operator* $\mathcal{D}_z^\mu$ defined by (see, for example [10, p. 181] [11] and [12, p. 70 *et seq.*])

$$\mathcal{D}_z^\mu \{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-t)^{-\mu-1} \; f(t)\, dt & \big(\Re(\mu) < 0\big) \\ \dfrac{d^m}{dz^m} \Big\{ \mathcal{D}_z^{\mu-m} \{f(z)\} \Big\} & \big(m-1 \leqq \Re(\mu) < m \;\; (m \in \mathbb{N})\big) \end{cases} \tag{1.10}$$

it is known that

$$\mathcal{D}_z^\mu \left\{ z^\lambda \right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} \, z^{\lambda-\mu} \qquad \big(\Re(\lambda) > -1\big) \tag{1.11}$$

where, and in what follows,

$$\mathbb{N} := \{1, 2, 3, \cdots\} \qquad \text{and} \qquad \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\} \tag{1.12}$$

The path of integration in the definition (1.10) is a line in the complex t-plane from $t = 0$ to $t = z$.

By introducing a new parameter p of the type which is involved in (for example) the Definitions (1.1) and (1.2), Özarslan and Özergin [1] defined the correspondingly *extended* Riemann–Liouville fractional derivative operator $D_z^{\mu,p}$ by

$$\mathcal{D}_z^{\mu,p} \{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-t)^{-\mu-1} \; \exp\left(-\frac{pz^2}{(z-t)t}\right) f(t)\, dt & \big(\Re(\mu) < 0\big) \\ \dfrac{d^m}{dz^m} \Big\{ \mathcal{D}_z^{\mu-m,p} \{f(z)\} \Big\} & \big(m-1 \leqq \Re(\mu) < m \;\; (m \in \mathbb{N})\big) \end{cases} \tag{1.13}$$

where, as before, $\Re(p) \geqq 0$. The path of integration in the Definition (1.13), which immediately yields the definition (1.10) when $p = 0$, is also a line in the complex t-plane from $t = 0$ to $t = z$. The argument $-\frac{pz^2}{(z-t)t}$ in the Definition (1.13) and elsewhere in this paper is obviously necessitated by the applicability of the definition (1.1) of the extended Beta function $B(\alpha, \beta; p)$ when we set

$$t = z\tau \qquad \text{and} \qquad dt = z d\tau \qquad (0 \leqq \tau \leqq 1)$$

In the case when (for example) $\mu = 0$, we find from the second part of the Definition (1.13) (*with* $m = 1$) that

$$\begin{aligned} \mathcal{D}_z^{0,p} \{f(z)\} &= \frac{d}{dz} \Big\{ \mathcal{D}_z^{-1,p} \{f(z)\} \Big\} \\ &= \frac{d}{dz} \left\{ \int_0^z \exp\left(-\frac{pz^2}{(z-t)t}\right) f(t)\, dt \right\} \end{aligned}$$

which obviously yields the function $f(z)$ when $p = 0$. Thus, in general, the natural connection of the Riemann–Liouville fractional derivative operator $\mathcal{D}_z^\mu$ defined by (1.10) with *ordinary* derivatives when the order μ is zero or a positive integer is lost by the extended fractional derivative operator in Definition (1.13) and its further generalizations which we have considered in our present investigation.

1.3. Extended Appell Hypergeometric Functions in Two Variables

In analogy with the Definition (1.7), and motivated by their Definition (1.13) of the *extended* Riemann–Liouville fractional derivative $D_z^{\mu,p}$, Özarslan and Özergin [1] extended the familiar Appell hypergeometric functions F_1 and F_2 in two variables as follows:

$$F_1(a,b,b';c;x,y;p) = \sum_{m,n=0}^{\infty} (b)_m(b')_n \frac{B(a+m+n,c-a;p)}{B(a,c-a)} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.14}$$

$$\big(\max\{|x|,|y|\} < 1;\ \Re(p) \geqq 0\big)$$

and

$$F_2(a,b,b';c,c';x,y;p) = \sum_{m,n=0}^{\infty} (a)_{m+n} \frac{B(b+m,c-b;p)B(b'+n,c'-b';p)}{B(b,c-b)B(b',c'-b')} \frac{x^m}{m!} \frac{y^n}{n!} \tag{1.15}$$

$$\big(|x|+|y| < 1;\ \Re(p) \geqq 0\big)$$

which, in the special case when $p=0$, yield the familiar Appell hypergeometric functions F_1 and F_2 in two variables (see [13, p. 14]). For each of these extended Appell hypergeometric functions, such properties as their integral representations and relationships with the extended Riemann–Liouville fractional derivative operator $\mathcal{D}_z^{\mu,p}$ defined by (1.13) can also be found in the work of Özarslan and Özergin [1].

The aim of this paper is to investigate the various properties of a *further* generalization of the extended fractional derivative operator $\mathcal{D}_z^{\mu,p}$ defined by (1.13) and apply the generalized operator to derive generating relations for hypergeometric functions in one, two and more variables. We first introduce, in Section 2, the following *further* generalizations of the extended Appell's hypergeometric functions:

$$F_1^{(\rho,\sigma)}(a,b,b';c;x,y;p) \qquad \text{and} \qquad F_2^{(\rho,\sigma)}(a,b,b';c,c';x,y;p)$$

in two variables and the extended Lauricella's hypergeometric function

$$F_{D,(\rho,\sigma;p)}^{(r)}(a,b_1,\cdots,b_r;c;x_1,\cdots,x_r)$$

of r variables $x_1,\cdots,x_r$ are defined and their integral representations are obtained. In Section 3, we introduce and study the properties and relationships associated with the above-mentioned *further* generalization of the extended fractional derivative operator $\mathcal{D}_z^{\mu,p}$ defined by Definition (1.13) and apply the generalized operator in order to obtain various generating relations in terms of the generalized extended Appell and Lauricella hypergeometric functions in two and more variables. Section 4 contains some results related to the Mellin transforms and the extended fractional derivative operator. In Section 5, some generating relations for generalized extended hypergeometric functions are obtained via the above-mentioned further generalized fractional derivative operator by following the lines which are detailed in the monograph by Srivastava and Manocha [14]. Finally, in Section 6, we conclude this paper by presenting a number of remarks and observations pertaining to our investigation here.

