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Communication

Hasse-Schmidt Derivations and the Hopf Algebra of Non-Commutative Symmetric Functions

Michiel Hazewinkel

Burg. 's Jacob Laan 18, NL-1401BR BUSSUM, The Netherlands; E-Mail: michhaz@xs4all.nl

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Abstract: Let **NSymm** be the Hopf algebra of non-commutative symmetric functions (in an infinity of indeterminates): **NSymm=Z**$\langle Z_1, Z_2, ... \rangle$. It is shown that an associative algebra *A* with a Hasse-Schmidt derivation $d = (id, d_1, d_2, ...)$ on it is exactly the same as an **NSymm** module algebra. The primitives of **NSymm** act as ordinary derivations. There are many formulas for the generators Z_i in terms of the primitives (and vice-versa). This leads to formulas for the higher derivations in a Hasse-Schmidt derivation in terms of ordinary derivations, such as the known formulas of Heerema and Mirzavaziri (and also formulas for ordinary derivations in terms of the elements of a Hasse-Schmidt derivation). These formulas are over the rationals; no such formulas are possible over the integers. Many more formulas are derivable.

Keywords: non-commutative symmetric functions; Hasse-Schmidt derivation; higher derivation; Heerema formula; Mirzavaziri formula; non-commutative Newton formulas

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1. Introduction

Let *A* be an associative algebra (or any other kind of algebra for that matter). A derivation on *A* is an endomorphism ∂ of the underlying Abelian group of *A* such that

$$\partial(ab) = a(\partial b) + (\partial a)b \text{ for all } a, b \in A \tag{1.1}$$

A Hasse-Schmidt derivation is a sequence $(d_0 = id, d_1, d_2, ..., d_n, ...)$ of endomorphisms of the underlying Abelian group such that for all $n \geq 1$.

$$d_n(ab) = \sum_{i=0}^{n} (d_i a)(d_{n-i} b) \tag{1.2}$$

Note that d_1 is a derivation as defined by Equation 1.1. The individual d_n that occurs in a Hasse-Schmidt derivation is also sometimes called a higher derivation.

A question of some importance is whether Hasse-Schmidt derivations can be written down in terms of polynomials in ordinary derivations. For instance, in connection with automatic continuity for Hasse-Schmidt derivations on Banach algebras.

Such formulas have been written down by, for instance, Heerema and Mirzavaziri in [1,2]. They also will be explicitly given below.

It is the purpose of this short note to show that such formulas follow directly from some easy results about the Hopf algebra **NSymm** of non-commutative symmetric functions. In fact this Hopf algebra constitutes a universal example concerning the matter.

2. Hopf Algebras and Hopf Module Algebras

Everything will take place over a commutative associative unital base ring *k*; unadorned tensor products will be tensor products over *k*. In this note *k* will be the ring of integers **Z**, or the field of rational numbers **Q**.

Recall that a Hopf algebra over *k* is a *k*-module *H* together with five *k*-module morphisms $m : H \otimes H \longrightarrow H$, $e : k \longrightarrow H$, $\mu : H \longrightarrow H \otimes H$, $\varepsilon : H \longrightarrow k$, $\iota : H \longrightarrow H$ such that (H, m, e) is an associative *k*-algebra with unit, (H, μ, ε) is a co-associative co-algebra with co-unit, μ and ε are algebra morphisms (or, equivalently, that *m* and *e* are co-algebra morphisms), and such that ι satisfies $m(\iota \otimes \mathrm{id})\mu = \varepsilon e$, $m(\mathrm{id} \otimes \iota)\mu = \varepsilon e$. The antipode ι will play no role in what follows. If there is no antipode (specified) one speaks of a bi-algebra. For a brief introduction to Hopf algebras (and co-algebras) with plenty of examples see Chapters 2 and 3 of [3].

Recall also that an element $p \in H$ is called primitive if $\mu(p) = p \otimes 1 + 1 \otimes p$. These form a sub-*k*-module of *H* and form a Lie algebra under the commutator difference product (p, p') a $pp' - p'p$. I shall use $Prim(H)$ to denote this *k*-Lie-algebra.

Given a Hopf algebra over *k*, a Hopf module algebra is a *k*-algebra *A* together with an action of the underlying algebra of *H* on (the underlying module of) *A* such that:

$$h(ab) = \sum_{(h)} (h_{(1)} a)(h_{(2)} b) \text{ for all } a, b \in A \text{ , and } h(1) = \varepsilon(h)1 \text{ where } \mu(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} \tag{2.1}$$

and where I have used Sweedler-Heynemann notation for the co-product.

Note that this means that the primitive elements of *H* act as derivations.

3. The Hopf Algebra NSymm of Non-Commutative Symmetric Functions

As an algebra over the integers **NSymm** is simply the free associative algebra in countably many (non-commuting) indeterminates, **NSymm=Z**$\langle Z \rangle = \mathbf{Z}\langle Z_1, Z_2, ... \rangle$. The comultiplication and counit are given by

$$\mu(Z_n) = \sum_{i+j=n} Z_i \otimes Z_j \text{ , where } Z_0 = 1 \text{ , } \varepsilon(1) = 1, \ \varepsilon(Z_n) = 0 \text{ for } n \geq 1 \tag{3.1}$$

As **NSymm** is free as an associative algebra, it is no trouble to verify that this defines a bi-algebra. The seminal paper [4] started the whole business of non-commutative symmetric functions, and is now a full-fledged research area in its own right.

Now consider an **NSymm** Hopf module, algebra *A*. Then, by Equations 2.1 and 3.1 the module endomorphims defined by the actions of the Z_n, $n \geq 1$, $d_n(a) = Z_n a$, define a Hasse-Schmidt derivation. Conversely, if *A* is a *k*-algebra together with a Hasse-Schmidt derivation one defines a **NSymm** Hopf module algebra structure on *A* by setting $Z_n a = d_n(a)$. This works because **NSymm** is free as an algebra.

Thus an **NSymm** Hopf module algebra *A* is precisely the same thing as a *k*-algebra *A* together with a Hasse-Schmidt derivation on it and the matter of writing the elements of the sequence of morphisms that make up the Hasse-Schmidt derivation in terms of ordinary derivations comes down to the matter of finding enough primitives of **NSymm** so that the generators, Z_n, can be written as polynomials in these primitives.

4. The Newton Primitives of NSymm

Define the non-commutative polynomials P_n and P'_n by the recursion formulas

$$\begin{aligned} P_n &= nZ_n - (Z_{n-1}P_1 + Z_{n-2}P_2 + \ldots + Z_1 P_{n-1}) \\ P_n' &= nZ_n - (P_1'Z_{n-1} + P_2'Z_{n-2} + \ldots + P'_{n-1}Z_1) \end{aligned} \tag{4.1}$$

These are non-commutative analogues of the well known Newton formulas for the power sums in terms of the complete symmetric functions in the usual commutative theory of symmetric functions. It is not difficult to write down an explicit expression for these polynomials:

$$P_n(Z) = \sum_{\substack{i_1+\ldots+i_m=n \\ i_j \in \mathbf{N}}} (-1)^{m+1} i_m Z_{i_1} Z_{i_2} \ldots Z_{i_m} \tag{4.2}$$

Nor is it difficult to write down a formula for the Z_n in terms of the *P*'s or *P*''s. However, to do that one definitely needs to use rational numbers and not just integers [5]. For instance

$$Z_2 = \frac{P_1^2 + P_2}{2}$$

The key observation is now:

4.3. Proposition

The elements P_n and P'_n are primitive elements of the Hopf algebra **NSymm**.

The proof is a straightforward uncomplicated induction argument using the recursion Formulas 4.1. See e.g., [3], page 147.

Using the P'_n an immediate corollary is the following main theorem from [2].

4.4. Theorem

Let A be an associative algebra over the rational numbers **Q** and let $(id, d_1, d_2, ..., d_n, ...)$ be a Hasse-Schmidt derivation on it. Then the δ_n defined recursively by

$$\delta_n = nd_n - \delta_1 d_{n-1} - ... - \delta_{n-1} d_1 \tag{4.5}$$

are ordinary derivations and

$$d_n = \sum_{\substack{r_1+r_2+...+r_m=n \\ r_j \in \mathbf{N}}} c_{r_1,r_2,...,r_m} \delta_{r_1} \delta_{r_2} ... \delta_{r_m} \tag{4.6}$$

where

$$c_{r_1,r_2,...,r_m} = \frac{1}{r_1 + r_2 + ... + r_m} \frac{1}{r_2 + ... + r_m} ... \frac{1}{r_{m-1} + r_m} \frac{1}{r_m} \tag{4.7}$$

4.8. Comment

Because

$$P_n' \equiv nZ_n \bmod(Z_1, Z_2, ..., Z_{n-1})$$

the formulas expressing the Z_n in terms of the P_n' are unique and so denominators are really needed.