2. The Generalized Extended Appell and Lauricella Functions

Let a function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ be analytic within the disk $|z|<R \quad (0<R<\infty)$ and let its Taylor–Maclaurin coefficients be *explicitly* denoted (for convenience) by the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$. Suppose also that the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ can be continued analytically in the right half-plane $\Re(z)>0$ with the asymptotic property given as follows:

$$\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right):=\begin{cases}\displaystyle\sum_{\ell=0}^{\infty}\kappa_\ell\,\frac{z^\ell}{\ell!} & (|z|<R;\ 0<R<\infty;\ \kappa_0:=1)\\[2ex] \mathfrak{M}_0\ z^\omega\ \exp(z)\left[1+O\left(\dfrac{1}{z}\right)\right] & \big(\Re(z)\to\infty;\ \mathfrak{M}_0>0;\ \omega\in\mathbb{C}\big)\end{cases}\tag{2.1}$$

for some suitable constants $\mathfrak{M}_0$ and ω depending essentially upon the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$. Here, *and in what follows*, we assume that the series in the first part of the Definition (2.1) converges absolutely when $|z|<R$ for some $R \ (0<R<\infty)$ and represents the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ which is assumed to be analytic within the disk $|z|<R \quad (0<R<\infty)$ and which can be appropriately continued analytically elsewhere in the complex z-plane with the order estimate provided in the second part of the Definition (2.1). For example, if we choose the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ to be a suitable quotient of several Γ-products with arguments linear in ℓ so that the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ becomes identifiable with the familiar Fox–Wright Ψ-function, we can easily determine the radius R of the above-mentioned disk and, moreover, we can then appropriately continue the resulting function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ analytically by means of a suitable Mellin–Barnes contour integral (see, for details [12, p. 56 *et seq.*]). Such functions as $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ can indeed be specified on an *ad hoc* basis.

In terms of the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ defined by (2.1), we now introduce a natural further generalization of the extended Gamma function $\Gamma_p^{(\rho,\sigma)}(z)$, the extended Beta function $B^{(\rho,\sigma)}(\alpha,\beta;p)$ and the extended hypergeometric function $F_p^{(\rho,\sigma)}(a,b;c;z)$ by

$$\Gamma_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(z)=\int_0^\infty t^{z-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-t-\frac{p}{t}\right)dt\tag{2.2}$$
$$\big(\Re(z)>0;\ \Re(p)\geqq 0\big)$$

$$\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta;p):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{p}{t(1-t)}\right)dt\tag{2.3}$$
$$\big(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \Re(p)\geqq 0\big)$$

and

$$\mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b;c;z):=\frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\,\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b+n,c-b;p)\,\frac{z^n}{n!}\tag{2.4}$$
$$\big(|z|<1;\ \Re(c)>\Re(b)>0;\ \Re(p)\geqq 0\big)$$

provided that the defining integrals in Definitions (2.2–2.4) exist.

Remark 1. For various special choices of the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$, the definitions in (2.2–2.4) would reduce to (known or new) extensions of the Gamma, Beta and hypergeometric functions. In particular, if we set

$$\kappa_\ell = \frac{(\rho)_\ell}{(\sigma)_\ell} \qquad (\ell \in \mathbb{N}_0) \tag{2.5}$$

the definitions (2.2–2.4) immediately yield the definitions in (1.5–1.7) for the extended Gamma function $\Gamma_p^{(\rho,\sigma)}(z)$, the extended Beta function $B^{(\rho,\sigma)}(\alpha,\beta;p)$ and the extended hypergeometric function $F_p^{(\rho,\sigma)}(a,b;c;z)$, respectively.

In terms of the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ defined by definition (2.1), it is not difficult to generalize the integral representation definition (1.8) to the following form:

$$\mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,(1-zt)^{-a}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{p}{t(1-t)}\right)dt \tag{2.6}$$

$$\big(\Re(p)>0;\ p=0 \quad \text{and} \quad |\arg(1-z)|<\pi;\ \Re(c)>\Re(b)>0\big)$$

For suitably constrained (real or complex) parameters ρ and σ, we propose these *further* generalizations of the extended Appell's hypergeometric functions:

$$F_1^{(\rho,\sigma)}(a,b,b';c;x,y;p) \qquad \text{and} \qquad F_2^{(\rho,\sigma)}(a,b,b';c,c';x,y;p)$$

in two variables, and the extended Lauricella's hypergeometric function:

$$F_{D,(\rho,\sigma;p)}^{(r)}(a,b_1,\cdots,b_r;c;x_1,\cdots,x_r)$$

of r variables $x_1,\cdots,x_r$, which are defined by

$$F_1^{(\rho,\sigma)}(a,b,b';c;x,y;p) := \sum_{m,n=0}^{\infty}(b)_m(b')n\;\frac{B^{(\rho,\sigma)}(a+m+n,c-a;p)}{B(a,c-a)}\;\frac{x^m}{m!}\;\frac{y^n}{n!} \tag{2.7}$$

$$\big(\max\{|x|,|y|\}<1;\ \Re(p)\geqq 0\big)$$

$$F_2^{(\rho,\sigma)}(a,b,b';c,c';x,y;p) := \sum_{m,n=0}^{\infty}(a)_{m+n}\;\frac{B^{(\rho,\sigma)}(b+m,c-b;p)B^{(\rho,\sigma)}(b'+n,c'-b';p)}{B(b,c-b)B(b',c'-b')}\;\frac{x^m}{m!}\;\frac{y^n}{n!} \tag{2.8}$$

$$\big(|x|+|y|<1;\ \Re(p)\geqq 0\big)$$

and

$$F_{D,(\rho,\sigma;p)}^{(r)}(a,b_1,\cdots,b_r;c;x_1,\cdots,x_r)$$
$$:= \sum_{m_1,\cdots,m_r=0}^{\infty}(b_1)_{m_1}\cdots(b_r)_{m_r}\;\frac{B^{(\rho,\sigma)}(a+m_1+\cdots+m_r,c-a;p)}{B(a,c-a)}\;\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \tag{2.9}$$

$$\big(\max\{|x_1|,\cdots,|x_r|\}<1;\ \Re(p)\geqq 0\big)$$

where the generalized extended Beta function $B^{(\rho,\sigma)}(\alpha,\beta;p)$ is given by Definition (1.6). Clearly, the Definition (2.7) corresponds to the special case of the Definition (2.9) when $r=2$. Moreover, in view of the relationship Definition (1.9), the Definitions (2.7) and (2.8) immediately yield the definitions in (1.14) and (1.15) when $\rho=\sigma$. More generally, in terms of the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ defined involved in (2.1), we have the following definitions:

$$\mathfrak{F}_1^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(a,b,b';c;x,y;p) := \sum_{m,n=0}^{\infty} (b)_m (b')n \, \frac{\mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(a+m+n,c-a;p)}{B(a,c-a)} \, \frac{x^m}{m!} \, \frac{y^n}{n!} \tag{2.10}$$

$$\big(\max\{|x|,|y|\}<1;\ \Re(p)\geqq 0\big)$$

$$\begin{aligned}\mathfrak{F}_2^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}&(a,b,b';c,c';x,y;p)\\ &:= \sum_{m,n=0}^{\infty} (a)_{m+n} \, \frac{\mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(b+m,c-b;p)\mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(b'+n,c'-b';p)}{B(b,c-b)B(b',c'-b')} \, \frac{x^m}{m!} \, \frac{y^n}{n!}\end{aligned} \tag{2.11}$$

$$\big(|x|+|y|<1;\ \Re(p)\geqq 0\big)$$

and

$$\begin{aligned}\mathfrak{F}_{D,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};p)}^{(r)}&(a,b_1,\cdots,b_r;c;x_1,\cdots,x_r)\\ &:= \sum_{m_1,\cdots,m_r=0}^{\infty} (b_1)_{m_1}\cdots(b_r)_{m_r} \, \frac{\mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(a+m_1+\cdots+m_r,c-a;p)}{B(a,c-a)} \, \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!}\end{aligned} \tag{2.12}$$

$$\big(\max\{|x_1|,\cdots,|x_r|\}<1;\ \Re(p)\geqq 0\big)$$

where the generalized extended Beta function $\mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha,\beta;p)$ is given by Definition (2.3).

We now proceed to derive integral representations for the above-defined hypergeometric functions in two and more variables.

Theorem 1. *For the generalized extended Appell functions*

$$F_1^{(\rho,\sigma)}(a,b,b';c;x,y;p) \qquad \text{and} \qquad \mathfrak{F}_1^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(a,b,b';c;x,y;p)$$

defined by (2.7) *and* (2.10), *the following integral representations hold true*:

$$\begin{aligned}F_1^{(\rho,\sigma)}&(a,b,b';c;x,y;p)\\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}(1-yt)^{-b'} \, {}_1F_1\left(\rho;\sigma;-\frac{p}{t(1-t)}\right) dt\end{aligned} \tag{2.13}$$

$$\big(\Re(p)>0;\ p=0 \quad \text{and} \quad \max\{|\arg(1-x)|,|\arg(1-y)|\}<\pi;$$
$$\Re(c)>\Re(a)>0;\ \min\{\Re(\rho),\Re(\sigma)\}>0\big)$$

and

$$\mathfrak{F}_1^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b,b';c;x,y;p) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}(1-yt)^{-b'} \cdot \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{p}{t(1-t)}\right)dt \tag{2.14}$$

$$\big(\Re(p)>0;\ p=0 \quad \textit{and} \quad \max\{|\arg(1-x)|,|\arg(1-y)|\}<\pi;\ \Re(c)>\Re(a)>0\big)$$

Proof. For convenience, we denote the second member of the Assertion (2.13) by $\Lambda_p(x,y)$ and assume that $\max\{|x|,|y|\}<1$. Then, upon expressing

$$(1-xt)^{-b} \qquad \text{and} \qquad (1-yt)^{-b'}$$

as their Taylor–Maclaurin series, if we invert the order of summation and integration (which can easily be justified by absolute and uniform convergence), we find that

$$\begin{aligned}\Lambda_p(x,y) &:= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}(1-yt)^{-b'}\ {}_1F_1\left(\rho;\sigma;-\frac{p}{t(1-t)}\right)dt\\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{m,n=0}^{\infty}(b)_m(b')_n\,\frac{x^m}{m!}\,\frac{y^n}{n!}\\ &\quad\cdot\int_0^1 t^{a+m+n-1}(1-t)^{c-a-1}\ {}_1F_1\left(\rho;\sigma;-\frac{p}{t(1-t)}\right)dt \end{aligned} \tag{2.15}$$

$$\big(|x|+|y|<1;\ \Re(c)>\Re(a)>0\big)$$

which, in view of the Definitions (1.6) and (2.7), yields the first member of the Assertion (2.13). Our demonstration of the integral Representation (2.13) is completed by applying the principle of analytic continuation, since the integral for $\Lambda_p(x,y)$ above in (2.13) exists under the constraints which are listed already with (2.13).

The proof of the Assertion (2.14) runs parallel to that of (2.13) and is based similarly upon the definitions (2.3) and (2.10) instead. The details involved are being omitted. □

Theorems 2 and 3 below follow easily from the Definitions (1.6) and (2.3) in conjunction with the Definitions (2.8) and (2.11) and the Definitions (2.9) and (2.12), respectively.

Theorem 2. *For the functions*

$$F_2^{(\rho,\sigma)}(a,b,c;d,e;x,y:p) \qquad \textit{and} \qquad \mathfrak{F}_2^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b,b';c,c';x,y;p)$$

defined by (2.8) *and* (2.11), *respectively*, *the following integral representations hold true*:

$$F_2^{(\rho,\sigma)}(a,b,b';c,c';x,y:p) = \frac{1}{B(b,c-b)B(b',c'-b')}\int_0^1\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}\,u^{b'-1}(1-u)^{c'-b'-1}}{(1-xt-yu)^a} \cdot {}_1F_1\left(\rho;\sigma;-\frac{p}{t(1-t)}\right)\ {}_1F_1\left(\rho;\sigma;-\frac{p}{u(1-u)}\right)dt\,du \tag{2.16}$$

$$\big(\Re(p) > 0;\ p = 0 \quad \textit{and} \quad |x| + |y| < 1;\ \Re(c) > \Re(b) > 0;\ \Re(c') > \Re(b') > 0;\ \min\{\Re(\rho), \Re(\sigma)\} > 0\big)$$

and

$$\mathfrak{F}_2^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a, b, b'; c, c'; x, y; p) = \frac{1}{B(b, c-b)B(b', c'-b')} \int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}\ u^{b'-1}(1-u)^{c'-b'-1}}{(1-xt-yu)^a} \cdot \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\frac{p}{t(1-t)}\right)\ \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\frac{p}{u(1-u)}\right) dt\ du \tag{2.17}$$

$$\big(\Re(p) > 0;\ p = 0 \quad \textit{and} \quad |x| + |y| < 1;\ \Re(c) > \Re(b) > 0;\ \Re(c') > \Re(b') > 0\big)$$

Proof. Since [14, p. 52, Equation 1.6(2)]

$$\sum_{m,n=0}^{\infty} f(m+n)\ \frac{x^m}{m!}\ \frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N)\ \frac{(x+y)^N}{N!} \tag{2.18}$$

it is easily seen that

$$(1-xt-yu)^{-a} = \sum_{N=0}^{\infty} \frac{(a)_N}{N!}\ (xt+yu)^N = \sum_{m,n=0}^{\infty} (a)_{m+n}\ \frac{(xt)^m}{m!}\ \frac{(yu)^n}{n!} \tag{2.19}$$

$$(|x| + |y| < 1;\ \max\{|t|, |u|\} < 1)\,,$$

which is rather instrumental in our demonstration of Theorem 2 along the lines of the proof of Theorem 1. □