4.9. Comment and Example

There are many more primitive elements in **NSymm** than just the P_n' and P_n. One could hope that by using all of them, integral formulas for the Z_n in terms of primitives would become possible. This is not the case. The full Lie algebra of primitives of **NSymm** was calculated in [6]. It readily follows from the description there that $\mathbf{Z}\langle Prim(\mathbf{NSymm})\rangle$, the sub-algebra of **NSymm** generated by all primitive elements is strictly smaller than **NSymm**. In fact much smaller in a sense that is specified in locus citandi. Thus the theorem does not hold over the integers.

A concrete example of a Hasse-Schmidt derivation of which the constituting endomorphisms cannot be written as integral polynomials in derivations can be given in terms of **NSymm** itself, as follows: The Hopf algebra **NSymm** is graded by giving Z_n degree *n*. Note that each graded piece is a free **Z**-module of finite rank. Let **QSymm**, often called the Hopf algebra of quasi-symmetric functions, be the graded dual Hopf algebra. Then each Z_n defines a functional $\alpha_n : \mathbf{QSymm} \longrightarrow \mathbf{Z}$. Now define an endomorphism d_n of **QSymm** as the composed morphism

$$\mathbf{QSymm} \xrightarrow{\mu_{\mathbf{QSymm}}} \mathbf{QSymm} \otimes_{\mathbf{Z}} \mathbf{QSymm} \xrightarrow{\mathrm{id} \otimes \alpha_n} \mathbf{QSymm}$$

Then the d_n form a Hasse-Schmidt derivation of which the components cannot be written as integer polynomials in ordinary derivations.

5. The Hopf Algebra LieHopf

In [1] a formula for manufacturing Hasse-Schmidt derivations from a collection of ordinary derivations is shown that is more pleasing—at least to me—than 4.6. This result from locus citandi can

be strengthened to give a theorem similar to Theorem 4.4 but with more symmetric formulae. This involves another Hopf algebra over the integers which I like to call **LieHopf**.

As an algebra **LieHopf** is again the free associative algebra in countably many indeterminates $\mathbf{Z}\langle U\rangle = \mathbf{Z}\langle U_1, U_2, ...\rangle$. However, this time the co-multiplication and co-unit are defined by

$$\mu(U_n) = U_n \otimes 1 + 1 \otimes U_n\ ,\ \varepsilon(U_n) = 0 \tag{5.1}$$

so that all the U_n are primitive. Also, in fact the Lie algebra of primitives of this Hopf algebra is the free Lie algebra on countably many generators.

Over the integers **LieHopf** and **NSymm** are very different but over the rationals they become isomorphic. There are very many isomorphisms. A particularly nice one is given in considering the power series identity

$$1 + Z_1 t + Z_2 t^2 + Z_3 t^3 + ... = \exp(U_1 t + U_2 t^2 + U_3 t^3 + ...) \tag{5.2}$$

which gives the following formulae for the *U*'s in terms of the *Z*'s and vice versa.

$$Z_n(U) = \sum_{r_1+...+r_m=n} \frac{U_{r_1} U_{r_2} ... U_{r_m}}{m!} \tag{5.3}$$

$$U_n(Z) = \sum_{r_1+...+r_m=n} (-1)^{m+1} \frac{Z_{r_1} Z_{r_2} ... Z_{r_m}}{m} \tag{5.4}$$

For two detailed proofs that these formulas do indeed give an isomorphism of Hopf algebras see [7]; or see Chapter 6 of [3]. In terms of derivations, reasoning as above in Section 4, this gives the following theorem.

5.5. Theorem

Let *A* be an algebra over the rationals and let $(id, d_1, d_2, ...)$ be a Hasse-Schmidt derivation on it. Then the ∂_n defined by

$$\partial_n = \sum_{r_1+...+r_m=n} (-1)^{m+1} \frac{d_{r_1} d_{r_2} ... d_{r_m}}{m} \tag{5.6}$$

are (ordinary) derivations and

$$d_n = \sum_{r_1+...+r_m=n} \frac{\partial_{r_1} \partial_{r_2} ... \partial_{r_m}}{m!} \tag{5.7}$$

5.8. Comment

Perhaps I should add that for any given collection of ordinary derivations, Formula 5.7 yields a Hasse-Schmidt derivation. That is the theorem from [1] with which I started this section.

6. Conclusions

Hasse-Schmidt derivations on an associative algebra *A* are exactly the same as Hopf module algebra structures on *A* for the Hopf algebra **NSymm**. This leads to formulas connecting ordinary derivations to

higher derivations.

It remains to explore this phenomenon for other kinds of algebras.

The dual of **NSymm** is **QSymm**, the Hopf algebra of quasi-symmetric functions. It remains to be clarified what a coalgebra comodule over **QSymm** means in terms of coderivations. There are also other (mixed) variants to be further explored.

References and Notes

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