Theorem 3. *For the functions*

$$F_{D,(\rho,\sigma;p)}^{(r)}(a, b_1, \cdots, b_r; c; x_1, \cdots, x_r) \qquad \textit{and} \qquad \mathfrak{F}_{D,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};p\right)}^{(r)}(a, b_1, \cdots, b_r; c; x_1, \cdots, x_r)$$

defined by (2.9) *and* (2.12), *respectively, the following integral representations hold true*:

$$F_{D,(\rho,\sigma;p)}^{(r)}(a, b_1, \cdots, b_r; c; x_1, \cdots, x_r) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-x_1t)^{-b_1} \cdots (1-x_rt)^{-b_r} \cdot\ {}_1F_1\left(\rho; \sigma; -\frac{p}{t(1-t)}\right) dt \tag{2.20}$$

$$\big(\Re(p) > 0;\ p = 0 \quad \textit{and} \quad \max\{|\arg(1-x)|, |\arg(1-y)|\} < \pi;\ \Re(c) > \Re(a) > 0;\ \min\{\Re(\rho), \Re(\sigma)\} > 0\big)$$

and

$$\mathfrak{F}_{D,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};p\right)}^{(r)}(a, b_1, \cdots, b_r; c; x_1, \cdots, x_r) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-x_1t)^{-b_1} \cdots (1-x_rt)^{-b_r} \cdot \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\frac{p}{t(1-t)}\right) dt \tag{2.21}$$

$$\big(\Re(p) > 0;\ p = 0 \quad \textit{and} \quad \max\{|\arg(1-x_1)|, \cdots, |\arg(1-x_r)|\} < \pi;\ \Re(c) > \Re(a) > 0\big)$$

Proof. The proof of Theorem 3 is much akin to that of its special (two-variable) case (that is, Theorem 1) when $r = 2$. We, therefore, omit the details involved. $\square$

3. Applications of the Generalized Extended Riemann–Liouville Fractional Derivative Operator

Earlier investigations by various authors dealing with operators of fractional calculus and their applications are adequately presented in the recent monograph [12] (see also [15]). The use of fractional derivative in the theory of generating functions is explained reasonably satisfactorily by Srivastava and Manocha (see, for details, [14, Chapter 5]). Here, in this section, we first introduce the following generalizations of the extended Riemann–Liouville fractional derivative operator $\mathcal{D}_z^{\mu,p}$ defined by (1.13):

$$\mathcal{D}_{z,(\rho,\sigma)}^{\mu,p}\left\{f(z)\right\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1}\ {}_1F_1\left(\rho;\sigma;-\frac{pz^2}{(z-t)t}\right) f(t)\,dt & \big(\Re(\mu)<0\big) \\[2ex] \dfrac{d^m}{dz^m}\left\{\mathcal{D}_{z,(\rho,\sigma)}^{\mu-m,p}\left\{f(z)\right\}\right\} & \big(m-1 \leqq \Re(\mu) < m\ \ (m\in\mathbb{N})\big) \end{cases} \tag{3.1}$$

and

$$\mathfrak{D}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}^{\mu,p}\left\{f(z)\right\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1}\ \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{pz^2}{(z-t)t}\right) f(t)\,dt & \big(\Re(\mu)<0\big) \\[2ex] \dfrac{d^m}{dz^m}\left\{\mathfrak{D}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}^{\mu-m,p}\left\{f(z)\right\}\right\} & \big(m-1 \leqq \Re(\mu) < m\ \ (m\in\mathbb{N})\big) \end{cases} \tag{3.2}$$

where, as also in (1.13), $\Re(p) \geqq 0$ and the path of integration in each of the Definitions (3.1) and (3.2) is a line in the complex t-plane from $t = 0$ to $t = z$.

Remark 2. The Definition (3.1) is easily recovered from (3.2) by specializing the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ as in (2.5). Moreover, by using the specialization indicated in (1.9), the Definition (3.1) reduces immediately to (1.13). For $p = 0$, the Definitions (1.13), (3.1) and (3.2) would obviously reduce at once to the familiar Riemann–Liouville Definition (1.10). Each of these and the aforementioned other specializations are fairly straightforward. Henceforth, therefore, we choose to state our results in their general forms only and leave the specializations as an exercise for the interested reader.

Making use of the Definition (3.2), we can easily derive the following analogue of the familiar fractional derivative Formula (1.11):

$$\mathfrak{D}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}^{\mu,p}\left\{z^\lambda\right\} = \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda+1,-\mu;p)}{\Gamma(-\mu)}\ z^{\lambda-\mu} \qquad \big(\Re(\lambda)>-1;\ \Re(\mu)<0\big) \tag{3.3}$$

which would readily yield Theorem 4 below.

Theorem 4. *In terms of a suitably bounded multiple sequence* $\{\Omega(m_1,\cdots,m_r)\}_{m_j\in\mathbb{N}_0\ (j=1,\cdots,r)}$*, let the multivariable function* $\Phi(x_1,\cdots,x_r)$ *be defined by*

$$\Phi(x_1, \cdots, x_r) = \sum_{m_1,\cdots,m_r=0}^{\infty} \Omega(m_1, \cdots, m_r) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!} \qquad (\max\{|x_1|, \cdots, |x_r|\} < \mathfrak{R}) \tag{3.4}$$

Then

$$\begin{aligned} &\mathfrak{D}^{\mu,p}_{z,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})} \left\{ z^{\lambda-1}\, \Phi(x_1 z^{\omega_1}, \cdots, x_r z^{\omega_r}) \right\} \\ &\quad = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{m_1,\cdots,m_r=0}^{\infty} \mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\lambda + \omega_1 m_1 + \cdots + \omega_r m_r, -\mu; p) \\ &\qquad \cdot \Omega(m_1, \cdots, m_r) \frac{(x_1 z^{\omega_1})^{m_1}}{m_1!} \cdots \frac{(x_r z^{\omega_r})^{m_r}}{m_r!} \end{aligned} \tag{3.5}$$

$$\big(\Re(p) \geqq 0;\ \Re(\lambda) > 0;\ \Re(\mu) < 0;\ \omega_j > 0\ \ (j = 1, \cdots, r);\ \max\{|x_1 z^{\omega_1}|, \cdots, |x_r z^{\omega_r}|\} < \mathfrak{R}\big)$$

provided that each member of (3.5) *exists.*

Proof. The Assertion (3.5) of Theorem 4 follows easily from the Definitions (3.2) and (2.3). We, therefore, skip the details involved. □

An interesting particular case of the fractional derivative Formula (3.5) asserted by Theorem 4 would occur when we specialize the sequence $\{\Omega(m_1, \cdots, m_r)\}_{m_j\in\mathbb{N}_0\ (j=1,\cdots,r)}$ as follows:

$$\Omega(m_1, \cdots, m_r) = (b_1)_{m_1} \cdots (b_r)_{m_r} \qquad \big(m_j \in \mathbb{N}_0\ \ (j = 1, \cdots, r)\big) \tag{3.6}$$

We thus obtain the following interesting generalization of a known result [14, p. 303, Problem 1]:

$$\begin{aligned} &\mathfrak{D}^{\mu,p}_{z,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})} \left\{ z^{\lambda-1} (1 - x_1 z^{\omega_1})^{-b_1} \cdots (1 - x_r z^{\omega_r})^{-b_r} \right\} = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{m_1,\cdots,m_r=0}^{\infty} (b_1)_{m_1} \cdots (b_r)_{m_r} \\ &\qquad \cdot \mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\lambda + \omega_1 m_1 + \cdots + \omega_r m_r, -\mu; p) \frac{(x_1 z^{\omega_1})^{m_1}}{m_1!} \cdots \frac{(x_r z^{\omega_r})^{m_r}}{m_r!} \end{aligned} \tag{3.7}$$

$$\big(\Re(p) \geqq 0;\ \Re(\lambda) > 0;\ \Re(\mu) < 0;\ \omega_j > 0\ \ (j = 1, \cdots, r);\ \max\{|x_1 z^{\omega_1}|, \cdots, |x_r z^{\omega_r}|\} < \mathfrak{R}\big)$$

provided that each member of (3.7) exists.

Since $\kappa_0 := 1$ in the Definition (2.1), in its *further* special case when

$$\omega_j = 1 \quad (j = 1, \cdots, r) \qquad \text{and} \qquad \mu \mapsto \lambda - \mu$$

this last result (3.7) can be written, in terms of the generalized extended Lauricella function $\mathfrak{F}^{(r)}_{D,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};p)}$ defined by (2.12), as follows:

$$\begin{aligned} &\mathfrak{D}^{\lambda-\mu,p}_{z,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})} \left\{ z^{\lambda-1} (1 - x_1 z)^{-b_1} \cdots (1 - x_r z)^{-b_r} \right\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \\ &\qquad \cdot \mathfrak{F}^{(r)}_{D,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};p)}(\lambda, b_1, \cdots, b_r; \mu; x_1 z, \cdots, x_r z) \end{aligned} \tag{3.8}$$

$$\big(\Re(p) \geqq 0;\ 0 < \Re(\lambda) < \Re(\mu);\ \max\{|x_1 z|, \cdots, |x_r z|\} < \mathfrak{R}\big)$$

which, for $p = 0$ or (alternatively) for

$$\kappa_\ell = 0 \qquad (\ell \in \mathbb{N}),$$

immediately yields the aforementioned known result [14, p. 303, Problem 1].

Yet another result would emerge when, in the *two-variable* ($r = 2$) case of the Definition (3.4), we set

$$\Omega(m_1, m_2) = (\alpha)_{m_1}(\alpha)_{m_2} \frac{\mathfrak{B}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\beta + m_1, \gamma - \beta)}{B(\beta, \gamma - \beta)}, \quad x_1 = \frac{x}{1-z}, \quad x_2 = z \quad \text{and} \quad \omega_1 = \omega_2 = 1$$

so that, by using the definition (2.4), we have

$$\Phi(x_1, x_2) = \Phi\left(\frac{x}{1-z}, z\right) = (1-z)^{-\alpha}\, \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left(\alpha, \beta; \gamma; \frac{x}{1-z}\right) \tag{3.9}$$

Now, just as in our demonstration of the Assertion (3.5) of Theorem 4, if we apply the fractional derivative formula (3.3) (with $\mu \mapsto \lambda - \mu$) to $z^{\lambda-1}$ times the Φ-function given by (3.9), we are led to the following result:

$$\begin{aligned} &\mathfrak{D}^{\lambda-\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)} \left\{ z^{\lambda-1}\,(1-z)^{-\alpha}\, \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left(\alpha, \beta; \gamma; \frac{x}{1-z}\right)\right\} \\ &\qquad = \frac{\Gamma(\lambda)}{\Gamma(\mu)}\, z^{\mu-1}\, \mathfrak{F}_2^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha, \beta, \lambda; \gamma, \mu; x, z; p) \\ &\qquad \big(\Re(p) \geqq 0;\ 0 < \Re(\lambda) < \Re(\mu);\ |x| + |z| < 1\big) \end{aligned} \tag{3.10}$$

where we have also used the Definition (2.11) for the generalized extended Appell function $\mathfrak{F}_2^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}$.

For $p = 0$ or (alternatively) for

$$\kappa_\ell = 0 \qquad (\ell \in \mathbb{N})$$

this last Formula (3.10) immediately yields a known result [14, p. 289, Equation 5.1(18)].

Remark 3. The Beta function $B(\alpha, \beta)$ defined (for $\min\{\Re(\alpha), \Re(\beta)\} > 0$) by

$$B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt =: B(\beta, \alpha) \qquad \big(\min\{\Re(\alpha), \Re(\beta)\} > 0\big) \tag{3.11}$$

can be continued analytically for $\max\{\Re(\alpha), \Re(\beta)\} < 0$ as follows (see, for example, [14, p. 26, Equation 1.1(48)]):

$$B(\alpha, \beta) := \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt & \big(\min\{\Re(\alpha), \Re(\beta)\} > 0\big) \\[3ex] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases} \tag{3.12}$$

$$\big(\mathbb{Z}_0^- := \{0, -1, -2, \cdots\} = \mathbb{Z}^- \cup \{0\}\big)$$

Thus, clearly, in their special cases when

$$p = 0 \qquad \text{or} \qquad \kappa_\ell = 0 \qquad (\ell \in \mathbb{N})$$

such *additional* constraints as $\Re(\mu) < 0$ in (3.3), (3.5) and (3.7), and $\Re(\lambda) < \Re(\mu)$ in (3.8) and (3.10), can be dropped fairly easily by applying *both* cases of the definition in (3.2).

4. Mellin Transforms of the Generalized Extended Fractional Derivatives

The Mellin transform of a suitably integrable function $f(t)$ with index s is defined, as usual, by

$$\mathcal{M}\{f(\tau):\tau\to s\}:=\int_0^\infty \tau^{s-1}\,f(\tau)\,d\tau \tag{4.1}$$

whenever the improper integral in (4.1) exists.

Theorem 5. *In terms of the generalized extended Gamma function* $\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(z)$ *defined by* (2.2), *the Mellin transforms of the following generalized extended fractional derivatives defined by* (3.2) *are given by*

$$\mathcal{M}\left\{\mathfrak{D}^{\mu,p}_{z,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}\left\{z^\lambda\right\}:p\to s\right\}=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\,\Gamma_0^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(s)B(\lambda+s+1,s-\mu) \tag{4.2}$$

$$\big(\Re(s)>0;\ \Re(\lambda)>-1;\ \Re(\mu)<0\big)$$

and

$$\begin{aligned}\mathcal{M}\left\{\mathfrak{D}^{\mu,p}_{z,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}\left\{z^\lambda(1-xz)^{-\alpha}\right\}:p\to s\right\}&=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\,\Gamma_0^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(s)\\&\cdot B(\lambda+s+1,s-\mu)\,{}_2F_1\left(\alpha,\lambda+s+1;\lambda-\mu+2s+1;xz\right)\end{aligned} \tag{4.3}$$

$$\big(\Re(s)>0;\ \Re(\lambda)>-1;\ \Re(\mu)<0;\ |xz|<1\big)$$

and, more generally, by

$$\begin{aligned}&\mathcal{M}\left\{\mathfrak{D}^{\mu,p}_{z,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}\left\{z^\lambda(1-x_1z^{\omega_1})^{-\alpha_1}\cdots(1-x_rz^{\omega_r})^{-\alpha_r}\right\}:p\to s\right\}\\&\quad=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\,\Gamma_0^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(s)\sum_{m_1,\cdots,m_r=0}^\infty(\alpha_1)_{m_1}\cdots(\alpha_r)_{m_r}\\&\quad\quad\cdot B\left(\lambda+\omega_1m_1+\cdots+\omega_rm_r+s+1,s-\mu\right)\,\frac{(x_1z^{\omega_1})^{m_1}}{m_1!}\cdots\frac{(x_rz^{\omega_r})^{m_r}}{m_r!}\end{aligned} \tag{4.4}$$

$$\big(\Re(s)>0;\ \Re(\lambda)>-1;\ \Re(\mu)<0;\ \max\{|x_1z^{\omega_1}|,\cdots,|x_1z^{\omega_1}|\}<1\big)$$

provided that each member of the Assertions (4.2), (4.3) *and* (4.4) *exists,* ${}_2F_1$ *being the Gauss hypergeometric function.*

Proof. Using the Definition (4.1) of the Mellin transform, we find from (3.2) that

$$\begin{aligned}&\mathcal{M}\left\{\mathfrak{D}^{\mu,p}_{z,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}\left\{z^\lambda\right\}:p\to s\right\}\\&\quad=\int_0^\infty p^{s-1}\left[\frac{1}{\Gamma(-\mu)}\int_0^z(z-t)^{-\mu-1}\ \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{pz^2}{(z-t)t}\right)t^\lambda\,dt\right]\,dp\\&\quad=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^\infty p^{s-1}\left[\int_0^1u^\lambda(1-u)^{-\mu-1}\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{p}{u(1-u)}\right)\,dt\right]\,dp\end{aligned} \tag{4.5}$$

where we have also set $t = zu$ and $dt = zdu$ in the inner t-integral. Upon interchanging the order of integration in (4.5), which can easily be justified by absolute convergence of the integrals involved under the constraints state with (4.2), we get

$$\begin{aligned} \mathcal{M}&\left\{\mathfrak{D}^{\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left\{z^\lambda\right\}:p\to s\right\} \\ &= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^1 u^\lambda(1-u)^{-\mu-1}\left[\int_0^\infty p^{s-1}\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{p}{u(1-u)}\right)\,dp\right]\,du \\ &= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^1 u^{\lambda+s}(1-u)^{s-\mu-1}\left[\int_0^\infty v^{s-1}\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-v\right)\,dv\right]\,du \end{aligned} \tag{4.6}$$

where we obviously have set

$$\frac{p}{u(1-u)} = v \qquad \text{and} \qquad dp = u(1-u)\,dv$$

in the inner p-integral. We now interpret the v-integral and the u-integral in (4.6) by means of the Definitions (2.2) (with $p = 0$) and (3.12), respectively. This evidently completes our derivation of the Mellin transform Formula (4.2) asserted by Theorem 5.

Alternatively, by substituting from (3.3) into the left-hand side of (4.2), we have

$$\mathcal{M}\left\{\mathfrak{D}^{\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left\{z^\lambda\right\}:p\to s\right\} = \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^\infty p^{s-1}\,\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda+1,-\mu;p)\,dp$$

$$\big(\Re(s)>0;\ \Re(\lambda)>-1;\ \Re(\mu)<0\big)$$

which would lead us once again to the Assertion (4.2) of Theorem 5.

In order to prove the Mellin transform Formula (4.3), we first write

$$\begin{aligned} \mathcal{M}&\left\{\mathfrak{D}^{\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left\{z^\lambda(1-xz)^{-\alpha}\right\}:p\to s\right\} \\ &= \sum_{n=0}^\infty(\alpha)_n\,\frac{x^n}{n!}\,\mathcal{M}\left\{\mathfrak{D}^{\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left\{z^{\lambda+n}\right\}:p\to s\right\} \\ &= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\,\Gamma_0^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(s)\,\sum_{n=0}^\infty(\alpha)_n\,\frac{(xz)^n}{n!}\,B(\lambda+n+s+1,s-\mu) \end{aligned} \tag{4.7}$$

$$\big(\Re(s)>0;\ \Re(\lambda)>-1;\ \Re(\mu)<0\big)$$

where we have used the already proven Assertion (4.2) of Theorem 5. The Assertion (4.3) of Theorem 5 would now follow upon interpreting the n-series in the last member of (4.7) as a Gauss hypergeometric function ${}_2F_1$.

Except for the obvious fact that the single n-series is to be replaced by the multiple $(m_1,\cdots,m_r)$-series, the demonstration of the third Assertion (4.4) of Theorem 5 would run parallel to that of the second Assertion (4.3). The details involved may thus be omitted here. □

The Mellin transform Formula (4.3) corresponds to the case $r = \omega_1 = 1$ of the general Result (4.4). Moreover, in its special case when $\alpha = 0$ (or when $x = 0$), (4.3) would reduce at once to the Mellin transform Formula (4.2).

In terms of the Lauricella hypergeometric function $F_D^{(r)}$ of r variables (see, for details, [14, p. 60, Equation 1.7(4)], the special case of the assertion (4.4) of Theorem 5 when $\omega_j = 1 \quad (j = 1, \cdots, r)$ yields the following Mellin transform formula:

$$\mathcal{M}\left\{\mathfrak{D}^{\mu,p}_{z,(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}\left\{z^\lambda(1-x_1z)^{-\alpha_1}\cdots(1-x_rz)^{-\alpha_r}\right\}:p\to s\right\} = \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\,\Gamma_0^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(s)$$
$$\cdot\, B(\lambda+s+1, s-\mu)\, F_D^{(r)}(\lambda+s+1,\alpha_1,\cdots,\alpha_r;\lambda-\mu+2s+1;x_1z,\cdots,x_rz) \tag{4.8}$$

$$\big(\Re(s)>0;\ \Re(\lambda)>-1;\ \Re(\mu)<0;\ \max\{|x_1z|,\cdots,|x_1z|\}<1\big)$$

which provides a multivariable hypergeometric extension of the Assertion (4.3) of Theorem 5. In particular, upon setting

$$\lambda = 0, \quad x = 1 \quad \text{and} \quad \kappa_\ell = 1 \quad (\ell\in\mathbb{N}_0)$$

in (4.2), if we make use of the Definition (3.1) (*with* $\rho = \sigma$), we obtain

$$\mathcal{M}\left\{\mathcal{D}^{\mu,p}_z\left\{(1-z)^{-\alpha}\right\}:p\to s\right\} = \frac{\Gamma(s)\,z^{-\mu}}{\Gamma(-\mu)}$$
$$\cdot\, B(s+1,s-\mu)\,{}_2F_1(\alpha,s+1;2s-\mu+1;z) \tag{4.9}$$

$$\big(\Re(s)>0;\ \Re(\lambda)>-1;\ \Re(\mu)<0;\ |z|<1\big)$$

which provides the *duly-corrected* version of a known result asserted recently by Özarslan and Özergin [1, p. 1832, Theorem 4.2].

5. A Set of Generating Functions

In this section, we derive linear and bilinear generating relations for the generalized extended hypergeometric functions in one, two and more variables (see Section 2) by following the methods which are described fairly adequately in the monograph by Srivastava and Manocha [14, Chapter 5]. Our main results are contained in Theorem 6 below.

Theorem 6. *Each of the following generating relations holds true for the generalized extended hypergeometric functions in one and more variables*:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,\mathfrak{F}_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\lambda+n,\alpha;\beta;z)\;t^n$$
$$= (1-t)^{-\lambda}\,\mathfrak{F}_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}\left(\lambda,\alpha;\beta;\frac{z}{1-t}\right) \qquad (|t|<1) \tag{5.1}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,\mathfrak{F}_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\nu-n,\alpha;\beta;x)\;t^n$$
$$= (1-t)^{-\lambda}\,\mathfrak{F}_1^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}\left(\nu,\lambda,\alpha;\beta;x,-\frac{xt}{1-t};p\right) \qquad (|t|<1) \tag{5.2}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda+n,\alpha;\beta;x)\, \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(-n,\gamma;\delta;y)\ t^n$$
$$= (1-t)^{-\lambda}\, \mathfrak{F}_2^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left(\lambda,\alpha,\gamma;\beta,\delta;\frac{x}{1-t},-\frac{yt}{1-t};p\right) \qquad (|t|<1) \tag{5.3}$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, \mathfrak{F}^{(r)}_{D,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};p\right)}(\alpha,\lambda+n,\beta_2,\cdots,\beta_r;\gamma;x_1,\cdots,x_r)\ t^n$$
$$= (1-t)^{-\lambda}\, \mathfrak{F}^{(r)}_{D,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};p\right)}\left(\alpha,\lambda,\beta_2,\cdots,\beta_r;\gamma;\frac{x_1}{1-t},x_2,\cdots,x_r\right) \qquad (|t|<1) \tag{5.4}$$

provided that each member of the generating relations (5.1) *to* (5.4) *exists.*

Proof. Our demonstration of Theorem 6 is based upon the generalized extended fractional derivative operator $\mathfrak{D}^{\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}$ defined by (3.2). We first rewrite the elementary identity:

$$[(1-z)-t]^{-\lambda} = (1-t)^{-\lambda}\left(1-\frac{z}{1-t}\right)^{-\lambda} \tag{5.5}$$

in the following form:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\,(1-z)^{-\lambda-n}\ t^n = (1-t)^{-\lambda}\left(1-\frac{z}{1-t}\right)^{-\lambda} \qquad (|t|<1) \tag{5.6}$$

Now, upon multiplying both sides of (5.6) by $z^{\alpha-1}$, if we apply the generalized extended fractional derivative operator $\mathfrak{D}^{\alpha-\beta,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}$ on each member of the resulting equation, we find that

$$\mathfrak{D}^{\alpha-\beta,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left\{\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, z^{\alpha-1}\,(1-z)^{-\lambda-n}\ t^n\right\}$$
$$= (1-t)^{-\lambda}\, \mathfrak{D}^{\alpha-\beta,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left\{z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\lambda}\right\} \qquad (|t|<1) \tag{5.7}$$

Interchanging the order of fractional differentiation and summation in (5.7), which can be justified when

$$\Re(\alpha)>0 \qquad \text{and} \qquad |t|<|1-z|,$$

we find from (5.7) that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, \mathfrak{D}^{\alpha-\beta,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left\{z^{\alpha-1}\,(1-z)^{-\lambda-n}\ t^n\right\}$$
$$= (1-t)^{-\lambda}\, \mathfrak{D}^{\alpha-\beta,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left\{z^{\alpha-1}\left(1-\frac{z}{1-t}\right)^{-\lambda}\right\} \tag{5.8}$$

which, by means of some obvious special cases of (3.8), yields the first Assertion (5.1) of Theorem 6 under the constraint derivable by appealing finally to the principle of analytic continuation.

Since

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, t^n = (1-t)^{-\lambda} \qquad (|t| < 1) \tag{5.9}$$

a *direct* proof of the generating relation (5.1), *without* using the generalized extended fractional derivative operator $\mathfrak{D}^{\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}$ defined by (3.2), can be given along the following lines:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda+n,\alpha;\beta;z)\; t^n \\
&\quad = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\sum_{k=0}^{\infty} (\lambda+n)_k\, \frac{\mathfrak{B}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha+k,\beta-\alpha)}{B(\alpha,\beta-\alpha)}\, \frac{z^k}{k!} \right) t^n \\
&\quad = \sum_{k=0}^{\infty} (\lambda)_k\, \frac{\mathfrak{B}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha+k,\beta-\alpha)}{B(\alpha,\beta-\alpha)}\, \frac{z^k}{k!} \left(\sum_{n=0}^{\infty} \frac{(\lambda+k)_n}{n!}\, t^n \right) \\
&\quad = (1-t)^{-\lambda} \sum_{k=0}^{\infty} (\lambda)_k\, \frac{\mathfrak{B}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha+k,\beta-\alpha)}{B(\alpha,\beta-\alpha)}\, \frac{\{z/(1-t)\}^k}{k!} \\
&\quad = (1-t)^{-\lambda}\, \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left(\lambda,\alpha;\beta;\frac{z}{1-t}\right) \qquad (|t| < 1)
\end{aligned} \tag{5.10}$$

where we have only used the Definition (2.4) in conjunction with the expansion Formula (5.9).

The proof of the second Assertion (5.2) makes similar use of the generalized extended fractional derivative operator $\mathfrak{D}^{\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}$ defined by (3.2) together with the following elementary identity:

$$[1-(1-z)t]^{-\lambda} = (1-t)^{-\lambda} \left(1+\frac{zt}{1-t}\right)^{-\lambda} \tag{5.11}$$

instead of the Identity (5.5).

Next, upon setting $z = x$ and $t \mapsto (1-y)t$ in (5.1), if we multiply the resulting equation by $y^{\gamma-1}$ and then apply the generalized extended fractional derivative operator $\mathfrak{D}^{\gamma-\delta,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}$ together with the elementary Identity (5.11), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\lambda+n,\alpha;\beta;x)\, \mathfrak{D}^{\gamma-\delta,p}_{y,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)} \left\{ y^{\gamma-1}\,[(1-y)t]^n \right\} \\
&\quad = (1-t)^{-\lambda}\, \mathfrak{D}^{\gamma-\delta,p}_{y,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)} \left\{ y^{\gamma-1} \left(1+\frac{yt}{1-t}\right)^{-\lambda} \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\left(\lambda,\alpha;\beta;\frac{x}{1-(1-y)t}\right) \right\}
\end{aligned} \tag{5.12}$$

which, in light of (3.10) as well as some obvious special cases of (3.8), leads us eventually to the bilinear generating Relation (5.3) asserted by Theorem 6.

Finally, the proof of the Assertion (5.4) is much akin to that of (5.1). In fact, the role played by the argument x_1 in (5.4) can be assumed instead by any of the other arguments $x_2, \cdots, x_r$. □

6. Concluding Remarks and Observations

In our present investigation, we have introduced and studied a *further* generalization of the extended fractional derivative operator related to a generalized Beta function, which was used in order to obtain some linear and bilinear generating relations involving the extended hypergeometric functions [1]. We have applied the generalized extended fractional derivative operator to derive generating relations for the generalized extended Gauss, Appell and Lauricella hypergeometric functions in one, two and more variables. Many other properties and relationships involving (for example) Mellin transforms and the generalized extended fractional derivative operator are also given.

It may be of interest to observe in conclusion that many of the definitions, which we have considered in this paper, can be *further* extended by introducing one *additional* parameter q $\big($*with* $\Re(q) \geqq 0\big)$. Thus, in terms of the Θ-function given by (2.1), we can introduce a *further* extension of the generalized extended Beta function in (2.3) as follows:

$$\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta;p,q) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\frac{p}{t}-\frac{q}{1-t}\right)dt \tag{6.1}$$

$$\big(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \min\{\Re(p),\Re(q)\}\geqq 0\big)$$

The corresponding *further* extensions of the Definitions (2.4) and (2.10) to (2.12) are given by

$$\mathfrak{F}_{p,q}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b;c;z) := \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\,\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b+n,c-b;p,q)\,\frac{z^n}{n!} \tag{6.2}$$

$$\big(|z|<1;\ \Re(c)>\Re(b)>0;\ \min\{\Re(p),\Re(q)\}\geqq 0\big)$$

$$\mathfrak{F}_1^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b,b';c;x,y;p,q) := \sum_{m,n=0}^{\infty}(b)_m(b')n\,\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a+m+n,c-a;p,q)}{B(a,c-a)}\,\frac{x^m}{m!}\,\frac{y^n}{n!} \tag{6.3}$$

$$\big(\max\{|x|,|y|\}<1;\ \min\{\Re(p),\Re(q)\}\geqq 0\big)$$

$$\mathfrak{F}_2^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a,b,b';c,c';x,y;p,q)$$
$$:= \sum_{m,n=0}^{\infty}(a)_{m+n}\,\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b+m,c-b;p,q)\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b'+n,c'-b';p,q)}{B(b,c-b)B(b',c'-b')}\,\frac{x^m}{m!}\,\frac{y^n}{n!} \tag{6.4}$$

$$\big(|x|+|y|<1;\ \min\{\Re(p),\Re(q)\}\geqq 0\big)$$

and

$$\mathfrak{F}^{(r)}_{D,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};p,q\right)}(a,b_1,\cdots,b_r;c;x_1,\cdots,x_r)$$
$$:= \sum_{m_1,\cdots,m_r=0}^{\infty}(b_1)_{m_1}\cdots(b_r)_{m_r}\,\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a+m_1+\cdots+m_r,c-a;p,q)}{B(a,c-a)}\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \tag{6.5}$$

$$\big(\max\{|x_1|,\cdots,|x_r|\}<1;\ \min\{\Re(p),\Re(q)\}\geqq 0\big)$$

respectively. Moreover, the fractional derivative operator $\mathfrak{D}^{\mu,p}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}$ defined by (3.2) can be *further* extended as follows:

$$\mathfrak{D}^{\mu,p,q}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1}\ \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\dfrac{pz}{t} - \dfrac{qz}{z-t}\right) f(t)\,dt & \left(\Re(\mu) < 0\right) \\ \dfrac{d^m}{dz^m}\left\{\mathfrak{D}^{\mu-m,p,q}_{z,\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}\{f(z)\}\right\} & \left(m-1 \leqq \Re(\mu) < m\ (m\in\mathbb{N})\right) \end{cases} \tag{6.6}$$

where $\min\{\Re(p), \Re(q)\} \geqq 0$ and, as also in (1.10), (1.13), (3.1) *and* (3.2), the path of integration in the Definition (6.6) is a line in the complex t-plane from $t = 0$ to $t = z$.

Since

$$-\frac{p}{t} - \frac{q}{1-t} = -\left(\frac{p+(q-p)t}{t(1-t)}\right) \qquad \left(\min\{\Re(p), \Re(q)\} \geqq 0\right)$$

the definitions in (6.1) to (6.6) would obviously coincide with the corresponding definitions in the preceding sections when we set the *additional* parameter $q = p$. Most (*if not all*) of the properties and results, which we have investigated in this paper in the $p = q$ case, can indeed be considered analogously for the $p \neq q$ case in a rather simple and straightforward manner. The details involved may, therefore, be left as an exercise for the interested reader.

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